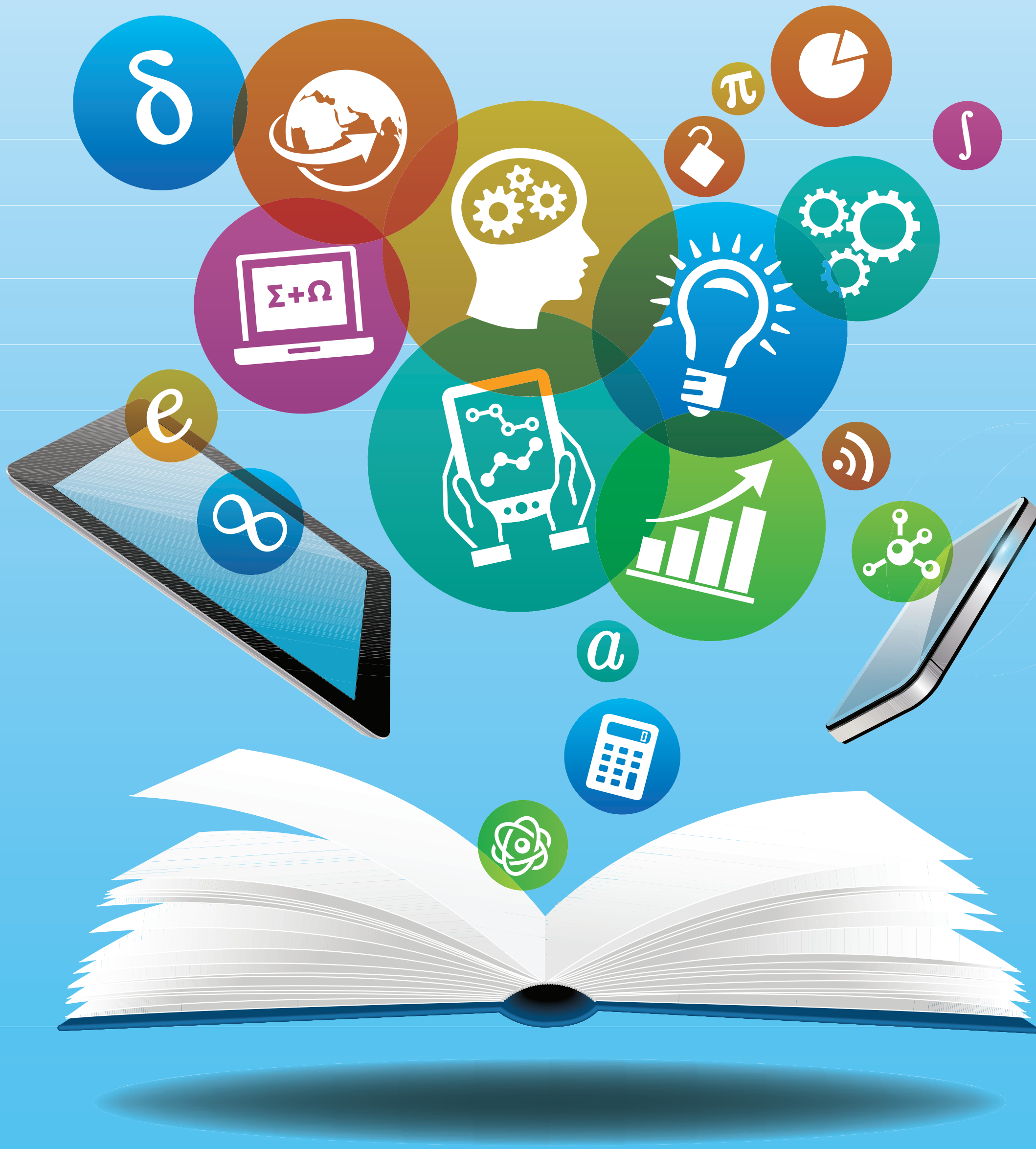
How to Teach Mathematics

THIRD EDITION

Steven G. Krantz



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Third Edition

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10 9 8 7 6 5 4 3 2 1 20 19 18 17 16 15

To Robert L. Borrelli, teacher and friend.

Contents

Preface to the Third Edition

It has been fifteen years since the appearance of the Second Edition of this book. A lot has happened since then. The teaching reform movement has become a fact of life. Many of us have had occasion to rethink how and why we teach. Many of us have taken time to learn about the myriad of new (often OnLine) teaching devices that are available today. Among these are MOOCs (massive open online courses), the Khan Academy, flipped classrooms, clickers, smartboards, and the list goes on at some length.

Our goal with this new edition is to present a streamlined approach to our teaching philosophy. Many found the First Edition of this book to be attractive because it was only 80 pages. The Second Edition was 300 pages. It offered much more, but was correspondingly more cumbersome. One could easily read the First Edition during a long lunch hour. The Second Edition represented more of an investment of time and effort.

This Third Edition will be a slimmed-down version of the key ideas in the first two editions. We still want to emphasize the nuts and bolts of good teaching: prepare, respect your students, be flexible, be knowledgeable, be of good spirit, be a role model, and prepare some more. We have eliminated several sections which, from today's perspective, appear to be redundant. And we have revised and modified several other sections.

The other goal of this new version is to explore many of the new OnLine learning tools that are now available. Some of these will make little sense to the traditionally trained instructor. Others will be fascinating, and will give us new ideas of things to try.

One of the innovations in the Second Edition was the inclusion of ten Appendices by other mathematicians with strong views about mathematics teaching. Some of these scholars agreed with me, and some of them (very politely) disagreed with me. In this new edition, I omit these Appendices. But they are available at the Web site

`www.math.wustl.edu/~sk/teachapps.pdf`

These Appendices still have value, and offer many ideas of intrinsic interest. But, in the pursuit of brevity, we have consigned them to an ancillary venue.

It is a pleasure to thank Lynn Apfel and James Walker for a careful reading of various versions of this new edition, and for offering innumerable sage comments and suggestions. Dave Bressoud worked assiduously to bring me up to speed on everything that has been happening in math teaching in the past fifteen years. I have Bressoud to thank for much of what is interesting and modern in this new edition. I also thank my Editors Edward Dunne and Sergei Gelfand for their wisdom and guidance.

It is our hope that this new edition of *How to Teach Mathematics* will speak to a new generation of budding mathematics instructors, and inspire them to new strata of excellence in teaching.

Steven G. Krantz
St. Louis, Missouri

Preface to the Second Edition

"[When a mathematician speaks about teaching], colleagues smile tolerantly to one another in the same way family members do when grandpa dribbles his soup down his shirt." Herb Clemens wrote these words in 1988. They were right on point at the time. The amazing fact is that they are no longer true.

Indeed the greatest single achievement of the so-called "teaching reform" movement is that it has enabled, or compelled, all of us to be concerned about teaching. Never mind the shame that in the past we were *not* concerned about teaching. Now we are all concerned, and that is good.

Of course there are differing points of view. The "reform" school of thought favors discovery, cooperative and group learning, use of technology, higher-order skills, and it downplays rote learning and drill. The traditionalists, by contrast, want to continue giving lectures, want the students to do traditional exercises, want the students to take the initiative in the learning process, and want to continue to drill their students.[1] Clearly there are merits in both points of view. The good news is that the two sides are beginning to talk to each other. The evidence? **(1)** A conference held at MSRI in December, 1996 with the sole purpose of helping the two camps to communicate (see the Proceedings in [**GKM**]); **(2)** The observation that basic skills play a new role, and are positioned in a new way, in the reform curriculum; **(3)** The observation that standard lectures—the stock-in-trade of traditionalists—are not the final word on engaging students in the learning process; **(4)** The fact that studies indicate that neither method is more effective than the other, but that both have strengths; **(5)** The new wave of calculus books (see [**STEW**]) that attempt a marriage of the two points of view.

The reader of this book may as well know that I am a traditionalist, but one who sees many merits in the reform movement. For one thing, the reform movement has taught us to reassess our traditional methodologies. It has taught us that there is more than one way to get the job done. And it has also taught us something about the sociological infrastructure of twentieth-century mathematics. We see that our greatest pride is also our Achilles heel. In detail, the greatest achievement of twentieth-century mathematics is that we have (to the extent possible) fulfilled the Hilbert/Bourbaki program of putting everything on a rigorous footing; we have axiomatized our subject; we have precise definitions of everything. The bad news is

[1] As I will say elsewhere in the book, the reformers constitute a heterogeneous group, just like the traditionalists. There is no official reform dogma, just as there is no official traditionalist dogma. Some reformers tell me that they strongly favor drill, but that drill should be built atop a bedrock of understanding. Many traditionalists seem to prefer to give the drill first—asking the students to take it on faith—and *then* to develop understanding. George Andrews has asked whether, if instead of calling it "mere rote learning" we called it "essential drill", would people view it differently?

that these accomplishments have shaped our world view, all the way down into the calculus classroom. Because we have taught ourselves to think strictly according to Occam's Razor, we also think that that should be the mode of discourse in the calculus classroom. This view is perhaps shortsighted.

First, students (and others, too!) do not generally learn *axiomatically* (from the top down). In many instances it is more natural for them to learn *inductively* (from the bottom up). Of course the question of how people learn has occupied educational theorists as far back as Beth and Piaget [**BPI**], and will continue to do so. But, as I say elsewhere in this book, the mathematics instructor must realize that a student cannot stare at a set of axioms and "see what is going on" in the same way that an experienced mathematician can. Often it is more natural for the student to first latch on to an example.

Second, we must realize that the notion of "proof" is a relative thing. Mathematical facts, or theorems, are freestanding entities. They have a life of their own. But a proof is largely a psychological device for convincing someone that something is true. A trained mathematician is taught a formalism for producing a proof that will be acceptable to his colleagues. But a freshman in college is not. What constitutes a believable proof for a freshman could easily be a good picture, or a plausibility argument. This insight alone can turn an ordinary teacher into a good one. What is the sense in showing a room full of freshmen a perfectly rigorous proof (of the fundamental theorem of calculus, say), secure in the knowledge that you have "done the right thing," but also knowing unconsciously that the students did not understand a word of it? Surely it is more gentle, as a didactic device, to replace "*Proof:*" with "Here is an idea about why this is true." In doing the latter, you have not been dishonest (i.e., you have not claimed that something was a strictly rigorous proof when in fact it was not). You have instead met the students half way. You have spoken to them in their own language. You have appealed to their collective intuition. Perhaps you have taught them something. Always keep in mind that persuasion has many faces.

I have witnessed discussions in which certain individuals were adamant that, if you give an explanation in a calculus class that is not strictly a proof, then you must say, "This is not a proof; it is an informal explanation." Of course such a position is a consequence of twentieth-century mathematical values, and I respect it. But I do not think that it constitutes good teaching. In the first place, such a mantra is both tiresome and discouraging for the students. The instructor can instead say, "Let's think about why this is true ..." or "Here is a picture that shows what is going on ..." and thereby convey the same message in a much friendlier fashion.

In my own mathematics department we have a "transitions" course, in which students are taught first-order logic, naive set theory, equivalence relations and classes, the constructions of the number systems, and the axiomatic method. They are also taught—at a very rudimentary level—how to construct their own proofs. Typically a student takes this course *after* calculus, linear algebra, and ordinary differential equations but *before* abstract algebra and real analysis. I think of the transitions course as a bellwether. Before that course, students are not ready for formal proofs. We should adapt our teaching methodology to *their* argot. After the transitions course, the students are more sophisticated. Now they are ready to learn *our* argot.

I have decided to write this new edition of *How to Teach Mathematics* in part because I have learned a lot about teaching in the five-year interval since this book first appeared. The teaching reform movement has matured, and so have the rest of us. I believe that I now know a lot more about what constitutes good teaching. I regularly teach our graduate student seminar to help prepare our Ph.D. candidates for a career in teaching, and I have an ever better understanding of how to conduct such training. I would like to share my new insights in this edition.

One of the best known mathematical errors, particularly in the study of an optimization problem, is to assume that the problem has a solution. Certainly Riemann's original proof of the Riemann mapping theorem is a dramatic example of this error, but the calculus of variations (for instance) is littered with other examples. Why can we not apply this hard-won knowledge to other aspects of our professional lives? Why do we assume that there is a "best" way to teach calculus? Or a "best" textbook? Teaching is a very personal activity, and different individuals will do it differently. Techniques that work for one person will not work for another. (Also, techniques that work in one class will not necessarily work in another.) I believe that we need, as a group, to acknowledge that there is a pool of worthwhile teaching techniques, and we should each choose those methods that work for us and for our students.

Ever since the first edition of this book appeared, mathematicians have approached me and asked, "OK, what's the secret? Students these days drive me crazy. I can't get through to them. They won't talk to me. How do you do it?" I wish that I had a simple answer. I would like to be able to say, "Take this little green pill." or "Say this prayer in the morning." or "Hold your mouth this way." But in fact there is no simple answer. Even so, I have invested considerable time analyzing the situation as well as talking to other successful teachers about how to make the teaching process work. I have come to the following conclusion.

Students are like dogs: They can smell fear. (I do not mean to say here that we should think of our students as attack dogs. Rather, they are sensitive to body language and to nuances of behavior. See also Section 2.9 on teaching evaluations.) When you walk into your classroom, the students can tell right away whether you really want to be there, whether you have something interesting to tell them, whether you respect them as people. If they sense instead that you are merely slogging through this dreary duty, just writing the theorems and proofs on the blackboard, refusing to answer questions for lack of time, then they will react to you in a correspondingly lackluster manner.

When I walk into my calculus class, I look forward to seeing the students perk up, with a look on their faces that says "Showtime!" In the few minutes before the formal class begins, I chat with them, joke around, find out what is going on in their lives. I relate to them as people. It will never happen that a student will go to the chair or the dean and complain about me. Why? Because they know that they can come and talk to *me* about their concerns. If a student is not doing well in my class, that student is comfortable coming to me. And he knows that the fault for his poor performance is as likely *his* as it is mine, because he realizes that I am doing everything that I can. If you believe what I am describing here, then perhaps you can also understand why I enjoy teaching, and why I find the process both stimulating and fulfilling.

I recently taught a fairly rigorous course in multivariable calculus—a subject in which students usually have a lot of trouble. The main reason that they have so much trouble is that there are so many ideas—vectors, cross products, elements of surface area, orientation, conservative vector fields, line integrals, tangent planes, etc.—and they are all used together. Just understanding how to calculate both sides of the equation in Stokes's theorem, or the divergence theorem, requires a great deal of machinery. The way that I addressed their difficulties is that I worked the students hard. I gave long, tough homework assignments. A day or two before any given assignment was due, I would begin a class discussion of the homework. If necessary, I would work out the bulk of a problem on the board for them. But I would add that I expected each of them to write up the problem carefully and completely—with full details. And I would give them a few extra days so that they could complete the assignment. But I did not stop there. Next class, I would ask how the homework was going. If necessary, we would discuss it again. If necessary, I would give them another extension. The point here is that I made it absolutely clear to the students that the most important thing to me was that they would complete the assignment. I would give them whatever time, and whatever help, was needed to complete the work. During the long fifteen-week semester, attendance in the class was virtually constant, and always exceeded 95%. At the end, I gave them a long, tough final exam. And the average was 85%. I can only conclude that I set a standard for these students, and they rose to it. Both they and I came away from the course with a feeling of success. They had worked hard, and they had learned something.

You may be thinking, "Well, Krantz teaches at a fancy private school with fancy private students. I could never get away with this at Big State University." That is a defeatist attitude. If you expect your students to try, then you must try. I have taught at big state universities. I understand the limitations that teaching a large class of not particularly select students imposes. But you can adjust the techniques described in the last paragraph to most any situation. If you wonder how I can afford to spend valuable class time going over homework, my answer is this: I am an experienced teacher, and fourteen weeks is a long time. I can always adjust future classes, leave out a few examples, give short shrift to some ancillary topics. I never worry about running out of time.

I have gone on at some length in this Preface to give the uninitiated reader a glimpse of where I am coming from. I hope that on this basis you can decide whether you want to read the remainder of the book. This is a self-help book in the strongest sense of the word. It is a kit that will allow you to build your own teaching methodology and philosophy. I certainly cannot do it for you. What I *can* do is provide you with some tips, and advice, and the benefit of my own experience. Nothing that I say here is "correct" in any absolute sense. It is just what I know.

One of my disappointments pursuant to the first edition of this book is that nobody has taken it as an impetus to write his own book espousing his own teaching philosophy. There have been some reviews of this book—several of them rather strong and critical both of the book and of its author (see [**MOO**], [**BRE1**]). I welcome such discussions, and would only like to see further discourse. I am delighted to be able to say that several distinguished scholars, who have been active in exploring and discussing teaching issues, have agreed to write Appendices to this new edition of *How to Teach Mathematics*. Let me stress that these are not all people who agree with me. In fact some of us have had spirited public disagreements. But we all share some common values. We want to discover how best to teach our students. The new Appendices help to balance out the book, and to demonstrate that any teaching question has many valid answers.

When I teach the teaching seminar for our graduate students, the first thing I tell them is this: "In this course, I am *not* going to tell you how to teach. You have to decide that for yourselves. What I intend to do is to sensitize you to certain issues attendant to teaching. Then you will have the equipment so that you can build your own teaching philosophy and style." I would like to suggest that you read this book in the same spirit. You certainly need not agree with everything I say. But I hope you will agree that the issues I discuss are ones that we all must consider as we learn how to teach.

When I was a graduate student—in one of the best math graduate programs in the country—I never heard a single word about teaching. Actually, that's not true. Every once in a while we would be talking about mathematics and someone would look at his watch and say, "Damn! I have to go teach." But that was the extent of it. Six years after I received my PhD, I returned to that same Ivy League school as a visiting faculty member. Times had changed, and one of the senior faculty members gave a twenty-minute pep talk to all new instructors. He said, "These days, you can either prove the Riemann hypothesis or you can learn how to teach." He went on to tell us to speak up during lectures, and to write neatly on the blackboard. This was not the most profound advice on teaching that I have ever heard, but it certainly represented progress.

The truth is that, as a graduate student, I was so hellbent on learning to be a mathematician that I probably gave little thought to teaching. I would have felt quite foolish knocking on my thesis advisor's door and asking his advice on how to teach the chain rule. I shudder to think what he might have replied. But we have all evolved. It makes me happy that my own graduate students frequently consult me on **(i)** mathematics, **(ii)** teaching, and **(iii)** the profession. Though I secretly may relish **(i)** a bit more than **(ii)** or **(iii)**, I do enjoy all three.

Teaching is an important part of what we do. Because of economic stringencies, and new societal values, university administrations are monitoring every department on campus to ensure that the teaching is (better than) adequate and is working. My university is known nationwide for its good teaching. Yet an experienced administrator here said recently that 80% of the tenured faculty (campus-wide) could *not* get tenure today on the basis of their teaching.

We simply cannot get away with the carelessness that was our hallmark in the past. Thanks in part to the teaching reform movement, we have all come to understand this change in values, and we are beginning to embrace it. A book like [**CAS**], which offers advice to a fledgling instructor, could not have existed twenty years ago. Now it is a valuable part of our literature.

Teaching is a regimen that we spend our entire lives learning and revising and honing to a sharp skill. This book is designed to help you in that pursuit.

I am happy to acknowledge the advice and help that I have received from many friends and colleagues in the preparation of this new edition. I would like particularly to mention Joel Brawley, David Bressoud, Robert Burckel, John B. Conway, Ed Dubinsky, Len Gillman, David Hoffman, Gary Jensen, Meyer Jerison, Kristen Lampe, Vladimir Maşek, Chris Mahan, Deborah K. Nelson, Hrvoje Sikic, Nik Weaver, Stephen Zemyan, and Steven Zucker. Lynn Apfel was good enough to read several drafts of this manuscript with painstaking care, and to share with me her cogent insights about teaching; I am most grateful for her contributions. Jennifer Sharp of the American Mathematical Society gave me the benefit both of her editing skills and of her knowledge of language and meaning. Her help has been invaluable.

Last, but not least, Josephine S. Krantz is a constant wellspring of inspiration; her Mom, Randi Ruden, is a source of solace.

Of course the responsibility for all remaining errors or foolishness resides entirely with me.

Steven G. Krantz
St. Louis, Missouri

Preface to the First Edition

While most mathematics instructors prepare their lectures with care, and endeavor to do a creditable job at teaching, their ultimate effectiveness is shaped by their attitudes. As an instructor ages (and I speak here of myself as much as anyone), he finds that he is less in touch with his students, that a certain ennui has set in, and (alas) perhaps that teaching does not hold the allure and sparkle that it once had. Depending on the sort of department in which he works, he may also feel that hotshot researchers and book writers get all the perks and that "mere teachers" are viewed as drones.

As a result of this fatigue of enthusiasm, a professor will sometimes prepare for a lecture *not* by writing some notes or by browsing through the book but by lounging in the coffee room with his colleagues and bemoaning **(a)** the shortcomings of the students, **(b)** the shortcomings of the text, and **(c)** that professors are overqualified to teach calculus. Fortified by this yoga, the professor will then proceed to his class and give a lecture ranging from dreary to arrogant to boring to calamitous. The self-fulfilling prophecy having been fulfilled, the professor will finally join his cronies for lunch and be debriefed as to **(a)** the shortcomings of the students, **(b)** the shortcomings of the text, and **(c)** that professors are overqualified to teach calculus.

There is nothing new in this. The aging process seems to include a growing feeling that the world is going to hell on a Harley. A college teacher is in continual contact with young people; if he feels ineffectual or alienated as a teacher, then the unhappiness can snowball.

Unfortunately, the sort of tired, disillusioned instructors that I have just described exist in virtually every mathematics department. A college teacher who just doesn't care anymore is a poor role model for the novice instructor. Yet that novice must turn somewhere to learn how to teach. You cannot learn to play the piano or to ski by watching someone else do it. And the fact of having sat in a classroom for most of your life does not mean that you know how to teach.

The purpose of this book is to set down the traditional principles of good teaching in mathematics—as viewed by this author. While perhaps most experienced mathematics instructors would agree with much of what is in this book, in the final analysis this tract must be viewed as a personal polemic on how to teach.

Teaching is important. University administrations, from the top down, are today holding professors accountable for their teaching. Both in tenure and promotion decisions and in the hiring of new faculty, mathematics (and other) departments must make a case that the candidate is a capable and talented teacher. In some departments at Harvard, a job candidate must now present a "teaching dossier" as well as an academic dossier. It actually happens that good mathematicians who are really rotten teachers do not get that promotion or do not get tenure or do not get the job that they seek.

The good news is that it requires no more effort, no more preparation, and no more time to be a good teacher than to be a bad teacher. The proof is in this book. Put in other words, this book is not written by a true believer who is going to exhort you to dedicate every waking hour to learning your students' names and designing seating charts. On the contrary, this book is written by a pragmatist who values his time and his professional reputation, but is also considered to be rather a good teacher.

I intend this book primarily for the graduate student or novice instructor preparing to sally forth into the teaching world; but it also may be of some interest to those who have been teaching for a few or even for several years. As with any endeavor that is worth doing well, teaching is one that will improve if it is subjected to periodic re-examination.

Let me begin by drawing a simple analogy: By the time you are a functioning adult in society, the basic rules of etiquette are second nature to you. You know instinctively that to slam a door in someone's face is **(i)** rude, **(ii)** liable to invoke reprisals, and **(iii)** not likely to lead to the making of friends and the influencing of people. The keys to good teaching are at approximately the same level of obviousness and simplicity. But here is where the parallel stops. We are all *taught* (by our parents) the rules of behavior when we are children. Traditionally, we (as mathematicians) are not taught anything, when we are undergraduate or graduate students, about what constitutes sound teaching.

In the past we have assumed that either

(i) Teaching is unimportant.

or

(ii) The components of good teaching are obvious.

or

(iii) The budding professor has spent a lifetime sitting in front of professors and observing teaching, both good and bad; surely, therefore, this person has made inferences about what traits define an effective teacher.

I have already made a case that **(i)** is false. I agree wholeheartedly with **(ii)**. The rub is **(iii)**. If proof is required that at least some mathematicians have given little thought to exposition and to teaching, then think of the last several colloquia that you have heard. How many were good? How many were inspiring? This is supposed to be the stuff that matters—getting up in front of our peers and touting our theorems. Why is it that people who have been doing it for twenty or thirty years still cannot get it right? Again, the crux is item **(iii)** above. There are some things that we do not learn by osmosis. How to lecture and how to teach are among these.

Of course the issue that I am describing is not black and white. If there were tremendous peer support in graduate school and in the professorial ranks for great teaching, then we would force ourselves to figure out how to teach well. But often there is not. The way to make points in graduate school is to ace the qualifying exams and then to write an excellent thesis. It is unlikely that your thesis advisor wants to spend a lot of time with you chatting about how to teach the chain rule. After all, he has tenure and is probably more worried about where his next theorem or next grant or next raise is coming from than about such prosaic matters as calculus.

The purpose of this book is to prove that good teaching requires relatively little effort (when compared with the alternative), will make the teaching process a positive part of your life, and can earn you the respect of your colleagues. In large part I will be stating the obvious to people who, in theory, already know what I am about to say.

It is possible to argue that we are all wonderful teachers, simply by *fiat*, but that the students are too dumb to appreciate us. Saying this, or thinking it, is analogous to proposing to reduce crime in the streets by widening the sidewalks. It is doubletalk. If you are not transmitting knowledge, then you are not teaching. We are not hired to train the ideal platonic student. We are hired to train the particular students who attend our particular universities. It is our duty to learn how to do so.

This is a rather personal document. After all, teaching is a rather personal activity. But I am not going to advise you to tell jokes in your classes, or to tell anecdotes about mathematicians, or to dress like Gottfried Wilhelm von Leibniz when you teach the product rule. Many of these techniques only work for certain individuals, and only in a form suited to those individuals. Instead I wish to distill out, in this book, some universal truths about the teaching of mathematics. I also want to go beyond the platitudes that you will find in books about teaching *all* subjects (such as "type all your exams", "grade on a bell-shaped curve") and talk about issues that arise specifically in the teaching of mathematics. I want to talk about principles of teaching that will be valid for all of us.

My examples are drawn from the teaching of courses ranging from calculus to real analysis and beyond. Lower-division courses seem to be an ideal crucible in which to forge teaching skills, and I will spend most of my time commenting on those. Upper-division courses offer problems of their own, and I will say a few words about those. Graduate courses are dessert. You figure out how you want to teach your graduate courses.

There are certainly differences, and different issues, involved in teaching every different course; the points to be made in this book will tend to transcend the seams and variations among different courses. If you do not agree in every detail with what I say, then I hope that at least my remarks will give you pause for thought. In the end, you must decide for yourself what will take place in your classroom.

There is a great deal of discussion these days about developing new ways to teach mathematics. I'm all for it. So is our government, which is generously funding many "teaching reform" projects. However, the jury is still out regarding which of these new methods will prove to be of lasting value. It is not clear yet exactly how `Mathematica` notebooks or computer algebra systems or interactive computer simulations should be used in the lower-division mathematics classroom. Given that a large number of students need to master a substantial amount of calculus during the freshman year, and given the limitations on our resources, I wonder whether alternatives to the traditional lecture system—such as, for instance, Socratic dialogue—are the correct method for getting the material across. Every good new teaching idea should be tried. Perhaps in twenty years some really valuable new techniques will have evolved. They do not seem to have evolved yet.

In 1993 I must write about methods that I know and that I have found to be effective. Bear this in mind: Experimental classes are experimental. They usually lie outside the regular curriculum. It will be years before we know for sure whether students taught with the new techniques are understanding and retaining the material satisfactorily and are going on to successfully complete their training. Were I to write about some of the experimentation currently being performed then this book would of necessity be tentative and inconclusive.

There are those who will criticize this book for being reactionary. I welcome their remarks. I have taught successfully, using these methods, for twenty years. Using critical self examination, I find that my teaching gets better and better, my students appreciate it more, and (most importantly) it is more and more effective. I cannot in good conscience write of unproven methods that are still being developed and that have not stood the test of time. I leave that task for the advocates of those methods.

In fact I intend this book to be rather prescriptive. The techniques that I discuss here are ones that have been used for a long time. They work. Picasso's revolutionary techniques in painting were based on a solid classical foundation. By analogy, I think that before you consider new teaching techniques you should acquaint yourself with the traditional ones. Spending an hour or two with this book will enable you to do so.

I am grateful to the Fund for the Improvement of Post-Secondary Education for support during a part of the writing of this book. Randi D. Ruden read much of the manuscript critically and made decisive contributions to the clarity and precision of many passages. Josephine S. Krantz served as a valuable assistant in this process. Bruce Reznick generously allowed me to borrow some of the ideas from his book *Chalking It Up*. I also thank Dick Askey, Brian Blank, Bettye Anne Case, Joe Cima, John Ewing, Mark Feldman, Jerry Folland, Ron Freiwald, Paul Halmos, Gary Jensen, John McCarthy, Alec Norton, Mark Pinsky, Bruce Reznick, Richard Rochberg, Bill Thurston, and the students in our teaching seminar at Washington University for many incisive remarks on different versions of the manuscript. The publications committees of the Mathematics Association of America and of the American Mathematical Society have provided me with detailed reviews and valuable advice for the preparation of the final version of this book.

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CHAPTER 1

Guiding Principles

1.0. Chapter Overview

Good teaching is a product of preparation, effort, knowledge, insight, and a good attitude on the part of the instructor. If you treat your teaching as analogous to having root canal treatments, then it's a certainty that you will not enjoy it. And your students probably will not either.

This chapter discusses broad issues connected with the teaching process: respect, preparation, time management, voice control, and so forth. The ideas presented here are ones that you will want to internalize. You want them to be a natural part of your teaching habit.

1.1. Respect

You cannot be a good teacher if you do not respect yourself. If you are going to stand up in front of thirty people or three hundred people and try to teach them something, then you had better

- Believe that you are well qualified to do so.
- Want to do so.
- Be *prepared* to do so.
- Make sure that these characteristics are evident to your audience.

It is a privilege to stand before a group of people—whether they be young adults or your own peers—and to share your thoughts with them. You should acknowledge this privilege by **(a)** dressing appropriately for the occasion, **(b)** making an effort to communicate with your audience, **(c)** respecting the audience's point of view.

One completely obvious fact is that you should have your material *absolutely mastered* before you enter the classroom.[1] If you *do* possess this mastery, then you can expend the majority of your effort and attention on conveying the material to the audience. If, instead, you have a proof or an example that is not quite right, and if you stand in front of the group trying to fix it, then you will lose all but the diehards quickly. (Note that this last statement is true both when you are giving a calculus lecture and when you are giving a colloquium talk at Harvard.)

When I was a novice teacher I often spent only a few minutes preparing my lectures. I would go to class and wing it. And, as you might predict, I made mistakes. Now that I am sliding into my dotage, I in fact prepare every class *very carefully.* I write out complete notes. And now I hardly ever make mistakes in class.

[1] I am going to emphasize this point throughout the book—almost *ad nauseum.* After more than forty years of teaching, I can tell you confidently that this above all else is the key to success in the classroom. The students need to know that you are in control of the situation. That you are on top of the material. That you can answer any question. They want to believe in you and trust you. So you *must* know what you are talking about.

On the rare occasions when I do, I can make the class feel like it's "our" mistake, fix it quickly, and move on. This gives the student the very strong impression that I am in complete control of the situation. The fact that I can *always* answer *all of* their questions (I only rarely have to say, "let me get back to you on that one") only serves to reinforce that impression.[2] And that is what I want.

It is easy to rationalize that, if the students were more able, then they could roll with the ups and downs of your lecture. This is strictly illogical. How do you behave when you are listening to a colloquium or seminar and the lecturer goes off into orbit—either to fix an incorrect argument or into a private conversation with his buddy in the front row or, worse, into a private conversation with himself? All right then, now that you have admitted honestly how *you* behave, then how can you expect unseasoned freshmen to be tolerant when you do not seem to be able to do the examples that *they* are expected to do? How about more seasoned sophomores?

One of the best arguments for even elementary college mathematics courses to be taught by people with advanced degrees is this: Because the material is all trivial and obvious to the professor, he/she can maintain a broad sense of perspective, he/she will not be thrown by questions, and he/she can concentrate on the act of *teaching.*

If you respect yourself then, it follows logically, you will respect your audience. You should prepare for your class. That way you will not be surprised by gaps in your thinking, you will not have to cast around for a necessary idea, and you will not lose your train of thought in class. You will be receptive to questions. You will sense immediately when the students are not understanding, and you will do something about it.

Throughout this book, I will repeatedly exhort you to prepare your classes. I do not necessarily have in mind that you should spend an inordinate amount of *time* preparing. Consider by analogy the psychology of sport. Weightlifters, for example, are taught to meditate in a certain fashion before a big lift. Likewise, preparing is a way to collect your thoughts and put yourself in the proper frame of mind to give a class. You might prepare by walking to the student union and buying a cup of coffee (thinking on the way about your lecture). Or you might prepare by browsing through some calculus books (or even—horror of horrors—browsing through the *actual text* for that particular class). I have sometimes prepared for a class by staring bleakly out the window pondering the notion that I have nothing to say. No matter—this is a yoga. It helps you to pull yourself together. It makes good sense. See also Section 1.2.

To me, preparation is the core of effective teaching. While this may sound like a tautology, and not worth developing, there is in fact more here than meets the eye. Just as being a bit organized relieves you of the stress and nuisance of spending hours looking for a postage stamp or a pair of scissors when you need them, so being the master of your subject gives you the ability to cope with the unexpected, to handle questions creatively, and to give proper stimulus to your students. An experienced and knowledgeable teacher who is comfortable with his/her craft is constantly adjusting the lecture, in real time, to suit the expressions on the students' faces, to suit their responses to queries and prods, to suit the rate and thoroughness

[2] It will happen very occasionally that a student will ask (for instance) for a counterexample which, even if you know it off the top of your head, is too recondite to present in class. Then the thing to do is to tell the student to see you after class.

with which they are absorbing the material. Just as a good driver (of a car) is constantly (and unconsciously) making little adjustments in steering in response to road conditions, to other drivers, to weather, and so forth, it is also the case that a good teacher (just as unconsciously) is engaging in a delicate give and take with the audience. Complete mastery is the unique tool that gives you the freedom to develop this skill.

You should treat questions with respect. I go into every class that I teach knowing full well that I am probably much smarter, and certainly much better informed, than most of the people in the room. But I do not need to use a room full of eighteen-year-olds as a vehicle for bolstering my ego. If a student asks a question, even a stupid one, then I treat it as an event. A wrong question can be turned into a good one with a simple turn of phrase from the instructor (see Section 1.5). If the question requires a lengthy answer, then give a short one and encourage the student to see you after class. If you insult—even gently—the questioner, then you not only offend that person but you perhaps offend everyone else in the room. Once the students have turned hostile it is difficult to win them back, both on that day and on subsequent days.[3]

In fact one of the most dynamic and inspiring things that you can do for your students is to help them think that their questions are steering the class. Even though students know in their heart of hearts that you are the boss, and you are the person with all the knowledge, they deeply resent the notion that you are the high priest dispensing wisdom. It is so much more stimulating to respond to a student question by saying, "Yes, that was my next point ..." or "You are anticipating the ideas. That's very good ..." or "You have raised a crucial issue. That's great. Let me pursue it for a moment ..."

Sometimes you can actually develop an idea in dialogue with a student—right in front of the class. For example, you say, "This point is not a boundary point. Why not?" The student replies, "Some neighborhoods have no points from the complement." "Good," you say. "So, in a certain sense, this point is *away* from the complement of the set. Instead we might say it's ..." "Interior?" interjects the student. How could a student *not* pay attention if he/she thinks that his/her thoughts, his/her questions, have shaped the discussion?

1.2. Prepare

Some people rationalize not teaching well by saying (either to themselves or to others), "My time is too valuable. I am not going to spend it preparing my calculus lecture. I am so smart that I can just walk into the classroom and wing it. And the students will benefit from watching a mathematician think on his feet." (As a student, I actually had professors who announced this nonsense to the class on a regular basis. And, as you can imagine, these were professors who royally botched up their lectures on a regular basis.)

It is true that most of us can walk into the room most of the time and mostly wing it. But most of us will not be very successful if we do so. Thirty minutes can be

[3] One of the legendary teachers of twentieth century mathematics was Walter Rudin. He began every class by telling the students that questions are important. Questions are wonderful. He liked all questions. There was no such thing as a dumb question. You should never hesitate to ask a question in my class. On one particular first day, after the lecture had been going on for a while, a student timidly raised his hand and posed a question. Walter looked him in the eye and said, "That's the stupidest question that I've ever heard in my life."

sufficient time for an experienced instructor to prepare a calculus lecture. A novice instructor, especially one teaching an unfamiliar subject for the first time, may need considerably more preparation time.[4] Make sure that you have the definitions and theorems straight. Read through the examples to make sure that there are no unpleasant surprises. It is a good idea to have a single page of notes containing the key points and also briefly listing your motivating ideas. To write out every word that you will say, write out a separate page of anticipated questions, have auxiliary pages of extra examples, have inspirational quotes drawn from the works of Thomas Carlyle, make up a new notational system, make up your own exotic examples, and so forth, is primarily an exercise in self-abuse. Over-preparation can actually stultify a lecture or a class. But you've got to know your stuff.

I cannot emphasize too strongly the fact that preparation is of utmost importance if you are going to deliver a stimulating class. However it is also true that the more you prepare the more you lose your spontaneity. You must strike a balance between **(i)** knowing the material cold and **(ii)** being able to "talk things through" with your audience.

My own experience is that there is a "right amount" of preparation that is suitable for each type of course. I want to be confident that I'm not going to screw up in the middle of lecture. But I also want to be actually thinking the ideas through as I present them. I want to feel that my lecture or class has an edge. It *is* possible to over-prepare. To continue to prepare after you have already prepared sufficiently is a bit like hitting yourself in the head with a hammer because it feels so good when you stop.

You must be sufficiently confident that you can field questions on the fly, can modify your lecture (again on the fly) to suit circumstances, can tolerate a diversion to address a point that has been raised. I always try to give the class the impression that I am more than willing to take time to address any question or issue. That I am *not* in a rush. That I really care about addressing their concerns and thoughts.

The ability to do these things well is largely a product of experience. But you can *cultivate* this ability too. You cannot learn to play the piano by accident. And you will not learn to teach well by accident. You must be aware—in detail—of what it is that you are trying to master and then consciously hone that skill.

If you do not prepare—I mean *really* do not prepare—and louse up two or three classes in a row, then you will experience one or more of the following consequences: **(i)** Students will take up your time after class and during your office hour (in order to complain and ask questions), **(ii)** Students will stop coming to class, **(iii)** Students will complain to the undergraduate director and to the chair, **(iv)** Students will (if you are *really bad*) complain to the Dean and write letters to the student newspaper, **(v)** Students will write bad teaching evaluations for your course. Now student teaching evaluations are not gospel (see Section 2.8). They contain some remarks that are of value and some that are not. Getting bad teaching evaluations does not necessarily mean that you did a bad job. And I know that the Dean will only slap me on the wrist if he gets a complaint about my teaching (however, if there are ten complaints, then I had better look out). Finally, I know that the chair will give me the benefit of the doubt and allow me every opportunity to put any

[4] A couple of years ago I taught an upper division course in graph theory. And I am *not* a graph theorist by any stretch of the imagination—even though I wrote a paper with Erdős about graph theory. I often had to spend a couple or more hours preparing any given lecture.

difficult situation in perspective. But if I spend thirty minutes preparing each of my classes then I will avoid all this grief and, in general, find the teaching experience pleasurable rather than painful. What could be simpler?

As well as preparing for a class, you would be wise to debrief yourself after class. Ask yourself how it went. Were you sufficiently well prepared? Did you handle questions well? Did you present that difficult concept as clearly as you had hoped? Did you draw that hyperboloid of one sheet clearly and efficiently? Was there room for improvement? Be as tough on yourself as you would be after any exercise that you genuinely care about—from playing the piano to engaging in a tennis match. It will result in real improvement in your teaching.

Read your teaching evaluations (Section 2.8). Many are insipid. Others are puerile. Most, however, are thoughtful and well-meant. If ten of your students say that your writing is unclear, or that you do too many calculations in your head, or that you talk too quickly, or that you are impatient with questions, then maybe there is a problem that you should address.[5] Teaching is a yoga. Your mantra is "Am I getting through to them?" Or it could be "Am I giving them something to take away?"

It is a good idea to try to anticipate questions that students will ask. But you cannot do this artificially, as a platonic exercise late at night over a cup of coffee. It comes with experience. Assuming that you have adopted the attitude that you actually care whether your students learn something, then after some experience of teaching you will know by instinct what points are confusing and why. This instinct enables you to prepare a cogent lecture—to know what to emphasize, where to slow down, where to provide extra examples. It helps you to be receptive to student questions. It helps you to have a good attitude in the classroom.

An easy way to cut down on your preparation time for a class is to present examples straight out of the book. The weak students will appreciate this repetition. Most students will not, and you will probably be criticized for this policy. On the other hand, it is rather tricky to make up good examples of maximum-minimum problems or graphing problems or applications of Stokes's theorem. It can be time-consuming as well. If you need more examples for your calculus class, then pick up another calculus book and borrow some. Develop a file of examples that you can dip into each time you teach calculus. You will learn quickly that making up your own examples is hard work. Do you ever wonder why most calculus books are so disappointing? All right, *you* try to make up eight good examples (with algebra that works out nicely) to illustrate the divergence theorem.

1.3. Speak Up

If you are going to be a successful teacher, then you have to find a way to fill the room with yourself. If you stand in front of the class (be it a class of ten or a class of a hundred) and just mumble and look at your shoes,[6] then you will not successfully convey the information. Even the most dedicated students will have

[5] A fact of life these days is the Web site *Rate My Professors* at `www.ratemyprofessors.com`. Going there, you can look up yourself or most any other professor in the country. You may be amused and surprised to see what sort of teachers many of your friends and colleagues are. You also may be amused or surprised or humiliated to see what sort of teacher you are. And this information is out there for everyone to see!!

[6] An introverted mathematician is one who looks at his/her shoes when he/she talks to you. An extroverted mathematician is one who looks at *your* shoes when he/she talks to you.

trouble paying attention. You will not have stimulated anyone to think critically. In fact you will have lost your audience.

I must stress that the way you comport yourself in *your* classroom should be consistent with your personality and your way of doing things. But you still need to get through to those kids. You will want to spend some time figuring out how to make this work.

You do not need to be a showoff or a ham or a joke teller to fill a room with your presence. You can be dignified and reserved and old-fashioned and still be a successful instructor with today's students. But you must let the students know that you are there. You must establish eye contact. You must let them know that you are *talking to them.*

Before I start a class session, especially with a large group of students, I engage some of them in informal conversation. I get them to talk about themselves. I ask them how they are doing on the homework assignment. I comment about the weather, or about something that is going on around campus. Then I make a smooth transition into the more formal class activities. That way I already have half a dozen people on my side. The others soon follow.

Some new instructors—especially those who are naturally soft-spoken or shy—may need some practice with voice modulation and projection. If you are such a person, then get together a group of friends and give a practice lecture for them. Ask for their criticism. Make a tape recording of your practice lecture and listen to it critically.

If there is any doubt in your mind as to whether you are reaching your audience during a particular class, then *ask* about it. Say "Can you hear me? Am I talking loud enough? Are there any questions?" This is one of many simple devices for changing the pace of a class, giving note-takers a break, allowing students to wake up.

Think of a good movie that you've seen recently. Now remove the changes of scenery; remove the voice modulation and changes of emotion; remove the changes in focal length; remove the skillful use of silence as a counterpoint to sound; remove the musical soundtrack. What would remain? Could you stay awake during a showing of what is left of this movie? Now think about your class in these terms.

1.4. Lectures

In an empty room sits a violin.

One person walks in, picks it up, draws the bow across the strings, and a horrible screeching results. He leaves in bewilderment.

A second person walks in, attempts to play, and the notes are all off key.

A third player picks up the instrument and produces heavenly sounds that bring tears to the eyes. He is Isaac Stern and the instrument is a Stradivarius.

Wouldn't it have been foolish to say, after hearing the first two players, that this instrument is outmoded, that it doesn't work? That it should be abandoned to the scrap heap? Yet this is what many are saying today about the method of teaching mathematics with lectures. Citing statistics that students are not learning calculus sufficiently well, or in sufficiently large numbers, government-sponsored projects nationwide assert that the lecture doesn't work, that we need new teaching techniques.

Whether you like lectures or not, we have to face some facts here. Most of us don't lecture very well. After all, when were we taught to lecture, and in what forum? How many excellent lecturers do you know? Where did they learn to lecture?

OK, so we admit it: The lecture doesn't work very well because most of us aren't very good at it. The trouble is that most of us aren't very good at any other method of teaching either. We are not skilled at any of these methods because we have received no training in them, and because we have not given them careful thought. While one possible solution to the problem is "The lecture is dead so let's move on to something else," another possible solution is "Let's learn how to lecture." See also Section 4.2 on classroom learning vs. Online learning.

Those who say that "the use of the lecture as an educational device is outmoded" rationalize their stance, at least in part, by noting that we are dealing with a generation raised on television and computers. Our students have spent a good deal of their lives with `Twitter` and `Google` and `Hulu`. They argue that today's students are too ready to fall into the passive mode when confronted with a television-like environment. The kids simply don't have the patience or the attention span to sit through a lecture. It follows that we must teach them interactively, or in groups, or using cooperative learning. Perhaps we should use computers and software to bring students to life.

Lectures have been used to good effect for more than 3000 years. I am hesitant to abandon them in favor of a technology (personal computers, videos) that has existed for just thirty years. In spite of popular rumors to the contrary, a lecture does not need to be a bone-dry desultory disquisition. It can have wit, erudition, and sparkle. It can arouse curiosity, inform, and amuse. It is a powerful teaching device that has stood the test of time.[7] The ability to give a good lecture is a valuable art, and one that you should cultivate.

However, you really have to work at making your lectures reach your students. It is true that mathematics teaching in this country is not, overall, a great success. The reason is not that the lecture method is "broken." Rather, we tend not to put a lot of effort into our teaching because the value-reward system is often not set up to encourage putting a lot of effort into it. Many of us spend semester after semester facing down rooms full of calculus students who are there only because the course is required for their major. And it is not required for their major because their department wants them to know calculus. Rather, it is required for their major because their department hopes that half of them will flunk. It is a sorry situation. And it is easy for us, the underpaid and overworked faculty, to become demoralized. Certainly "The lecture is dead." is one way to rationalize an already dreary reality.

Of course many of us content ourselves with internal rewards, with the sense of satisfaction in doing a good job. No matter what rewards you seek, you must identify and learn to use the tools that will make you an effective teacher. You must learn to develop eye contact with your audience, to fill the room with your voice and your presence, and to present your ideas with enthusiasm and clarity. Other sections of this book discuss these techniques in detail.

Turn on your television and watch a self-help program, or a television evangelist, or a get-rich-quick real estate huckster. These people are not using overhead

[7]And it goes without saying that an Online course will have none of these attributes.

projectors, or computer simulations, or `Mathematica`. (Incidentally, they are also not using group learning or self-discovery!) In their own way they are lecturing, and *very powerfully.* They can persuade people to donate money, to change religions, or to join their cause. Of course your calculus lecture should not literally emulate the methods of any of these television personalities. But these people and their methods are living proof that the lecture is not dead, and that the traditional techniques of Aristotelian rhetoric are as relevant and effective as ever. By watching even a charlatan do his/her stuff, you can learn something about how to engage an audience, how to answer questions, how to interact with people.[8]

I like to startle people by telling them that I have refined my teaching technique by watching David Letterman (of late night television fame). This is not an exaggeration. Letterman is a master at communication, at dealing with many different types of people, at taking a situation that is going sour and turning it around. *I am not talking here about telling jokes.* I am instead talking about skill at handling people. At taking a conversation which is going dead and bringing it to life. At taking a shy, reticent person and making him/her wake up and contribute.

My friend Glenn Schober was once teaching a class to help train graduate students to teach. For the first day he carefully crafted a lecture on elementary mathematics in which he purposely made 25 cardinal teaching errors. He walked into class on the first day, told the students he was going to give a sample lecture, and did so. At the end he said, "There were a number of important teaching errors that I intentionally committed in this lecture. See how many you can identify." The students found 32 errors.

This story is amusing, but it is also an important object lesson. There are two kinds of "important" in this world—your kind and my kind. Any classroom situation has a promulgator and an audience. If the lecturer and the audience work together, and share the same goals, then they will reach the same place at the same time, with mutual satisfaction as the result. If not, then they will be working at cross purposes, with mutual frustration as an outcome. These statements apply no matter *what* teaching method you are using. The key to success is that you must *communicate* with your audience.

There are other useful teaching environments besides lectures. Although less common in mathematics than in some humanities courses, group discussions can be useful. If you want to get students interested in what the boundary of a set in a metric space ought to be, then you can begin with a discussion in which students offer various suggestions. Before you define what a finite set is, ask the students to suggest a definition.

It is not difficult to see that putting a student with a group of four or five peers, so that they can conduct an intimate, one-on-one discussion of a mathematical topic, is bound to generate student interest. It also may tend to help timid students to open up, and to engage in communication. But we must acknowledge that educational activities that, in effect, make the classroom the venue where all learning takes place use time differently than learning activities that make the

[8]Some of the most popular and successful lecturers today are sports coaches or military officers who give standing-room-only motivational presentations about self-esteem and motivation and success. They often have the audience—usually consisting of mature adults who have solid, middle-class lives—weeping openly because they are so moved. Can you imagine having such an effect with your calculus lecture? If you have ever seen one of these people in action, as I have, you will be quite impressed. They really know something about how to connect with an audience.

student's dorm room (or the library) the place where learning takes place. To be specific, the old-fashioned paradigm for student learning was that the student would sit in class for an hour listening to a lecture and taking notes. Then he/she would go home and spend three to five hours deciphering the notes, filling in the gaps, and doing the homework. The new paradigm has the student learning *right now*, either before a computer screen or interacting with a group of other students. The reform teacher makes sure that students are engaged in the learning process because he/she actually *manages the engagement*. The traditional teacher leaves more of the responsibility for engagement with the student himself/herself.[9]

To repeat, many of the reform methods use time in a different, and less familiar, manner than do the traditional (lecture) methods (see also Section 3.6). When you are giving a lecture class, you know just what is going on during the allotted class time, and you also know what is *supposed* to be going on during out-of-class study time. If you instead teach with group work, self-discovery, computer labs, `Mathematica` notebooks, and other new devices, then you must be retrained as an instructor. You must learn anew how to monitor what is going on in your class and how to evaluate student progress.

Reformers, among them Ed Dubinsky, assert that they work their students much harder than do the traditionalists—and that the students are so fulfilled by the learning process that they are glad to do the work. Since reform teachers have many more student contact hours than do traditional teachers, they are probably well qualified to make this assertion (see [**ASI**] for more on this idea). The jury is still out on the question of whether students taught with reform methods or students taught with traditional (lecture) methods derive the most from their education. Which students learn more? Which retain more? Which have greater self-esteem? Which feel more empowered? Which have greater interest in the learning process? Which teaching method encourages more students to become math majors? How many students taught by the reform methods go on to graduate school and become professional mathematicians? Frankly, we don't know.

Of course it is as unfair to group "reformers" together as it is to group "liberals" or "anti-vivisectionists" together. I have heard reformers say in a public venue that, in the reform environment, only half as much material can be presented to the students (as is/was presented in the traditional lecture environment). At the same time, they assert that that's about as much as the students ever learned anyway. So nothing is really lost. Traditionalists, remembering the lectures they attended and the way that they worked when they were students, might bridle at this reasoning. A traditionalist will watch a reform class, perceive unmanaged use of time, and conclude that less learning is taking place. The reformer will argue that his/her students are working harder and internalizing more.

Clearly we must weigh any teaching method according to both its merits and its demerits. Traditional techniques (lectures) can have the effect on today's students of making learning and erudition seem to be dry, dusty, uninteresting, and

[9]It has been pointed out (by Steven Zucker among others) that the main difference between high school learning and college learning is this: In high school, learning takes place at the moment of impact, in the classroom, and the primary responsibility for learning lies with the teacher. In college, learning takes place (primarily) outside the classroom, and the primary responsibility for learning lies with the student. It is because students do not understand this distinction, and how to deal with it, that they have such a struggle with the first year of college. They simply do not know how to study.

irrelevant. However, they are efficient at promulgating a great deal of information. Some of the new techniques are terrific at getting students involved in the material, at making the ideas come alive, and perhaps at aiding student retention. But these methods use time in new ways, and it seems possible that less material will be taught when they are used. Do check out Section 4.5 on the flipped classroom, which suggests a new reform-style teaching method that does use time effectively and well.

It has been natural, in this section on the lecture, to digress about reform. For one of the battle cries of the reform movement has been that "The lecture is dead." Let us now conclude this section by mentioning some teaching devices that should be of interest both to reformers and to traditionalists.

A casual reading of the present discussion might mislead the reader into thinking that lectures cover more material but are boring while reform methods cover less material but are more engaging. There is more to it than that. Lectures do not need to be boring. They can be engaging, fascinating, and even exciting. Reform methods do not necessarily *have to* cover less material, but they can easily do so if they are not carefully managed. A smart math teacher will learn from both points of view, and craft his/her teaching methodology accordingly.

It can be instructive to have students volunteer to do problems at the blackboard. Once in a great while—when I am lecturing—if a student offers an alternative proof of a proposition or another point of view, I hand him/her the chalk. Everyone is usually quite surprised, but the results are generally pleasing and it provides a nice change of pace.

Computer labs (Section 4.6) can also be a useful instructional device. The subject of sophomore-level differential equations lends itself well to helping students explore the interface between what we can do by hand and what the machine can teach us. Let's be frank. We do not know how to solve most differential equations explicitly, or in closed form. Thus it is important for students to see how much analysis one can do with traditional methodologies and then to see how the machine can use phase plane analysis, numerical methods, and graphing to provide further concrete data.

We should continue to seek new and better methods and technologies for teaching. This author, and this book, has a built-in bias toward traditional methods, such as lectures. That is because he has watched them work and used them successfully for more than forty years. I hope that other writings will describe and explore some of the new teaching techniques. The new Chapter 4 of this edition of *How to Teach Mathematics* explores various uses of the Internet inside and outside the classroom.

1.5. Questions

In a programmed learning environment, whether the interface is with a PC or with `Mathematica` notebooks or with a MAC, the student cannot ask questions. The give and take of questions and answers is a critical aspect of the human part of the teaching process. Teachers are *supposed* to answer questions.

There is more to this than meets the eye. When I say that a teacher answers questions I do not envision the student saying, "What is the area of a circle?" and the teacher saying "πr^2." I instead envision the student struggling to articulate some confusion and the experienced teacher turning this angst into a cogent question and then answering it. To do this well requires experience and practice. I frequently find myself responding to a student by saying, "Let's set your question aside for a minute and consider the following." I then put the student at ease by quickly running through something that I know the student knows cold, and that serves as a setup for answering the original question. With the student on my side, I can answer the primary problem successfully. The point is that some questions are so ill-posed that they literally cannot be answered. It is the teacher's job to make the question an answerable one and then to answer it.[10]

A similar, but alternative, scenario is one in which the student asks a rather garbled question and I respond by saying, "Let me play the question back for you in my own words and then try to answer it ...." The point is that the responses "Your question makes no sense" or "I don't know what you mean" are both insulting and a cop-out. To be sure, it is the easy answer. But you will pay for it later. It takes some courage for the student to ask a question in class, in front of his/her peers. By treating questions with respect, you are both acknowledging this fact and helping someone to learn. If instead you stare at the student as though he has weevils in his eyebrows, then you will gain no allies and will most likely lose several friends and make a few enemies.

Yet another encouraging response to a student question is to say, "Thank you. That question leads naturally to our next topic ..." Of course you must be quick on your feet in order to be able to pull this off. It is worth the trouble. Students respond well when they are treated as equals—see the discussion of this technique at the end of Section 1.1.

There are complex issues involved here. A teacher does not just lecture and answer questions. A good teacher helps students to discover the ideas. There are few things more stimulating and rewarding than a class in which the students are anticipating the ideas because of seeds that you have planted. The way that you construct your lecture and your course is one device for planting those seeds. The way that you answer questions is another.

When I discuss teaching with a colleague who has become thoroughly disenchanted with the process, I frequently hear complaints of the following sort: "Students these days are impossible. The questions that they pose are unanswerable. Suppose, for example, that I am doing a problem with three components. I end up writing certain fractions with the number 3 in the denominator. Some student will ask 'Do we always put a 3 in the denominator when doing a problem from this section?' How am I supposed to answer a question like that?"

Agreed, it is not obvious how to answer such a question, since the person asking it either **(i)** has not understood the discussion, **(ii)** has not been listening, or **(iii)** has no aptitude for the subject matter. It is tempting to vent your spleen against the student asking such a question. Do not do so. The student asking this question probably needs some real help with analytical thinking, and you cannot give the required private tutorial in the middle of a class hour. But you can provide guidance. Say something like "When a problem has three components it is logical

[10] It is almost always a good idea to repeat the student question and/or to write it on the board. Because a lot of people in the room will not have heard the question, or will not have understood the question. So you just answering it cold will omit much of the class from the discussion. You must take control of the situation.

that factors of $1/3$ will come up. This can happen with certain problems in this section, or in any section. But it would be wrong to make generalizations and to say that this is what we do in all problems. If you would like to discuss this further, please see me after class." In a way, you are making the best of a bad situation. But at least you are doing something constructive, and providing an avenue for further help if the student needs it.

The Dalai Lama once visited the headquarters of *Time* magazine in Chicago. He was given the chef's tour, and then there was a grand formal lunch at which the various executives of the enterprise pontificated *ad nauseum.* The Dalai Lama—an elfin man—sat swathed in his saffron robe, an inscrutable smile on his face, saying nothing. After about an hour, the CEO of *Time* turned to the Dalai Lama and said, "Do you have any questions about *Time*, the nation's premiere news magazine? Go ahead, ask us anything at all." The Dalai Lama bowed his head for a moment, apparently deep in thought. Then he looked up and said, "Why do you publish it?"

We mathematicians are very much like the executives of *Time* in that story. We are wrapped up in our own world, we all speak the same language, and we suffer intruders with pained resignation. Sadly, our students are like intruders. They come to us with questions we would never have dreamed of, often expressed in language that is obscure at best, and they expect answers. They do not speak our language, and they do not necessarily respect our mores. Yet it is our job to talk to our students, to engage them in discourse, to answer their questions. We must exercise patience in order to gain their trust. And we must try to speak to them in their language, rather than in our own.

Let us consider some further illustrations of the principle of making a silk purse from a sow's ear—that is, answering unexpected or awkward questions in a constructive manner. The first example is a simple one.

> **Q:** Why isn't the product rule $(f \cdot g)' = f' \cdot g'$?

The answer is *not* "Here is the correct statement of the product rule and here is the proof." Consider instead how much more receptive students will be to this answer:

> **A:** Leibniz, one of the fathers of calculus, thought that this is what the product rule should be. He recorded this thought in his diary. Ten days later, he gave the correct form—with a proof and the cryptic statement that he had known this to be the correct form "for some time." Because we have the language of functions, we can see quickly that Leibniz's first idea for the product rule could not be correct. If we set $f(x) = x^2$ and $g(x) = x$ then we can see rather quickly that $(f \cdot g)'$ and $f' \cdot g'$ are unequal. So the simple answer to your question is that the product rule that you suggest gives the wrong answer. Instead, the rule $(f \cdot g)' = f' \cdot g + g' \cdot f$ gives the *right* answer and can be verified mathematically.

The second example is more subtle.

> **Q:** Why don't we divide vectors in three-space?

The *wrong* answer is to tell about Stiefel-Whitney classes and that the only Euclidean spaces with a division ring structure are $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^4$, and $\mathbb{R}^8$. A better answer is as follows.

> **A:** J. Willard Gibbs invented vectors to model physical forces. There is no sensible physical interpretation of "division" of physical forces. The nearest thing would be the operations of projection and cross product, which we will learn about later.

Notice that in both illustrations an attempt is made to turn the question into more than what it is—to make the questioner feel that he has made a contribution to the discussion.

> **Q:** Why isn't the concept of velocity in two and three dimensions a number, just like it is in one dimension?

If you are in a bad mood, you will be tempted to think that this person has been dreaming for the past hour (or the past week!) and has understood absolutely nothing that you have been saying. Bear up. Resist the temptation to voice your frustrations. Instead try this:

> **A:** Let me rephrase your question. Instead let's ask, "Why don't we use vectors in one dimension to represent velocity just as we do in two and three dimensions?" One of the most important features of vector language is that a vector has *direction* as well as *magnitude*. In one dimension there are only two directions—right and left. We can represent those two directions rather easily with either a plus sign or a minus sign. Thus positive velocity represents motion from left to right and negative velocity represents motion from right to left. The vector language is *implicit* in the way that we do calculus in one dimension, but we need not articulate it because positivity and negativity are adequate to express the directions of motion.
>
> In dimensions two and higher there are infinitely many different directions and we therefore require the explicit use of vectors to express velocity.

As the author of this book, I have the luxury of being able to sit back and drink coffee and think carefully about how to formulate these "ideal" answers to poor questions. When you are actually teaching you must be able to do this on your feet, either during your office hour or in front of a class. At first you will not be so articulate. This is an acquired skill. But it is one *worth acquiring.* It is a device for showing respect for your audience, and in turn winning *its* respect.

Large lectures pose special problems with the issue of student questions. Obviously you cannot let each student ask his/her little question. You cannot let your lecture get bogged down with questions like "How do you do problem 6?" or "Will this be on the test?" See Section 2.13 for a discursive consideration of questions in the large lecture context.

A final note about questions. Even though you are an authority in your field, there are certainly things that you do not know. Occasionally these lacunae in your knowledge will be showcased by a question asked in class or during your office hour (it does not happen often, so don't get chills). The sure and important attribute of an intelligent, educated individual is an ability to say, "I don't know the answer to that question. Let me think about it and tell you next time." On the (rare) occasions when you have to say this, be sure to follow through. If the item that you don't know is an integral part of the class—and this had better not be the case very often—get it down cold because the question is liable to come up again in a different guise later in the course. If it is not an integral part of the course, then you have no reason to feel bad. Just get it straight and report back.

The main point is that you should never, under any circumstances, try to fake it. If you do, then you will look bad, your interlocutor will be frustrated and annoyed, and you will have served no good purpose. If there is any circumstance in which honesty is the best policy, this is it. Professor of Economic History Jonathan R. T. Hughes was wise to observe that "There is no substitute for knowing what you are talking about."

1.6. Time

There are several aspects of teaching that require time management skills. When you are giving a lecture, you must cover a certain amount of material in the allotted time—and at a reasonable rate. When you give a course, you must cover a certain amount of material in one semester or term. When you give an exam, it must be doable by an average student in the given time slot. When you answer a question, the length of the answer should suit the occasion.

All of these topics will be touched upon in other parts of this book. They require some thought, and some practice and experience, so that they become second nature to you.

Nobody can design a class so that (as in the movies) the last 'QED' is being written on the blackboard just as the bell rings. There are certain precepts to follow in this regard:

- Have some extra material prepared to fill up extra time.
- If you finish your lesson with five minutes to spare, don't rocket into a new topic. You will have to repeat it all next time anyway, and students find this practice confusing. You can use the spare five minutes to summarize what you have covered that day, or to give the students a "thought" problem to take home, or to just tell them to have a nice day. That is a much more gracious way to end the class.
- If the clock shows that just five minutes remain, and you have ten or fifteen minutes of material left to present, then you will have to find a comfortable place to quit. Don't race to fit all the material into the remaining time. At the same time—if possible—don't just stop abruptly and in mid-thought, thinking that you can pick up a calculation cold in the following class.

  An experienced teacher will know which will be the last example or topic in the hour, and that he/she might get caught for time. Therefore the instructor will plan in advance for this eventuality and think of several junctures at which he might bring the hour to a graceful close. With enough experience, you will know intuitively how to identify the comfortable places to stop; thus end-of-the-hour time management problems can be handled on the fly. In particular, if five minutes remain then *do not begin* a ten-minute example!
- If you prepare (the last part of) your lesson in units of five minutes' duration, and if you are on the ball, then you should never have to run over by more than two minutes nor finish more than two minutes ahead of time. (The idea here is if there are three minutes remaining, then you

can include another five-minute chunk without running over by more than two minutes. If there are just two minutes remaining, then you should stop.)

- *If you run out of time*, do not keep lecturing past the end of the period—at least not by more than a minute or two. Students have other classes to attend, and they will not be listening. If the time is gone, then just quit. Make up for your lapse in the next class (this will require some careful planning on your part). Best is to plan your class so that you do not get caught short of time.

  A special note about buzzers: Some math buildings have a loud buzzer or bell that sounds at the end of the class period.[11] Once that buzzer sounds, all is lost. Most students will instantly start packing up their books and heading for the exit. If there is a clock on the wall, then you will know when the buzzer is going to sound, and you can be adjusting your lecture and zeroing in on a suitable conclusion as you go along. At a school without a buzzer (especially one that also has no clocks in its classrooms!), you have a bit of slack since no two wristwatches are in agreement. You may want to interpret the advice in this section according to the physical environment in which you are teaching.

  Some students have the annoying habit of setting the alarms on their electronic watches to chime precisely on each hour. Worse, some students set their alarms for five minutes before the hour or three minutes before the hour or two minutes after the hour. Since no two watches agree, what you experience at the end of class is a cacophony of electronic beeping. This distracts the students, it disrupts your class, and it is a damned nuisance. Tell the students in advance—on the first day of class—that you want all electronic alarms turned off during your class. Be stern about it, and nail those who fail to comply.
- If a student asks a question that requires a long answer, don't let your answer eat up valuable class time. Tell the student that the question is a good one, but ancillary to the main subject matter of the course (it had better be, or else you evidently forgot to cover an important topic) and that the question can best be treated after class. However, do not let the student get the impression that the question is being given the brush-off.
- On the other hand, if a student asks a question for which a brief answer is appropriate (such as "Shouldn't that 2 be a 3?" or "When is the next homework assignment due?") then do give a suitably brief answer. Anecdotes about your childhood in Shropshire are probably out of place.

By the way, this last is more than a frivolous remark. As we slide into our golden years, we seem to be irretrievably moved to share with our students various remembrances of things past—"It seems to me that twenty years ago students worked much harder than you people are willing to work." or "When I was a student, we put in five hours of study for each hour of class time." or "I used to

[11] I once taught in a classroom in which there was a buzzer that went off three minutes before the end of the class period, and another that went off at the end of the class period. This was just the pits. Because, as soon as the first buzzer went off, students started packing up their books and turning off their brains. It was really counterproductive.

walk six miles barefoot through the snow to attend calculus class, and it was uphill both ways." Trust me: Students hate this sort of emotional slobbering. You will defeat all the other good things that you do by giving in to this temptation to prattle.

If you have the time problem under control at the level of individual class meetings, then you will have the ability to pace your course in the large as well. You should have a good idea how much material you want to cover. And when you plan the course you should allot a certain number of class periods for each topic. If you are teaching undergraduates, they depend on your course for learning a certain body of material (that may be prerequisite for a later course). Don't shortchange them.

A test should be designed for the allotted time slot. You can rationalize giving a two-hour exam in a one-hour time slot by saying to yourself that there is so much material in the course that you simply *had* to make the test this long. This is nonsense. The point of the exam is not to *actually test* the students on every single point in the course, but to make the students *think* that they are being tested on every point in the course. Ideally, the students will study everything—but your test amounts to a spot check. Even if you had a four-hour time slot in which to give the exam, you couldn't really test them on everything, now could you? See Section 2.9 on exams.

If you give a two-hour exam in a one-hour time slot, then you run several risks: that students will become angry, demoralized, alienated, or all three. Telling a student not to worry about his/her grade of 37/100 because the average was 32/100 does not work. Students are unable to put such information into perspective.

1.7. Applications

One of the most chilling things that can happen to an unprepared, unseasoned faculty member is to have a belligerent (engineering) student raise his/her hand and say, "What is all this stuff good for?" And one of the most irresponsible things that a faculty member can say in response is, "I don't know. That is not my problem." If you do not have an answer for this student question, then you are not doing your job.

I have found it useful in all of my undergraduate classes to tell the students about applications of the techniques being presented *before* the aforesaid chilling question ever comes up. This requires a little imagination. If I am lecturing about matrix theory, then I tell the students about Markov processes, or I say a little bit about image processing and data compression and the fast Fourier transform. Or I tell them about eigenvalue asymptotics for clamped beams and applications to the building of a space station (not coincidentally, this is a problem that I have worked on in my research). If I am lecturing about surfaces then I tell them about the many applications of surface design problems. If I am lecturing about uniform continuity or uniform convergence, then I tell them about some of the applications of Fourier analysis. The pedagogical technique that I am describing in effect defuses any potential belligerence from engineering or other students who have no patience for mathematical abstraction.

Carrying out this teaching technique requires a little forethought and a little practice. After a while it becomes second nature, and you will find yourself thinking of potential answers while taking a shower or walking to class. If it suits your style, keep a file of clever applications of elementary mathematics. It is not true that the concept of uniform convergence is used on a daily basis by civil engineers to construct bridges. Do *not* use this facile line of reasoning to talk yourself into abandoning the effort to acquaint lower-division students with the applications of mathematics. Instead, reason that uniform convergence is a bulwark of the theory behind the practical applications of mathematics. It is important. Act as though you believe it.

Try to be flexible and to reach out for up-to-date and striking applications. Uniform convergence is a basic idea in the convergence of series of functions. One of the most interesting uses of series is in Fourier analysis. And what is Fourier analysis good for? Mention the hot new theory of wavelets and some of its uses. Wavelets were used to clean up a recording of Brahms; they are used every day by the FBI to analyze and compress fingerprint files; they are used to analyze electrocardiogram readings; they are used to analyze biological neural systems. A few years ago two mathematicians were awarded the Grammy for producing (using wavelets) a clean version of the only known live recording of Woody Guthrie. Don't you think this is impressive? Wouldn't your students agree?

It is a matter of personal taste (and much debate) as to how much should be done with applications in, say, a calculus class. For most of us the problem is solved by the very nature of the undergraduate curriculum. There is little opportunity to do any but the most routine applications. But times are changing and many mathematics departments are re-evaluating their curricula. There is considerable enthusiasm for infusing the freshman-sophomore curriculum with more applied material. Examples of new approaches to the calculus, by way of applications and technology, can be seen in the Amherst project materials [**CAL**] and the Harvard materials [**HAL**]. Other "reform" projects include [**ANG**], [**C4L**], [**DIP**], [**DSM**], [**OSZ**], [**SMM**], [**STEW**], [**WAT**]. Some more modern references include [**HALG**], [**STC**], and [**LAR**]. The project [**KOB**], available for free on the Worldwide Web, does not fit comfortably into the "reform" rubric. But it is a new approach to the calculus text, and it is noteworthy for its applications.

If you decide to work applications into your class then consider this: Mathematical modeling is complicated and difficult. If you take an already complex mathematical idea and spend an hour applying it to analyze a predator-prey problem, or to derive Kepler's third law, or to design the Wankel engine, then you are likely to lose all but the most capable students in the room. How will you test them on this material? Can you ask the students to do homework problems if their understanding is based on such a presentation?

Think of what it is like to teach the Divergence Theorem. There are almost too many ideas, layered one atop the other, for a freshman or sophomore to handle. Students must simultaneously keep in mind the ideas of vector field, gradient, surface, surface integral, curl, orientation, and so forth. Most cannot do it. The same phenomenon occurs when one is attempting to get students to understand a really meaty application. I am not advising you against doing these applications. But if you should choose to do one, go into it with your eyes open. If, after fifteen minutes, the students' eyes glaze over then you will have to shift gears. Be prepared with a physical experiment, or a computer simulation, or an overhead slide, or a transition into another topic.

A stimulating presentation of an application should be broken into segments: a little analysis, a little calculation, a little demonstration. Lower-division students cannot follow a one-hour analysis. Always bear in mind that, no matter how satisfying you find a particular application, your audience of freshmen may be somewhat less enthusiastic and may require help and encouragement.

On the other side of the coin, don't get sucked into doing just trivial, artificial applications. This cheapens our mission in the students' eyes and makes us seem disingenuous. The calculus and its applications are among the great achievements of western civilization. Be proud to share with your class the analytical power of calculus. Do so by presenting some profound applications, but put some effort into making the presentation palatable.

My experience is that, for freshmen, brief (*not* trivial) applications are the best. They can be modern, they can be interesting, but if the analysis entails layering too many levels of ideas on top of each other, then most students will be lost. What you should do, in preparing to present an application to your class, is to lay the whole thing out on a piece of paper and then examine each step. Consider which steps you can actually discuss in detail with the class, and which steps you should summarize. It is perfectly all right, in the middle of your presentation, to say, "Now I'm going to skip some difficult calculations. I have them available on a handout (or a Web site!). The calculations produce the following formula." Or you could say, "A delicate physical analysis, which you can view at thus and such a Web site, yields the following mathematical model." It is part of our training to want to show *every single dirty rotten detail*. When teaching your freshmen, learn to resist this temptation. Make your presentation fit into a fifteen minute window, and make it accessible to your audience.

No matter how you package your wares, there is always the danger that afterward some student will ask the question, "Will this be on the test?" What can you say? Will a long, elaborate application be on the test? No, but some of the analytical techniques that you used in the example could be. Or an example based on the applied analysis could appear on the test (but be sure to do such an example in class). You had better have an answer prepared for questions such as these. See also Sections 1.5, 3.7 on answering difficult questions.

Do not give applications that consist of a lot of terminology with a negligible mathematical kernel buried in it. For example, a recent calculus text has a problem about the destruction of trees in a tropical rain forest. After wading through dozens of pieces of unfamiliar terminology, the student finds that all he/she is required to do is to take the derivative of the given function and set it equal to zero. Doing such an example gives the students the message that applications are nothing but mindless drivel. And that is the *wrong* message. Another calculus text has a problem that begins (here I am paraphrasing) "The function $y = x^2$ is important in relativity theory and cosmology . . . " The author is of course alluding to Einstein's equation $E = mc^2$. It is easy to see that such a problem is thoroughly disingenuous, and that any student who knows a little physics will be insulted and turned off. This is *not* the way to gain credibility with your students.

Learn not to treat applications as something "apart" from the subject proper of the course. In fact applications can be a device for helping students to learn key mathematical ideas. As a simple instance, the notion of velocity helps students to understand what the derivative means. As a more sophisticated example, the notion of centrifugal force helps students to understand what curvature means. Applications should not be a tiresome appendage that is tacked onto the course. They should be an essential part of what they are learning.

Recently I taught a rather sophisticated course in multi-variable calculus. I say that it was sophisticated because it was preceded by a mandatory course in linear algebra, and linear algebra was used throughout the multi-variable calculus. Advanced ideas were considered frequently. When we got to Green's theorem, I took great pleasure in telling students something of the life of Green, and that one can visit his mill today and purchase (for fifteen pence) a pencil with Green's theorem emblazoned on the side. Then I told them, as an application of Green's theorem, that you can calculate the area inside a planar region Ω by calculating the following boundary integral:

$$(*) \qquad \oint_{\partial\Omega} \frac{1}{2}x\,dy - \frac{1}{2}y\,dx.$$

Most sophomores are not sufficiently experienced to appreciate what this statement says, so I told them about the "planimeter," which is a hardware device—commonly used by draftsmen—that actually implements equation $(*)$. When I said to them, "You take this little doohickey and run it around the edge of a figure on a piece of paper and it tells you the area inside," they just went wild. *This* practical piece of information they really understood. We looked at several examples of pairs of domains with equal perimeter but different areas and discussed how a mechanical device tracing the boundary might distinguish the two. The students had a great many questions. When class was over, several of them stormed over to our rather distinguished School of Architecture and demanded to see a planimeter. The rejoinder from the architects was, "What's that?" When told, the architects said, "That's impossible. You'd need a computer to do something like that." Well, there was nothing for it. I had to go out and buy a planimeter.

This cost me a pretty penny, but it was a terrific object lesson for the students. I cared enough to follow up on what I had told them. I stood my ground against the architects. And I triumphed!

William Thurston, in his article [**THU**] on the teaching of mathematics, points out that math is a "tall" subject and that math is a "wide" subject. The tallness articulates the fact that mathematics builds up and up, each new topic taking advantage of previous ones. It is wide in the sense that it is a highly diverse and interactive melange. It interfaces with all of the other sciences, with engineering, and with many other disciplines as well. It is our job as teachers of mathematics to introduce students to this exciting field, and to motivate students to want to study mathematics and to major in it. Applications are a device for achieving this end. Using them wisely and well in the classroom is a non-trivial matter. Talk to experienced faculty in your department about what resources are available to help you to present meaningful applications in your classes.

An unpublished biography [**TBJ**] of R. L. Moore (see Section 1.8) tells of Moore's rough-and-ready childhood schooling in a one-room schoolhouse in late nineteenth century East Texas. One Fall, the headmaster was overheard on the first day of the school year saying to a parent "Do you want me to teach the earth round or flat?—I can do it either way." This is an approach to applications that we should all best avoid. When you give applications, tell the truth. Or else don't give any.

1.8. The Moore Method

R. L. Moore was an important and influential faculty member at the University of Texas from the 1920s to the 1970s. Possessed of a strong and single-minded personality, he developed a special method of teaching that has been successfully practiced and disseminated by his students and followers even to the present day.

The Moore method worked like this—and it was practiced in all his classes, from the most elementary undergraduate to the most advanced graduate. Students would show up on the first day, usually with no notion of what Moore was like or what he expected. He would hand out a single mimeographed sheet with some definitions and some axioms and some theorems stated on it. And he would say, "Who would like to go to the board and prove the first theorem?" Of course Moore was usually greeted by bewildered silence. Many of his students did not know the meaning of the words "definition" and "axiom" and "theorem" and "proof," nor did they know the basic rules of logic. But Moore could take it. He sat and he waited. Often the first hour would go by without a word spoken. And so would the second. And many times the third as well. Eventually, some brave soul would go to the blackboard and attempt a "proof" of Theorem 1. Moore would rip that person apart. And that set the tone for the class.

In each session of R. L. Moore's class, some student would go to the board with a proof he/she had concocted, and everyone else in the class—student and professor alike—would examine it critically. There were no holds barred. Moore would not *assist* the students in creating a proof, but he would *respond* to whatever they said.

The good thing about Moore's classes is that his students came to *possess* the mathematics. If you put it incorrectly on the blackboard, then you were subjected to merciless criticism. If you got it right, then you received quiet but sincere approbation.

Moore did not allow his students to read books or papers (this included even his graduate students!). Indeed Moore saw to it that the mathematics library got no funds for new books! As legend had it, if he found a student of his in the library he would throw that person (bodily) out the door. Students were to create the mathematics themselves—from whole cloth. The class worked best if the students began with approximately equal abilities. They were not allowed to read, and they were not allowed to collaborate outside of class. Moore was merciless in weeding out those students who did not cooperate or who did not fit in.

The Educational Advancement Foundation (EAF) grew in recent years out of the Moore method. Generously funded by a former student of R. L. Moore, it seeks to promulgate a version of the Moore method that incorporates Inquiry-Based Learning. This in turn uses ideas from Cooperative Learning. EAF now has an academy at California Polytechnic University, and has a strong following.

I leave it to you, the reader, to determine the good and the bad points of the Moore method. Mary Ellen Rudin, one of Moore's most prominent and successful students, does not recommend it. She says, in part, "All you get [from this method] is a (probably overconfident) uneducated ignoramus." By contrast Paul Halmos, an eloquent avatar of good teaching, says, "The Moore method is, I am convinced, the right way to teach anything and everything—it produces students who can understand and use what they have learned." Moore's teaching method—and his techniques for selecting the students whom he wanted in *his classes*, and whom he would mentor—caused emotions to run high in the University of Texas mathematics department. It is said that "...the department was literally divided into two camps: the Moore and the anti-Moore. The Moores had the fourth floor, and it was armed: guns in the drawers up there." See [**BAP**], [**TBJ**], [**HALM**], [**WIL**] for my sources, and for more on the Moore method. A more recent source is [**COP**].

The Moore method is obviously a powerful way to teach any subject. Its success was as much a product of Moore's strong personality as it was of its essentially combative nature. And Moore's students—especially his graduate students—often became strong proponents of the method. Moore had a good many Ph.D. students who went on to become prominent mathematicians. His method spread far and wide, and it was difficult to grow up in American mathematics in the 1960s and 1970s without hearing of, and experiencing, the Moore method.

I took one class that was taught by Moore's method. Fittingly, it was point-set topology (Moore's pet subject). I both loved this class and hated it. I loved it because I loved the subject matter and I loved learning it; I hated it because I thought the professor was lazy and not doing his job. Moore would have smiled in approbation.

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CHAPTER 2

Practical Matters

2.0. Chapter Overview

Like many activities in life, fine teaching is composed of many technical components. When everything works properly, then the whole is considerably greater than the sum of its parts. However, if some of the crucial parts are rusty or, worse, non-functional, then the whole will creak and drag and not do a good job of it.

The novice instructor should probably read every section in this chapter. The more experienced instructor may wish to pick out particular sections for concentrated effort.

2.1. Voice

There is nothing more stultifying than a lecture in a reasonably large classroom on a hot day delivered by an oblivious professor mumbling to himself /herself at the front of the room. We are not all actors or comedians or even great public speakers. But we are teachers, and we must convey a body of material. We must capture the class's attention. We must *fill the room.*

If you are unlucky, then you may be assigned to teach in a classroom that works against you. Perhaps visibility is poor for the students in the back, or the acoustics are bad, or the blackboard is substandard. If you are burdened with such a teaching environment, try your best to get it changed. If you cannot, then think hard about how to get the best out of this classroom. If the blackboard is unusable, then consider lecturing with an overhead projector or a computer projector. It may be possible to get a portable SmartBoard (see Section 4.12). If the acoustics are bad, then consider using a microphone. If visibility is poor, then think about changing the seating arrangement. No matter what the liabilities, you must take charge.

Your voice is one of your primary tools. And you must use that tool in part to control the environment in your classroom. The most important presence in the room is not the blackboard, nor the desk, nor the text. It is *you.* You want the students' attention focused on you.

I am not saying that you must lose your dignity, or act silly, or show off. You must learn to use your voice and your eyes and your body and your presence as a tool. If you are going to say something important, then make a meaningful pause beforehand. *Say* that it is important. Repeat the point. Write it down. Give an example. Repeat it again.

You can gain the attention of a large group by lowering your voice. Or by raising it. Or by pausing. One thing is certain: You will not gain the audience's attention by rolling along in an uninflected monotone. Again, I am not suggesting that you undergo a personality change in order to be a sound teacher. What I am suggesting is that you find ways to talk to them as a *person interacting with people*, rather than as an ill-at-ease, out-to-lunch egghead.

At a well-known university (of good quality!) in southern California they once tried bringing in actors from Hollywood to help professors spice up their delivery. One instructor of an Abnormal Psychology course was advised to use the line, "I never teach about any mental illness until I try it out myself." Such pandering is inappropriate, offensive, and childish. Can you imagine yourself using such patter? Who would want to?

What I am suggesting here is that you take just a little time and contemplate your lecture/classroom style. A lecture or class should be a controlled conversation between you and your audience. It is a trifle one-sided, of course. But there must be cerebral interaction between the teacher and the students. That means that you, the instructor, must grab and maintain the attention of the class. Your behavior in front of the group is a primary tool for keeping the lines of communication open.

When you are talking about a subject that you perceive to be trivial, and when you are nervous, you tend to talk too fast. Novice instructors find themselves barreling through their lectures. You must resist this tendency. If you are really new at the business of teaching, then practice your lectures. Get a friend to listen. In calculus, a fifty minute lecture with three or four good examples and some intermediate explanatory material is probably just about right (I'm thinking here of a lecture on max-min problems, for example). Try to make each class consist of about that much material, and make it fill the hour. If you finish early, that is fine (but it may mean that you talked too fast). You can quit early for that day, or do an extra example, or use the extra time to answer questions.

Part and parcel of the message in the last paragraph is that we instructors also tend to do too much in our heads—we don't write enough on the blackboard. This is a dreadful mistake. Students have a hard enough time keeping up even when you write *everything* on the board. Skipping steps will leave even the good students in the dust. Write out everything. Write it slowly, and write it clearly. But write it.

Don't give the students the impression that you are in a rush. It puts them off, and reflects a bad attitude toward the teaching process. If on Wednesday you plan to explain the chain rule, then do just that. If the chosen topic does not fill the hour, then do an extra example or field questions. Do not race on to the next topic. One idea per class, at the lower-division level, is about right. (Of course if you are teaching a multi-section class at a big university, then it is important to keep pace with the other instructors. This is yet another reason for keeping careful track of your use of time. See also Section 1.6.)

It is something of an oversimplification, but still true, that a portion of the teacher's role is as a cheerleader. You are, by example, trying to persuade the students that this ostensibly difficult material is doable. Part of the secret to success in this process is to have a controlled, relaxed voice, to appear to be at ease, and to be organized. Don't let a small error fluster you. Make it seem as though such a slip can happen to anyone, and that fixing it is akin to tying your shoelaces or pulling up your socks.

But, as with all advice in this book, you must temper the thoughts in the last paragraph with a dose of realism. If you make the material look easy, then students will infer that it *is* easy. The psychological processes at play here are not completely straightforward. Nobody would be foolish enough to go to an Isaac Stern concert and come away with the impression that playing the violin is trivial. Yet students attend my calculus classes, watch me solve problems, conclude that the material is easy and that they have it down cold. They then decide that in fact they *don't* need to do any homework problems or read the book, and then they flunk the midterm.

These are the same students who come to me after the exam and say, "I understand all the ideas. The material is absolutely clear when you talk about it in class. But I couldn't do the problems on the exam." How many times have you heard such a statement from your students? I like to tease my students by reminding them that this is like saying, "I really understand how to swim, but every time I get in the water I drown."

On the one hand, you don't want to make straightforward material look difficult. After 300 years, we've got calculus sewn up. There is no topic in the course that is intrinsically difficult. We merely need to train our students to do it. So *do* make each technique look straightforward. But remind the students that *they themselves need to practice*. Do this by telling them so, by giving quizzes, by varying the examples and introducing little surprises. Ask the class questions to make the students turn the ideas over in their own minds. Use your *voice* to encourage, to wheedle, to cajole, to question, to stimulate.

Mathematics instructors in general, whether they are "reformers" or "traditionalists" or "high techers" or "plug-and-chuggers," agree that each student must take each idea in the course and rebuild it in his/her own mind. This is nothing new. Go read Beth and Piaget [**BPI**]—they discuss this notion in detail. An awareness of this concept will help you to shape your teaching methods. If the students cannot understand what you are talking about, then it is unlikely that they will take the ideas home and think about them. If the students watch you state and prove a lot of abstruse theorems, and in the process become terminally depressed, then it is unlikely that they will take the ideas home and analyze them and internalize them. If the students watch you flounder around, unable to complete an example coherently or explain a concept neatly, then it is unlikely that they will take the ideas home and rebuild them in their own minds.

If instead you kindle the students' curiosity, plant in them a desire to learn, show them something they have never seen before and *make them realize that it is something they have never seen before*—and certainly never understood before—then there is a real likelihood that they will leave class turning the new thoughts over in their minds, talk among themselves about it, ask questions, and come back to you with their own ideas. *That* is teaching.

Even if you know how to use your voice well with a small audience, and to capture their attention and get them excited about learning, there are special problems with the large classes that are used in the teaching of calculus (for instance) at many universities. Refer to Section 2.13 for more on this matter.

2.2. Eye Contact

We all know certain people who invariably emerge as the leader of any group conversation. Such people seem to sparkle with wit, erudition, and presence. They usually pick the topic and they usually aim the discussion. They have a sense of humor, and they are intelligent. What is their secret?

It is partly a matter of attention and awareness. The sort of person I am describing has an inborn curiosity; he/she is aware of you, and interested in you, and genuinely eager to learn about your opinions and experiences and interests. When you ask yourself what makes another person interesting, an honest answer would have to entail that such a person is outer directed, and cares about others.[1]

This is obviously a talent that is partly inborn and partly cultivated. Some of the trick is to show genuine interest in what other people have to say before bounding ahead with what you have to say. Another part is to talk about subjects, and to tell anecdotes, that you know will interest other people. Being charming and witty helps too, but in this section I want to concentrate on more mechanical features of repartee.

Many of the devices that make for an engaging conversationalist also make for an engaging teacher. A review of the last paragraphs, and of the rest of this book, will bear out this assertion. In this section I will discuss the importance of *eye contact*.

Telling a good joke while staring at the floor with your thumb in your ear will not have the same effect as telling the joke while looking at your listener, engaging his/her attention, and reacting to the listener while the listener is reacting to you. A good joke teller has his/her audience starting to chuckle half way through the joke and just dying for the punch line. Getting a good laugh is then a foregone conclusion.

Another aspect of good joke telling—one for which Jack Benny was justifiably noted—is timing. You need to a tell a joke at the right pace, to know when to insert pauses, to know when to raise your voice or lower it, to know when to grimace or smile. Delivery is everything.

Giving a good lecture or class is serious business, and is not the same as telling a joke. But many of the moves are the same. If you want to hold your audience's attention then you must look at your audience. You must engage not one person but all. You must learn to use your body as a tool. Step forward and back. Force the eyes of the audience to follow you. A good lecturer speaks to individuals in the audience, to grouplets in the audience, and to the whole audience. Like a movie camera, you must zoom in and zoom out to get the effects that you wish to achieve. A ninety minute movie filmed at the same constant focal length would be dreadfully boring. Ditto for a lecture.

Some people are very shy about establishing eye contact. It is a device that you must consciously cultivate. The end result is worth it. The teacher who can establish eye contact is also the teacher who is confident, who is well prepared, and who conducts a good class.

2.3. Blackboard Technique

Make sure your blackboards are clean before you begin. Take extra time to erase the old stuff thoroughly.

Write neatly. Write either in very plain longhand or print. Be sure that your handwriting is large enough. Be sure that it is dark enough. Endeavor to write straight across the blackboard in a horizontal line. Proceed in a linear fashion. Don't have a lot of insertions, arrows, and diagonally written asides.

Don't put too much material on each board. The ideas stand out more vividly if they are not hemmed in by a lot of adjacent material. In particular, it is difficult

[1] A boring person is one who talks about himself. An interesting person is one who talks about you.

for students to pay attention when the teacher fills the board with long line after long line of print. An excellent guitarist once said that the silences in his music were at least as important as the notes. When you are laying material out on a blackboard, the same can be said of the blank spaces.

Some people find it useful to divide the blackboard into boxes. This practice makes it easier for the lecturer to organize what he/she is writing, and also makes it *much, much easier* for the students in the audience to organize the material in their minds and in their notes.

Label your equations so that you can refer to them verbally. Draw sketches neatly. Use horizontal and vertical lines to set off related bodies of material.

You can control your output more accurately by keeping the length of each line short. Think of the blackboard as being divided into several boxes and write your lecture by putting one idea in each box. To repeat: If necessary, *actually divide the blackboard into boxes.*

If the classroom has sliding blackboards, think ahead about how to use them so that the most (and most recent) material is visible at one time. For those combinatorial theorists among you, or those experts on the game of NIM, this should be fun.

If you are right-handed, consider writing first on the right-hand blackboard and then working left. The reason? That way you are never standing in front of what you've written. Good teaching consists in large part of a lot of little details like this. You shouldn't be pathological about these details, but if you are aware that they are there then you will pick up on them.

Every now and then during your presentation, you should stand aside and pause. Don't say *anything.* This gives you an opportunity to collect your thoughts and catch your breath. You can verify the accuracy of what you have written. It gives the students an opportunity to catch up and to ponder what they've heard. They may decide to formulate questions. If instead you are barging ahead at full speed for the entire hour, then students never have a moment to think about what they are hearing. They cannot interact with you because you are not interacting with them.

Try to think ahead. Material that needs to be kept—and not erased—should be written (probably in a box) on a blackboard to the far left or far right where it is out of the way but can be referred to easily. You may wish to reserve another box on the blackboard for asides or remarks. Some instructors put material that needs to be seen for the entire hour on an overhead slide. This frees up all the blackboards, and keeps those important equations or definitions front and center.

These ideas are another facet of the precept that you know the material cold so that you can concentrate on your delivery. Just as an actor knows his/her lines cold so that he/she can make bold entrances and exits, and not trip over his/her feet, so you must be able to focus a significant portion of your brain on the *conveying of the information*.

If your lesson will involve one or more difficult figures then practice them on a sheet of paper in advance. Remember that you are a mathematical role model for the students. If you make it appear that it is difficult for *you* to draw a hyperboloid of one sheet, then how are the students supposed to be able to do it? Of course you can prepare the figure ahead of time on an overhead slide (or even photocopy it straight out of a book, or straight from `Mathematica` or `Maple`, onto a transparency). This solves the problem of having a nice figure to show the students. It does not solve the problem of *showing the students how to draw the figure*. As a result, it puts a barrier between them and the ultimate goal of learning how to *read* the graph. If necessary, consult a colleague who is artistically adept for tips on how to draw difficult figures.

You will find students quite resistive to learning to graph—especially in three dimensions. I learned a useful teaching device when last I taught multi-variable calculus, and it became clear to me when I was showing my students how to graph. That device is *persistence*. I made it clear to them that anything that gave them a pain in the neck was going to appear repeatedly in subsequent work. For example, almost all of them did rather poorly on the graphing portion of the first midterm. So I gave them several followup quizzes to help them hone their graphing skills. After each quiz they all gave me their "Is that finally the end of graphing?" look. But after examining their work I said, "Nope; not good enough yet." And on we went. Over and over again I graphed functions in three dimensions. I went through every step. And I did it *at the blackboard*, just as I expected them to do it with pencil and paper. Graphing appeared again on the final. And I *told* them that it would so appear. In fact I told them that the best way to study for the final was to find everything on the first two midterms that they hated, and that this material would certainly be on the final. They believed me, and it worked.

If you cannot organize the steps of a maximum-minimum problem, then can you really expect the students to do so? In the best of all possible worlds, the students' work is but a pale shadow of your own. So your work should be the platonic ideal. Sometimes, in presenting an example or solving a problem at the blackboard, you may inadvertently gloss from one step to another. Or you might make a straightforward presentation look like a bag of tricks. Or you may do some of the steps in your head. This practice is very confusing for students, especially the ones who lack confidence. By organizing the solution in a step-by-step format you can avoid these slips.[2]

After you have filled a board, it should be neat enough and clear enough that you could snap a digital snapshot and read the presentation from the screen on your camera. In particular, you should not lecture by writing a few words, erasing those, and then writing some more words on top of the erased old words. Students cannot follow such a palimpsest. I cannot emphasize this point too strongly: Write from left to right and from top to bottom. *Do not erase.* When the first box is filled, proceed to the second. *Do not erase.* Only when all blackboards are full should you go back and begin erasing. Students must be given time to stare at what they've just seen as well as what is currently being written. Keep as much material as possible visible at all times.

BUT: When it is time to erase, be sure to erase thoroughly. It is well worth spending a few extra moments being sure that the blackboard is sparkling clean before you begin a new block of material. If you endeavor to write over a sloppily erased blackboard, then your writing will be obscure at best. Everything will look fuzzy. This is really a psychological issue. Of course the students can squint and strain and figure out what you are writing (even if it is a virtually unreadable

[2]When I solve a maximum-minimum problem, I have eight steps that I *always* follow. Every time I do an example I carefully enunciate each of the steps.

melange of overwrites), but it bums them out to have to do so. Try to make their job as easy as possible.

Do not stand in front of what you are writing. Either stretch out your arm and write to the side or step aside frequently. Read aloud to the class as you write. Make the mathematics happen before their eyes and *be sure that they can see everything.* Every once in a while, pause and step aside to catch your breath and to let them catch up.

Here is a common error that is made even by the most seasoned professionals. Imagine that you do an example that begins with the phrase "Find the local maxima and minima of the function . . . " And so forth. Say that you've worked the example. Now suppose that the next example begins with the same phrase. It is a dreadful mistake to erase all but that first phrase and begin the new example on the fly, as it were.

Why is this a mistake?—it *seems* perfectly logical. But the students are taking notes! How can they keep up when you pull a stunt like this? *Slow yourself down.* Write the words again. If a student gets two sentences behind then he/she may as well be two paragraphs behind. Give frequent respites for catch up.

And now a coda: How much of what you are saying should you write? In my experience, the answer is "As much as possible." When you are transmitting sophisticated technical ideas verbally, students have trouble keeping up. Many of them are not native English speakers. They need a little help. Write down everything except asides (actually, some asides are worth recording as well). Say the words as you write them. This is also a device for slowing yourself down. Most of us tend to talk far too fast—at least about mathematics. Slowing yourself down and writing deliberately will help you to keep your handwriting clear and will make the lesson as a whole appear to be neat and clean. If the *appearance* is neat and clean then perhaps the *ideas* are neat and clean—at least that's what you want the students to think.

The flip side of the last paragraph is that the tendency to talk too rapidly may cause you to write too rapidly (and therefore sloppily). Thus periodically checking the quality of your handwriting on the blackboard can serve as a means of telling whether either your verbal or written delivery is too speedy.

Let me reiterate one of my most fundamental precepts. There is a real psychological barrier for the instructor to overcome when learning blackboard technique, and voice control. When we understand very deeply what we are talking about, then it all seems quite trivial. We can convince ourselves rather easily—at least at a subconscious level—that it is embarrassing to stand in front of a group and enunciate whatever mundane material is the topic of the day. Thus we are inclined to race through it, both verbally and in the way that we render it on the blackboard. *Be conscious of this trap and do not fall into it.* I have never been criticized for being too clear, whether I was giving a calculus lecture to freshmen or a seminar lecture at the Mathematisches Forschungsinstitut Oberwolfach. *Slow down.* Be deliberate. Enunciate. Explain.

Many of us, at the beginning of the class, rattle on verbally at some length before we finally persuade ourselves that we had better start writing something on the board. Please don't do this. Start writing the material from the very outset. If you want the students to notice it, and write it down, and get it straight, then you had better set the example by writing it.

When I taught at UCLA we had a chair who was frequently engaged in delicate negotiations on international phone calls with nonnative English speakers. One instance was a case of trying to help a Russian mathematician expatriate. The chair would take copious notes during the phone conversation, so that he could later consult with his executive committee about the situation. But he was a very nervous sort of guy, and he usually found that he could not read his notes. Literally! So he could not report on his phone conversations and his negotations usually ended up being ineffective.

You do *not* want your teaching to be like what I described in the last paragraph. You must *force* yourself to write boldly and clearly. Usually it is best to print rather than to use longhand. You should write large and as plainly as possible. Put plenty of space between the words and between the lines. Slow yourself down and make it come out right. It should be possible for a student to take a digital photo of your blackboard (with his/her phone, for example) and take it home and read it.

Writing material neatly and slowly is a subtle way of telling the students that this material is important. If you are taking the trouble to write it down deliberately, then it must be worth writing deliberately. Conversely, if you scribble some incoherent gibberish, or scribble nothing at all, then what signal are you sending to the students?

2.4. Homework

In most lower-division courses, and many upper-division ones, it is by way of the homework that you have the greatest direct interaction with your students. When students waylay you after class or come to your office hour, it is usually to ask you about a homework problem. This is why the exercise sets in a textbook are often the most important part of the book (many textbook authors do not seem to have caught on to this observation yet) *and* why it is critical that homework assignments be sensibly constructed.

Let me stress again that I am not trying to sell you a time-consuming attitude or habit. If you take twenty minutes to compose a homework assignment then you are probably taking too much time. But consider the following precepts:

- Do not make the homework assignment too long.
- Do not make the homework assignment too short.
- Check over the problems you assign to confirm that there are no notational or obvious typographical errors. (Students can waste great amounts of time trying to fathom typos that are trivial to you and me. As a result, they become quite frustrated and angry. Doing this sort of checking shows them that you are on their side.)
- Be sure that the assignment touches on all of the most important topics.
- Be sure that the homework assignment drills the students on the material that you want them to learn and the material that you will be testing them on.
- Generally speaking, the homework problems should resonate with the material you present in class. If you lecture on A and give homework on B and test the students on C you will really create a world of hurt. Be consistent.
- *Make sure that at least some of the homework problems are graded.*

- Plan ahead. The exams that you give should be based only on material that the students have seen in the classes and in the homework.

If homework does not count and is not graded, then students will not do it. That is a fact. I realize that many of us have neither the time nor the inclination to spend long hours each evening grading homework. Many universities and colleges these days simply do not have the resources to provide enough graders for lower-division courses. But there are compromises that you can make. For example, you can tell the students that, of ten problems on the homework assignment, just three will be graded. But don't tell them which three.[3] This device will force most of the serious students to do *all* the homework problems, but it requires much less grader time to get the grading done.

If the last suggestion will not work for you, then you can give weekly quizzes that you yourself will grade. The amount of your time involved will be little, and it is a device to force students to keep up with the work. Incidentally, this device also gives you a gentle way to keep your finger on the pulse of the class.

Today there are Internet alternatives to traditional homework. One that is freely available is `WeBWorK` from the University of Rochester. This is an Online homework system that gives students instant feedback on the quality of their work. In some of my classes the for-credit homework assignments are done on `WeBWorK`. In other classes I use `WeBWorK` as a supplement to the traditional hardcopy homework. See Section 4.9 for more on `WebWork` and similar tools.

Consider implementing the following policy to help get your students more interested in doing the homework. Students can and do benefit from collaboration, just as we mathematicians do in our research. While you probably do not want to encourage collaboration on exams, you may wish to encourage it on homework. Of course I'm not talking about "I'll copy yours this week and you can copy mine next week." Instead, I'm talking about an intelligent exchange of information among equals.

Some studies have shown that one reason that Asian students in this country tend to do very well in their mathematics classes (and there are surely many reasons) is that they work in groups. More precisely, they first work hard as individuals. Then they get together and compare results. In short, they collaborate in much the same way that mature mathematicians collaborate. They are willing to say, "I can do this but I cannot do that. What can you contribute?" At the same time, the studies indicate that certain other elements of the student population are either loath to work in groups or are unaware of the benefits of this activity. These strata tend to do poorly in mathematics classes. See [**TRE**] for details.

Some of the more interesting teaching reform projects, including those from Harvard and Duke, are specifically designed to encourage students to learn mathematics through group activities. Reports on these experiments are encouraging.

If you do decide to encourage group work in your classes, then you will have to make peace between said collaboration and your grading policies. If homework is not collected, then there is no problem and you can separate the good students from the bad through exams and quizzes. If instead homework is collected, then

[3] The famous math teacher Ray Redheffer used to give *two* homework assignments in each class meeting. He told the students that he would be collecting one of them next time, but he would not tell them which one.

you will have to consider carefully how to tell whose work is whose, or at least how to divide up the credit.

2.5. Office Hours

At most universities the instructor is required to hold two or more office hours per week.[4] Choose three hours that are convenient for you or convenient for the students or both. Monday/Wednesday/Friday at 11:00 A.M. is, on most campuses, one of the most popular times for classes. If you schedule your office hour at that time then many students will not be able to attend. One good strategy is to stagger your office hours, so that they are at different times on different days. Another is to make an office hour from half past the hour to half past the hour, so that a student's class is likely to overlap only half of it rather than all of it.

Of course you cannot select a time for office hours that will please *everyone*, so don't even attempt to do so. Set your office hours, and announce them, and explain to the students that you can make appointments for those who cannot attend the regularly scheduled hours. Such an announcement will not appreciably increase the number of visitations from your students, and it is just good business to set such a policy.

Promise students that you will be there during your office hour. And be there. Students should be made to understand that they need not wait for a natural or personal disaster in order to come to your office hour. It is perfectly all right for a student to come to your office hour and say "I don't get problem 6." or "The chain rule makes no sense to me."

During your office hour, you will usually not be overwhelmed with students (except perhaps just before an exam). In fact it is a general rule of thumb that, the larger the class, the smaller the percentage of students who will come to your office hour. But those who do show up will appreciate your attentions. Of the hours that you have designated, you can spend some of them catching up on your correspondence, making up the next homework assignment, or reading the *Notices* or the *Monthly* or the *Mathematical Intelligencer*.

If you have sufficient space in your office, it is a good idea to have a table and a couple of chairs set up in a special part of the office—*away from your desk and your papers and books and personal artifacts*—where you will consult with students. What is the reason for this affectation? First of all, you don't want students inadvertently walking away with your papers or your correspondence. Second, you don't want them spilling coffee on your latest manuscript or your new book that you purchased at great expense from Marcel Dekker. Third, students are by nature careless. They may put their feet on your desk or use your telephone or grab your fountain pen. Rather than appear to be an old fuddy-duddy and constantly be scolding, it is so much easier to have a special venue in which to "hold court."

When a student comes to my office expressing befuddlement over a particular type of problem, I have a powerful and decisive weapon that I unleash. I begin by asking, "Do you have a half hour or so?" If the answer is "Yes", then I sit the student down and say, "Try a problem of the kind you are having trouble with. When you get stuck, tell me." Of course the student invariably gets stuck, and I

[4]I also know of schools—good schools—where teaching is the main activity and professors are expected to have their office doors open all day long. And the students really take advantage of this largesse.

give him/her a little help. I might need to intervene three or four times during the first problem that the student does for me. But the second problem may require only two interventions, and the third only one. By the time the fourth problem rolls around, the student's newfound confidence is irrepressible, and the transaction is a great success. The student goes away pleased and happy that he/she has now mastered a heretofore mystifying mathematical idea. Of course I always tell the student, "If you get home, and you find that you are still confused, then come back and we'll do this again."

On days when your office hour is not crowded, and you only have a couple of customers, I highly recommend that you try this teaching technique. It's good business, and it always produces satisfied customers. Word gets around in the class that the professor is not such a bad guy after all. Perhaps, as a result, a few other students will drop by for help.

You want to convey to students that the office hour is a particular time that you have set aside for them. If you consult with students while sitting at your desk and glancing at your mail, or scribbling notes for an upcoming seminar, or reading your email, or answering phone calls, then you in fact will *not* convince them of your dedication. Instead, if you hold court in the special part of your office that you have set aside for consultations, then your students will understand that this time is theirs. If you really want to do it right, then let your voice mail pick up on your phone calls during your office hour. For those sixty minutes, give yourself to your students.

The office hour is your opportunity to get to know at least some of your students personally. Of course I do not mean by this that you should get involved in their *personal lives*. Problems about their love lives or their parents or their social diseases should be referred to the professional counselors that are on the staff of every college. What I mean is that you should take the opportunity to get to know some of your students as people, and to let them get to know you as well.

This activity has several beneficial side effects, both for you and for them. When you are lecturing, you can have certain individuals in the room in mind as you formulate your remarks. You can make reference (*without* mentioning any names) to questions that came up during office hour. It is reassuring to the average student (the type that *does not* go to office hour) to know that conscientious students (the type that *do* go to office hour) have some of the same questions that they have.

This point is in fact worth developing. Some components of teaching may be compared with certain aspects of psychotherapy. One big feature of therapy—certainly an aspect that is exploited by popular psychology and self-help books—is to assure the patient that he/she is not alone. There are thousands of people with exactly the same problems, suffering in just the same ways. And they have been treated successfully.

Just so, when you teach you must give both tacit and explicit reassurances to students that their questions and confusions are not theirs alone. An eighteen-year-old is scared to death that he/she is the only person in the room who doesn't understand why the numerator in the quotient rule has the form that it has—or why it does not seem to be symmetric in its arguments. Such a student would not dare ask about it in front of a room full of his/her peers. The student may not even be sure how to articulate the question, so surely will not want to flounder about in front of the entire class. At the same time the student may be afraid to come to your office hour and, alone but in *your* august presence, ask for a clarification.

Thus you must signal to students that questions are a good thing. When a student asks a question in class that might be of general interest, I not only repeat it but I often state that I am glad this question was raised. I carefully record the question on the blackboard. Several people have visited me privately, I add, and asked variants of the same question. If there is a question that should be asked but has not been, then I ask it myself. I say that if this point is unclear to them (the students) then they should come see me in my office hour and get it straightened out. You don't need to give away door prizes to drum up business at your office hour. However, it is psychologically important for students to know that you are available, whether they actually come to see you or not.

I have said repeatedly in this book that persistence is an important attribute for the successful teacher. Another such attribute is patience. If a student finds the nerve to ask a question in class or during office hour—*even if it is a question that I have answered before*, in fact *even if I have answered it several times before*—then I treat the questioner with respect and I pay due homage to the question and I answer it. I *never* say, "I've already answered that question. Go home and read your notes." Such a rejoinder would be counterproductive, and would discourage further question-asking in class.

I often announce to my classes that students may drop by my office even when it is not my office hour. If I am not busy, I'll be happy to talk to them. In practice, this charity does not appreciably increase the flow of business. There are always students who strictly respect your designated office hour and there are always those who drop by whenever they please. But making an announcement of this nature is one of those little details that contribute to a favorable student attitude. For it sends a signal to the class that you care, and that you truly want to help them learn mathematics. If you do make such an announcement, be courteous to those who take you up on it. If you are busy and must send the student away, do so with respect and suggest another time for the student to return.

I once had a colleague who, whenever a student would show up at his door, would crawl under his desk until the student went away. This is certainly a memorable way to deal with students, but not one that I would recommend. When a student comes to your office, make him/her feel welcome. Act as though you are happy that he/she cared to dropped by. Endeavor to adopt the same cheery tone that you would assume if a good friend paid a surprise visit. Such an action on your part will put the student at ease, and will make the transaction go smoothly and productively.

The office hour is a way to step out of your role as instructor and let the students know that you are a person. It is a way to become acquainted with some of your students. Any good public speaker "works the audience" before his/her speech. Holding your office hour is one way to work the audience. You will also get a feeling during the office hour for how the class is doing, what problems and concerns have arisen, how the pace is working. It is wrong, and self-defeating, to view your office hour as a dreary duty. It is a teaching tool that you should use wisely.

2.6. Designing a Course

Many of us never have the privilege of actually designing a course. Instead, we are assigned to teach prepackaged courses that the department has already assembled. This will especially be true if your teaching load is primarily "service courses": precalculus, calculus, linear algebra, ordinary differential equations. In non-service courses—upper-division courses or courses that are taught for majors—you may in fact have considerable discretion as to what you will include in the course, and how you will organize it. In what follows, I shall draw a sketch of what input you may have into the structure of a course, and also what input you may not have.

If you are teaching one of the prepackaged service courses, then certainly the content will be pre-specified. And, like it or not, you had better stick to the syllabus or outline that the department provides for you. Your students are taking this course *only because* it is required for their major. If you are an instructor who louses up this course for students (so that they must repeat it), then before you know it other departments will be designing their own course to substitute for this one. Such a consequence of your actions will not endear you to your department chair. Funding and hiring at many universities is linked rather directly to the number of courses offered and number of students taught. You do not want to be known as the instructor who killed Math 117.

Even if the course you are teaching seems to be good old-fashioned pure math, and is taught almost exclusively to mathematics majors, you may find that your department has rather rigid ideas about what should be included in the course. The department may also have a committee that selects the text.[5] If you are not sure what text to use, or what course outline to follow, don't be afraid to approach a more experienced faculty member for advice on this matter (see also Sections 2.11, 2.12).

If indeed you are teaching a course—perhaps an advanced topics course—where you will have a free hand, then the main precept is to *slow down*. It is too easy to fall into the trap of preparing the course for your fellow faculty (don't worry—they probably have other things to do). This is another good opportunity for obtaining advice from a more experienced colleague. Draw up a tentative outline for the course and show it to a friend. Design the course in such a way that there are several natural places to quit for the semester. Don't drive students away with a syllabus that reads like "Everything that I know, or wish that I knew, about mathematics." Generally it is better to slow down, to give more examples, and to achieve more depth, than to set up a situation that allows you to go to the next conference and brag about all the topics that you covered and all the others that you wanted to cover but could not.

If you have the opportunity to design a course, then approach the task logically. First write down the key ideas in the subject. Then think about how you want to flesh out each one. How will you illustrate the important techniques? What links will you construct between the big ideas? How will you order the topics? How will you examine the students on this material? What you are doing, in effect, is living through the course once in your head, so that when you do it with the students it

[5] The only "letter of reprimand" that is in my personnel file at my current university stems from my having selected my own text for a course for which an official text had already been selected by a committee.

will go smoothly. It is too easy for a mathematician to spend a couple of weeks of class time going down a (rather foolish) primrose path that is dictated by his/her momentary fancy. A little planning can help you to avoid such a reckless teaching error.

Draw up a syllabus (Section 2.11) that is reasonable, and that accurately reflects what you intend to cover in the course. Most students do not have the maturity or experience to look at a completely impossible course syllabus and reason that "The instructor will never cover all this. Of course he/she will slow down." You have to provide the leadership, both in the small and in the large.

Your reading list also sends important signals to your class. If your required reading is *Linear Operators*[6] by Dunford and Schwartz and the complete works of Leonhard Euler,[7] then (no matter what the merits of these books, or of your course), you will have no audience.

Use a little practical sense when selecting a text. Also remember that, although most of your students love knowledge, they will not want to spend $250 for a textbook. Section 2.12 offers detailed advice on how to select a text.

2.7. Handouts

It is tempting to write up a lot of handouts for your course. If you give a class hour on Stokes's theorem and feel that you have not made matters clear, then you might be inclined to draw up a handout to help students along. You also might suspect that this extra effort on your part will improve your teaching evaluations and, in particular, that students will appreciate all this additional work that you have put in. Well, it won't and they don't. Only prepare a handout when it will really make a difference. Students feel that they already have enough to read. Inundating them with handouts will only confuse them.

Some professors prepare handouts to expose the students to more meaty problems than those in the text, or to explain ancillary material, or to give summaries of key ideas. Sometimes a handout can supplement what is in the text. I cannot argue that in every circumstance at every college this is a bad thing to do.

What I can do is examine my own conscience and tell you what I see. If I give a lesson that is not up to snuff, or if I do a poor job explaining what curvature is, or if I goof up a proof in class, then I can salve my conscience by writing up a handout. It takes about an hour, it is a way of doing penance, and it is a way of working past the guilt of having screwed up in class. In my heart of hearts, however, I know that what I *should* do is strive to give better classes.

Just speaking for myself, I find it more comfortable and convenient to put ancillary material for the course on the Web. This seems to be less "in your face," and the students think of it as a resource that they can consult if and when convenient. Class emails are also a useful device for supplying extra information or small corrections.

Let me temper these remarks with one important exception. At many universities, it is common to distribute prepared lecture notes. At some, such as UCLA, the student association hires graduate students to prepare careful lecture notes of key courses (such as calculus) and sells them in the student store. This can be a real boon to the students. First, many a student is unable to take good notes and

[6]An important historical work comprising 2500 pages and three volumes.

[7]The complete works comprise over 70 volumes.

listen to the lecture (and think!) at the same time. Knowing that good notes are available for a modest price gives such a student the freedom to sit back and really listen. Second, having prepared notes available makes missing class a less onerous inconvenience than it would be otherwise.

Having an institutionalized lecture notes system is akin to providing students with a textbook. It does not really fly in the face of what I said in the first few paragraphs of this section. You will have to use my advice here in the context of what resources are available at your institution.

2.8. Teaching Evaluations

Mention teaching evaluations to most faculty members and you will get reactions ranging from horror to nausea: "Teaching evaluations are just a popularity contest." or "Kids these days don't want to be taught; they want to be entertained." or "I know that I'm a good/competent teacher, and I certainly don't need to be evaluated. I only distribute the damned forms because the administration makes me do it."

In this section I am going to tell you that, yes, individual student teaching evaluations can be irritating. They can reflect nothing more than the students' immaturity, or pique, or the sorry state of their digestion. But, taken as an aggregate, student teaching evaluations contain valuable information that will help you to become a better teacher.

The simple fact is that most people in most lines of work are regularly evaluated. The evaluation of physicians and lawyers is perhaps rather distant and painless. But it is done. It is a relatively recent development that tenured professors are evaluated—*by their students*—for teaching competence. How did this come about?

Part of the root of the teaching evaluation process comes from the student unrest—especially at U. C. Berkeley—in the 1960s. Students were partly distressed by free speech issues, and partly because they could never find their professors outside the classroom. In those days, professors were not required to hold office hours! The then chancellor at Berkeley, Clark Kerr, recalled wistfully that in the old days the duties of a chancellor were to provide "parking for the faculty, football for the alumni, and sex for the students." He could see the tides of change coming, however, and he helped to usher them in.

Today, there are manifold reasons for teaching evaluations. University administrations are taking a hard look at tenured university faculty and demanding accountability (in some contexts, this sort of examination operates under the rubric "Total Quality Management" or TQM—see Section 2.13 for more on TQM); a system of post-tenure review is being put in place at more and more universities; taxpayers are pressuring universities to hire faculty who can teach (and who can speak English); if a department chair wants to get a faculty member tenured or promoted, then he/she must provide ample evidence that the candidate is a good teacher.

You may find that reading your teaching evaluations is a gut-wrenching experience. As an instance, James R. Martino of Johns Hopkins University compiled the following pairs of quotations (as cited in [**MAR**]). Each pair comes from different students evaluating the same instructor for the same course.

> **1.** Does a good job of explaining new concepts and doing examples to demonstrate how it's actually done.

> **2.** Hopefully, he can explain the concepts at a more fundamental level; he assumes that some of the concepts are too trivial to be explained fully in class.

> **1.** Great in all aspects of communication ... is extremely intelligent when it comes to knowing [the] course.
> **2.** The man is a horrible teacher. I'm sorry to say that he reflects the math department.

> **1.** [The professor] is a giant. The man is humorous and very intelligent. He actually makes math interesting.
> **2.** I don't go to lecture anymore because I never learn anything from the instructor. He seems to be talking about irrelevant topics and phrases questions in ways which are hard to understand. I see no benefit and reason to go to lecture except to obtain the homework assignment.

Not logically consistent, are they?[8] Do these examples prove that teaching evaluations are worthless? That they are the puerile rantings of unformed minds? I don't think so. The extremes of opinion that you see here are no more bizarre than those that you would see in any public opinion survey. Any statistician will tell you that a sample of two is far too small for obtaining useful information.

When you read your teaching evaluations, don't let the outliers upset you. On the one hand, the isolated opinions may be those of thoughtful iconoclasts who really have something to say. You may indeed learn something from the musings of those independent thinkers. On the other hand, the outliers could be people who have been struggling all semester, who are too timid to get help, who are having personal problems, who have been ill, or who just like to complain. As award-winning teacher Tom Banchoff [**BAN**] says, "good teaching is not identical with perfect ratings." Instead, if you want to learn something about your teaching and improve it, look for trends in the student evaluations. If ten students say that you talk too softly or too rapidly, or if many of the evaluations say that you are a poor communicator, or if a plurality indicate that you cannot explain ideas at the students' level, then you should consider these criticisms carefully. Examine your conscience and determine whether you can learn something from your teaching evaluations and improve your technique.

If I have succeeded in convincing you that there may be something of value in teaching evaluations (taken as a group, not necessarily individually), now let me send chills up your spine. The article [**AMR**] describes the following experiment (which I present here in slightly simplified form), that was performed at an American college. Ten instructors were chosen from ten different departments. A random group of thirty undergraduates was assembled. Over the course of several days, each instructor gave a fifty minute lecture to this same group. At the end of each lecture, each student completed a standardized written teaching evaluation form for that instructor. The results were tabulated. Now each lecture was also filmed, and a ten second slice was taken from the beginning, from the middle, and from the end of each lecture. The sound was removed, and the thirty excerpts were

[8] Note that we are unlikely to see pairs of teaching evaluations that differ over whether the professor wore plaid socks. What they will differ on is *subjective matters*, like clarity or level.

spliced together in random order. The result was a five-minute silent film showing randomly arranged ten-second clips of each of the ten instructors.

Next, a hand-picked group of thirty sophomore women was assembled, and they viewed the five-minute silent movie. Each young woman completed the same standardized written teaching evaluation form for each instructor—based only on what she saw in the silent movie. The results were tabulated. The startling fact is that the data from the first group of students—who saw the original lectures—correlated extraordinarily well with the data from the second group—who only saw the silent film. What conclusions may we draw from these observations?

It is easy to focus on the flaws of this experiment. You may well ask, "How can a ten second slice show what a professor can really do?" If the professor is in the middle of the proof of the fundamental theorem of calculus, or explaining the more technical aspects of Marx's theory of commodity valuation, then he/she will not show well in a ten second slice. Perhaps, as in the review of a good restaurant, one should view the professor over several different days. Keeping these limitations in mind, let us see what we might learn from the study.

I think there is a nugget of insight buried in this exercise, and it may be this: Young people do not generally have the intellectual equipment to determine whether their calculus teacher or their genetics teacher really knows his/her stuff, or is doing an optimal job, or knows how to communicate the material most cogently. But young people have a lifetime of experience evaluating body language and nonverbal communication. The second group of students (the sophomore women) *only had* body language and nonverbal communication on which to base their evaluations.[9] Since their evaluations correlated very closely with those of the first group (the students who experienced *all* aspects of the lectures), one is tempted to conclude that the first group based its assessments also on the body language and nonverbal communication aspects of the lectures.

Cynics like to conclude from this study that students simply cannot evaluate your teaching. They are just reacting to the "vibes" that you give off. If you are hellbent on writing off teaching evaluations as worthless, this reasoning may appeal to you.

An informal study conducted at Cornell University [**CEW**] performed the following experiment. A single professor taught the same class, to similar audiences, two semesters in a row. The first time around the lecturer evinced a monotone, staid teaching style. The second time around, he showed great enthusiasm for his task, using many hand gestures and vocal diversity. The students reacted far more positively in the second semester than in the first; in fact in the second class they even gave the textbook more enthusiastic praise. What may we conclude from this information? One arguable conclusion is that it is almost impossible for a person to evaluate his/her own teaching. While the Cornell professor was being monotone and staid, he may also have discouraged student questions and done a poor job of answering those that he did receive. In the second semester, while being enthusiastic, he also may have interacted much more vigorously with the students. Even so, the experiment is worth considering. Studies show that students react more to the

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[9]Sophomore women were chosen because, in the opinion of the experimenters, young women are more sensitive to nonverbal communication than are young men.

instructor's enthusiasm than to anything else. The Cornell experiment only serves to affirm that finding.[10]

Students are not scholars. But they are people. They probably react to human input more than they react to intellectual input. That is just a given, and it is not likely to change. Even so, what they have to say has value. If a freshman says, "This instructor doesn't know what he is talking about." then you have reason to be skeptical. If instead that freshman says, "This instructor cannot present the material in a manner that is accessible to freshmen." then perhaps he/she is giving you information that you could not obtain in other ways. Frankly, a student is probably much more sensitive to this issue than you are. Your level of knowledge and understanding is likely so elevated that you simply *don't know* when you are pitching things at the wrong level.

A student certainly has something of value to say about whether he/she feels he/she is learning something, about whether the instructor can explain the material. You should give the matter careful thought so that you can put your teaching evaluations into context and learn something from them.

There are objective studies that suggest that student evaluations of a given instructor are consistent over time, that they correlate well with administrative and peer review, that (taken as a whole) they are independent of extraneous student characteristics, and that they correlate significantly with how much students actually learn (see [**CEN**], [**COH**], [**FEL1**], [**FEL2**], [**HCM**], and further references in). The paper [**KULM**] offers statistical evidence that student teaching evaluations measure (*accurately*) **(1)** instructional skill, **(2)** respect and rapport, and **(3)** instructional organization (remember that these are *statistics*; they do not speak of any one particular teaching evaluation form, but of the overall content of the aggregate of teaching evaluation forms). This information may or may not appeal to you. But there it is.

Studies of the teaching evaluation process are ongoing. Some current references are [**DAN**], [**KKP**], [**MAR**], and [**MAT**]. As the years go on, and your teaching experience accrues, you will develop your own ideas and values about teacher evaluation.

I repeat: It is a noteworthy observation that student teaching evaluations correlate well with peer evaluation (see [**DAV**]). They do *not*, however, correlate well with self-evaluation. On days when you are sitting around badmouthing your teaching evaluations as immature and worthless, bear this thought in mind.

In my experience, teaching evaluations (again taken as a group) have empirical value. Obviously, what information you glean from your teaching evaluations is up to you. If a significant number of your students say that you make too many mistakes in class, or you ridicule people who ask questions, or you don't prepare, or your tests are unfair, then I think it's a cop-out to just claim that they are reacting to your vibes. After all, this is your audience giving you feedback. Why not try to learn from it?

On the other hand, if a large number of your students say that you are inspiring, well-informed, creative, and an excellent pedant, then don't just say, "Aw shucks" and forget about it. This is positive reinforcement. You must be doing something right!

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[10] I generally get good teaching evaluations. The traits of mine that students cite repeatedly are my knowledgeability, my enthusiasm, and my easy-going nature.

Bear in mind that the evaluation of teaching need not take place only at the end of the semester. Mid-semester evaluations can be extremely useful. They tell you how the course is going, whether the students perceive that they are learning anything, what problems may have arisen.[11] It is one thing to get your evaluations at the end of the term and say to yourself, "Boy, there's something I did wrong. I'll try to fix it next time." It is quite another to realize after only half the course has gone by that corrections must be made *while the course is in progress*.[12] For then you have an opportunity to make things right, and your end-of-term evaluations should improve as a result. The book [**GOL**] offers a number of insights on course evaluation techniques.

Some math faculty are uncomfortable with the standardized teaching evaluation forms that the university provides. Usually these forms ask the student to answer twenty or so questions with a rating of "1" to "5" to indicate "Poor", "Fair", "Good", "Very Good", "Excellent". While some of the questions are about reasonable issues (such as "Was the text useful?", "Was the instructor prepared?", etc.), others are rather vague ("Evaluation of the Instructor Overall" and "Evaluation of the Course Overall"). It is a fact that the Dean is a busy person and wants a quick and dirty estimation of the teaching of any given faculty member. Often the numerical responses to these last two questions can give such a gauge, but it is not a gauge with which the faculty member being examined is necessarily copacetic.

You may feel, with some justification, that the evaluation process described in the last paragraph does not give the mathematics professor an adequate chance to show what he/she can do. If this describes you, then I encourage you to discuss the matter with your colleagues, your chair, and your dean. The deans I have met are not averse to individual departments developing their own methods of teaching evaluation. Possibilities to consider are

- **(1)** The department formulating and writing its own teaching evaluation form. This could be a traditional hardcopy form, or a newfangled Online form.
- **(2)** Videotaping of lectures.
- **(3)** Peer review.
- **(4)** Self-evaluation.
- **(5)** Consultation with other faculty experts.
- **(6)** Exit interviews by a third party professional.

Let me give a brief explanation of each of these techniques.

(1) Polling is not a trivial matter, and it requires a special skillset to put together a questionnaire that will elicit the information that you seek. Nonetheless, members of the department may find it cathartic to engage in this exercise. And there is little doubt that they will be more comfortable with the result. The chair may have a job convincing the dean that this is a teaching evaluation form that he/she can rely on for the information that he/she needs. But that will be a job worth doing.

(2) Few things are more honest, more stark, and sometimes more demoralizing, than a videotape of a lecture. The videotape will show all your awkward

[11] Note that, at many institutions, midterm teaching evaluations can be just between you and the students. Nobody else need see them after you have collected them.

[12] My university actually requires midterm evaluations for certain freshman courses.

mannerisms, your squeaky voice, your dandruff, your strange pauses and facial expressions. A lecture that you thought was ethereal will come across as peculiar when first viewed on videotape.

If you agree to have yourself videotaped, then view the tape with an experienced faculty member who can point out both what is good and what needs improvement. You will need some help in keeping the matter in perspective.

(3) Peer review consists of having some of one's colleagues attend some class lectures, and perhaps also review some class materials that were prepared, by the person being reviewed. Most intelligent people feel comfortable receiving cogent remarks from someone whom they respect and admire. Why not learn about teaching from a fellow mathematician?

(4) Self-evaluation might consist of the candidate preparing a "Teaching Portfolio." This portfolio would contain an enunciation of the candidate's overall teaching goals, plus a list of particular goals connected with particular courses that the candidate is going to teach. A mentor would review the portfolio regularly with the candidate, and help him/her assess whether he/she is achieving his/her objectives.

(5) Some people are shy, or uncomfortable in front of groups, or are poor speakers. Notable success has been had by having such individuals work with experts from the Communications Studies or Speech Department. While it may be uncomfortable to have a fellow mathematician tell the candidate that his/her teaching is inadequate, it may be more natural (akin to going to a fitness trainer or a podiatrist) to have the candidate consult with a well-meaning faculty member from another department. And I can tell you that this is a method that works in practice.

(6) I rather like this last method for evaluating teaching, but it is expensive both in terms of time and in terms of money. The idea is that a professional interviewer, perhaps someone with a background in psychology, will interview each student (or perhaps a sample of the students) at the end of the course. The interview can be brief—perhaps ten or fifteen minutes. But the interviewer can ask questions that will draw out the student's concerns. He/she can also zero in on important points that the student is trying to articulate and help him/her to develop them. Finally, the reviewer (ideally) will have statistical skills that will make it possible to amalgamate all the input and derive various trends and patterms.

At the end of the interview process, the interviewer will write up an in-depth report on the class, and the instructor, in question.

Once you start thinking creatively about ways to evaluate teaching, you will certainly develop ideas of your own. Bear in mind, as you do so, that the Dean has an affection for "Evaluation of the Course Overall" and "Evaluation of the Instructor Overall" because these simple questions give him two numbers. He/she can quickly assess whether a given candidate cuts the mustard or not. When you devise alternative assessment techniques, be sure that each one results in useful and accurate advice for the teacher being examined and also in a *quick and incisive* take on the candidate's teaching abilities. The Dean does not have the time to view videos, or read long position papers. He/she may not insist on a *number*, but he/she needs the evaluation to be of the nutshell variety.

I shall conclude this section with a story of which I am rather fond. My university has a teaching award (for faculty) that is entirely student-driven. That is to say, the students nominate the candidates, the students select the winners, and the students present the awards. One of my friends won one of these awards, so I went to the ceremony to show my support. The thing that struck me about the student presentations about the awardees—and these were faculty from a great variety of departments across the university—is that they all said the *same thing*. That is, what they appreciated about this particular faculty member is that he/she *cared*, he/she learned their names, he/she took an interest in the learning process, he/she was always available. These are points to ponder when you consider the nature and quality of your teaching.

2.9. Exams

In this section I will discuss how to compose an exam, how to formulate questions on an exam, how to judge the length of an exam, how to grade the exam, and how to evaluate the results of an exam.

I will also discuss larger issues: **(i)** How much should you tell your students about what is on the exam?, **(ii)** How should you handle student questions about how the exam was graded?, **(iii)** How comprehensive should you make your exam? I will also discuss, in some detail, the question of whether exams should be multiple choice or of the (more traditional) written-out variety.

Let me state my thesis quite plainly. Handwritten exams, in which students write out complete solutions to stated problems, are good. Multiple choice, machine-graded exams are not so good. Of course nothing is black and white. Handwritten exams have their down side and multiple choice exams have their up side. The relevant issues will be developed as the section unfolds.

In most elementary math classes (and many advanced ones) the principal device for determining grades is the examination. These are usually (but not always) given in class, or during a special time slot in the evening. There are a number of points of view about what constitutes a good exam.

Some professors attempt to put together elaborate exam problems, each of which synthesizes several of the concepts introduced in the course. This practice causes me to pose some questions which you should ask yourself frequently when you teach or write: "Who is my audience? Am I trying to teach eighteen year olds or am I trying to impress myself? Am I trying to effect an educational experience? Or am I trying to put together an exam that I can show to my cronies while crowing about how dumb it proves the students to be?"

By contrast, there is the "minimalist" exam. A famous old exam from M.I.T. consisted of the single problem

> You have a pile of warm metal shavings in the shape of a cone. Discuss.

There's a conversation stopper. On the other hand, a notable instructor at that same venerable institution for many years formulated final exams as follows

> There are fifteen important concepts in this course. Discuss any thirteen of them, outlining key ideas and providing proofs as time permits.

These types of exams may be suitable for certain students at M.I.T. some of the time. They are not appropriate at most universities today most of the time.

My practice is extreme in yet a third direction. I usually tell my students what will be on the exam. No, I don't write each exam problem on the blackboard during a review session. But if a student asks, "Will we be tested on the chain rule?", I give him/her an honest answer (with the understanding that if I say "yes" then this should be construed as "maybe"). If the student says "How many problems are on the exam?" then I tell. If a student wants to know how many questions are multiple choice and how many not, I give. To deny this information is just power tripping. It serves no good purpose.

To be honest, 95% of my exam questions (in an elementary course) are straightforward. They offer no surprises. They are similar, but not isomorphic to, homework exercises. With the other 5% I am more fast and loose. I use these as a vehicle to identify the really bright and able students in the class.

I know good teachers at first class universities who take the straightforward approach one step further. They have a blanket policy in all elementary classes (calculus and linear algebra and ODE, let's say) that *all* exam questions come directly from the homework. Literally. And they announce this on the first day of class and repeatedly throughout the course. It's an interesting policy. They tell the students exactly what will be on the test (in a sense), but on the other hand they really don't. This policy leaves students little room for complaining about the content of exams. On the other hand, it does not challenge them. And it encourages them to memorize (and perhaps to cheat!). Use this policy with caution.

Exam time is when you really have the students' attention. Get as much from it as you can. Drive home the important ideas of the course. Give a thumbnail sketch of the evolution of these ideas a few days before the exam. Such a review helps students to organize their thoughts.

Your exams are one of your most important tools for communicating with your class. The students may be at only half mast during some of your classes. But at exam time they are giving you their full and rapt attention. This is your big chance to tell them what this course is about, and how they are doing in it. There is no sense to use your exams as a device for alienating the class, and there are so many ways in which you can do so. If you are consciously going to give your students a killer exam, then you should ask yourself *why* you are doing it. What are you trying to accomplish? Whom are you trying to impress? Consider carefully before you give such an exam. If the class is already dead then giving a hairy exam will pound the final nail into the coffin's lid. If the class is instead on your side, then why make a conscious effort to drive the students away?

Put another way, the purpose of a class is to transmit knowledge and information. Any given class has a dozen or more key ideas in it. That is what the tests should be about. A midterm or final exam in a basic course should not be a repository for ancillary theorems. It should not be a forum for obscure results not covered in class, or touched upon only in passing. An exam should be about the *principal topics* in the course—ones that you have emphasized and illustrated and repeated (*ad nauseum* if necessary). Topics covered on the exam should be ones that the students have heard about in class and seen in the homework.

Make sure that the questions you ask elicit the basic information that you seek. If your question about the chain rule turns into an algebraic morass, then it does not test the students about the essential material that they are to have mastered. If your maximum-minimum problem involves arithmetically or algebraically complex expressions that obscure what is going on, then you are not really testing the students as you wish to do. Thus it is important that you, the instructor, work the test problems through in advance.[13] This takes some time, but less time than all the aggravation that ensues if you give a poorly formulated or carelessly prepared exam.

Multiple choice or show the full solution? There are arguments for and against both systems. From the professor's point of view, one argument for multiple choice is that the grading of the exams requires no effort (in many cases it can be done by machine). And the exam is completely objective. But these reasons are a bit self-serving, and there is another more interesting consideration.

If you give traditional exams on which students write out solutions to the problems, then you usually fall into the malaise when grading of giving a lot of partial credit. Since you are human, you may tend to give even more partial credit on the 75$^{\text{th}}$ examination paper than on the 5$^{\text{th}}$. The upshot is that it is actually possible for a student to get through the entire calculus sequence, with a grade of "C" or better, not knowing any particular calculus technique in its entirety. Because their grade is based on 40% of this problem and 55% of that problem and never 100% of *any* problem. By contrast, it can be argued, the multiple choice exam has the advantage of requiring the student to actually *get to the correct answer* on a number of problems. But there is more to mathematics than just getting the correct answer. So you must consider to what extent your multiple choice exam is exposing students to the wrong value system.

On the other side, it can be argued that multiple choice exams involve a lot of gamesmanship. A student who has not studied, but who is clever, can sometimes get a reasonable grade on such an exam just by guessing shrewdly. (Of course you can offset this feature with negative scores for wrong answers. Also, if you give about ten possible choices for each question, and if the exam is otherwise well constructed, then you can make this eventuality unlikely.) It can also be argued that it is easier for students to cheat on a multiple choice exam.

I think that a more serious point about multiple choice exams is similar to the liability of large lectures. They don't do a good job of engaging the student in the learning process (see Section 3.7). A handwritten exam is a form of discourse between the student and the teacher (Section 3.9). The student writes his/her thoughts, the teacher evaluates those thoughts, and the student (ideally) learns from the exchange. A multiple choice exam is more like getting money from an Automatic Teller Machine. The job gets done, but no nurturing or growth occurs.

You also have to ask yourself which type of exam really tests the students on what they should be learning. Are they learning problem-solving skills? Are they learning the key ideas? Can they state the theorems? Can they prove them? Do they understand the definitions? Can they reproduce, with comprehension, the

[13]Of course part of my job as professor in a course is to write up the exam solutions for posting on the Internet. I always do this right away, right after I have written the exam. That way I can verify that all the exam questions work out as they should, that there are no excessively tricky points, and no calculational messes. Having done so, I can give the exam with confidence and be reasonably sure that nothing can go wrong. Sometime I even have my TA work the exam too. That provides an extra reality check.

important examples? It seems patently clear that a written-out exam can do all of these. A multiple choice exam would be less informative in almost all instances.

A common student complaint about multiple choice exams, and one which I find difficult to gainsay, is that the student can do a problem almost completely correctly but have a small arithmetic slip, with the result that he/she cannot find the correct choice among those given. If, instead, the exam had allowed the student to submit his/her full solution for reading by the instructor, then the student would no doubt have received substantial partial credit. Instructors will argue that students should learn to be accurate. A small arithmetic slip will cause the bridge to fall down or the brain surgery to go awry.

Perhaps especially critical these days is that multiple choice exams do not appear to be a good vehicle for training students to do multi-step word problems. This is one aspect of mathematical training in which American students lag behind students in Japan and other countries. A well-crafted written-out exam can walk the student through six or more steps, beginning at square one and ending with the solution of some really interesting problem or phenomenon. This can be done with a multiple choice exam too, but it is much trickier to pull it off.

If I am teaching a large class (200 students or more) in which a hand-graded exam is infeasible, then I find it useful to compose my exams as follows: If there are twelve problems on the exam then ten of them are multiple choice and two are "short answer". The short answer problems are of the sort that I can grade instantly—just by glancing at them.

The students in large classes that I have taught are comfortable with an exam that is primarily multiple choice. But they appreciate the personal touch suggested by a couple of short-answer problems that are graded by hand.

It seems to me that, in a small class (60 students or fewer), the professor can write a traditional exam requiring full answers to questions and then spend some time grading the papers carefully. In this context you can not only attend to the grading yourself but you can make constructive comments. These comments can be brief, and they can be encouraging. The serious students do read them, and do benefit from them.

I have presented arguments in favor of machine-graded multiple choice exams and also arguments against them. Once again, I shall be prescriptive: Hand-graded exams are better. They keep you in touch with how the class is doing as a whole, and also with individuals in the class. They give you the opportunity to discern what topics require additional coverage in class. Your comments on the exam are a useful part of the teaching process. If it is at all feasible, even in a class of eighty or more students, endeavor to give traditional hand-graded exams (or at least an exam that has a hand-graded component).

It is tempting, especially for new instructors, to hold review sessions for exams. This is a way of making yourself feel generous, it is easier than doing something more productive, and it will make the students grateful. But it also makes exams seem more onerous than they really are. (If you do decide to hold a review session anyway, then read Section 2.14 on problem sessions.) And it makes the students who cannot attend the review session feel as though they are at a serious disadvantage. I find it more useful to write a practice exam that I distribute a week in advance of the real test. About two days before the test I post solutions to the practice problems (either on the class Web page or on a bulletin board or both). Of course there is always the danger that students will think that first reading the practice problems and then reading your solutions will constitute studying for the exam. I always caution the students strenuously against this trap. No system is perfect.

Tests that are too long, or too involved, do not work. Your exam should contain a reasonable number of questions of reasonable length, and they should not be interlinked. If problems are interconnected, and if a student makes a critical error in one of these, then all of the related problems are affected. If test problems are too involved then students can panic, mismanage their time, and turn in a performance that does not at all reflect their true abilities. Let me stress once again that you *need not* examine students on every topic that has been covered. Make it clear to them that they are *responsible* for all the topics; but the exam will be, in effect, a spot check.

Master teacher Tom Banchoff [**BAN**] recommends the following technique for dealing with student panic on exams. He gives regular, 50 minute, in-class exams—as we all do. But each student has the option of going home and re-working the exam at leisure to show what he/she really knows. Banchoff takes both performances into account when he does the end-of-term grading.

By the same token, it is sometimes appropriate to give a "take-home" exam. You will have to decide whether the particular class you are teaching can be trusted with such an exam. And then you will have to lay down some ground rules. Open book or closed? Timed or not? Can consult other people or not? A take-home exam gives the students an opportunity to really show what they can do. But it has many unmanaged aspects that can lead to trouble.

It is very easy to misjudge a test that you write. A problem that seems trivial at first blush may have complex arithmetic or algebra hidden in it. Thus *you must personally work the test out completely before you give it to your class.* An exam that you can do in twenty minutes—with all solutions written out neatly—is probably about right for a 50 minute exam for a class of freshmen. If it takes you 40 minutes, and you find yourself laboring over the algebra or arithmetic, then obviously this is not a suitable 50 minute exam for freshmen.

The point value of each exam question should be clearly exhibited on the exam. The total number of possible points on the exam should be displayed. It is tempting to make difficult problems worth a lot of points and trivial problems worth very few. But of course the end result, since many students will not do well on the hard problems, is that the class average is pushed down. On the other hand, you don't want to make the easy problems worth a lot of points and the hard ones worth just a few—this sends entirely the wrong message to the class about what is important. So you must strike a balance.

It is a useful device to break difficult exam questions up into steps. This practice helps the weaker students to get started, and to display what portion of the material they actually know. It also makes the exam easier to grade, and increases the consistency of your grading.

When you are grading exams, it is important to be as consistent as you can be. Begin by writing out the solution to each problem. Break the solution into pieces and assign a point value to each part. Thus, in a maximum-minimum problem, setting it up might be worth 3 points, doing the calculations another 3, and enunciating the answer another 3. One spare point for overall analysis makes a total of ten. Some instructors like to be even more precise than this. Refer to Section 2.13 for the concept of "horizontal" grading for insuring uniformity.

Remember that some students, the day the test is returned to them, will come to you with questions about how their individual exams were graded. In some cases, they will come with a friend and ask why two similar solutions were graded differently. If you are systematic, then you can handle such transactions with dispatch.

As a general rule, I would advise that you meet with students *privately* to discuss their exam grades. Do not do it after class, in a context where several other students can evesdrop. Do not do it in groups. This is a personal transaction, like a consultation with a physician, and you should respect both your and the student's privacy.

Should you write your exams out in longhand (with a pen), or should you word-process or TeX the exams? Even though it is a certain amount of extra trouble, I would recommend that you typeset your exams using TeX. The end result is crisp and clean and really cannot be misunderstood. Whereas a handwritten exam, no matter how wonderful your handwriting may be, can easily be misconstrued.

Anyone who knows anything about TeX will tell you that the hardest part of this high-level language is formatting. And, for an exam, you need to do a fair amount of just that. But it is really worth the effort, and both you and your students will appreciate the result. In addition, the chair, the dean, and the parents will see your efforts as professional and worthy. And that is probably a good thing.

Now back to the trenches. When teaching a big class, it is best to generate some statistics about each exam that you give. When you hand an exam back to 200 people and someone asks, "What is the average?" or "What is the cutoff for an 'A'?" then you had better have an answer ready. The alternative is chaos. Therefore consider calculating the mean, the mode, and the median (if you don't know these words then look them up). Calculate the standard deviation and use it as a guide in setting up your grading curve. Draw a histogram. When you are explaining to a student how the exam was graded, such statistics are a great help.

Incidentally, hand exams back at the *end* of the class period. For if you return them at the beginning of the hour then students will spend the period reading the exam and comparing grades rather than listening to your lecture.[14] For large classes, we actually set up a special room where students go to get their exams back.

Hand back exams just as soon as possible after the exam is given—in the next class period if possible. If you procrastinate, and do not return the exams to students for a couple of weeks or, worse, until right before the next exam, then much of the didactic value of the exam will be lost. Students will have put that material on the back burner while they are learning new material, and their overall interest in the exam and its contents will have waned. If you can return exams in the very next class, then you can bring that portion of the course to closure and move confidently into the next body of material that you must teach your students.

[14] Actually, new privacy laws put restrictions on how you can return exams to students. What you *cannot* do is just hand the stack of exams to a student in the first row and then tell him/her to find his/her exam and then pass it along. For then you are violating the privacy of the other students. You are also making it at least theoretically possible that a student can steal another student's exam and alter it (for instance by putting his/her name on it).

It seems natural to spend the class period following an exam actually working the exam at the board. Let me tell you decisively that this is not a good use of time. First, students resent your implicit statement that "Look at me—unlike you, I can do the exam quickly and easily." Second, what each student really cares about is how *he/she* performed on the exam. If a student did a problem incorrectly, it is possible that he/she will want to see you give the correct solution. Otherwise interest is minimal. Best is to make the written-out exam solutions available to all class members (in a posted copy, or in a hard copy that you distribute in class, or on the Web). Always insist that, before a student comes to you to complain about how a problem is graded, the student must first read your solution. You cannot imagine how much time and travail this practice will save.

You also need to get on with the course you are teaching. Make it clear to the students that you welcome their comments about the exam. Make yourself available during office hours, and even have extra office hours for discussing the exams if appropriate. But don't waste valuable class time hashing over an exam that has been taken and graded. Such an exam is basically a dead issue, and it is time to move on.

2.10. Grading

The pot of gold at the end of the rainbow, from the student's point of view, is the grade at the end of the course. Grading is a multi-parameter problem. The students want to be treated fairly, yet they want to to feel that the course has substance. They want to be enlightened, yet they want (to some degree) to be delighted, to be entertained. They want to respect you (the instructor), but they want to be your friend. There are a variety of devices for making your grading scheme palatable (without being essentially more lenient) to students. What is the most evenhanded and efficient way to determine grades?

I have used a number of grading schemes successfully, and some unsuccessfully. I would like to record a few of the former here—merely for the reader's delectation. My main goal in formulating my grading policies is to make the greatest number of students feel that they have been treated fairly (and, not incidentally, to reduce student complaints). This does not mean that I am a lenient grader, nor that I give away grades for no special reason.

Always tell students on the first day of class, and in your syllabus (see Section 2.11), how you will grade the course. You want this to be a matter of public record. If students complain about your grading practices, and there will occasionally be some who do, then you have your public statements to fall back on. And don't lie. If you say that you will grade according to a certain scheme—with exams worth so much and homework worth so much and so forth—then do so. If you say in your syllabus that you will grade on a curve, then do so. If you say in the syllabus that you have an absolute grading scale (90% is an "A", 80% is a "B", etc.), then stick to that.

You may wish to consider in advance how you will handle students who are distraught about their midterm grades. One method that usually works for me in borderline cases is to say to the complaining student, "If these few points really make a difference in your course grade at the end of the semester then let's discuss it at that time." This arrangement usually makes everyone happy, and very few students will take you up on the offer.

If a student comes to complain about a grade, then show the student some courtesy. If you cannot come up with a cogent reason for the way that you graded an exam or a problem, then that is *your fault*. Rethink your grading practices. Never fall back on your august position as your first line of defense. You show the student absolutely no respect by saying, "That's the way I graded your test and I'm the boss." That is not how you would wish to be treated. You can always turn a session of "Why was this test graded this way?" into a favorable transaction. What does it cost you to give the student a few extra points if the points are merited?[15]

However *never* penalize a student for being honest. If the student comes to you and says, "You added up my points incorrectly. I should have received an 87 instead of a 90." then just send the student home with a little praise for being so perceptive. Tell the student that if you ever inadvertently give him too few points then he should feel free to approach you at that time as well.

It is tempting, especially when you are a new instructor, to endeavor to take an "organic" approach toward grading. Students are very receptive to the instructor who says, "I try to grade on a subjective system. If your strong grades are on the midterm exams, then I downplay the homework and the final. If your work shows improvement, then I take that into account when I determine your grade. I try to emphasize everyone's strengths. I am your friend." This approach works well in the short run. It is a good opiate—for you as well as for the students.

One weakness of the organic grading method is that it is intrinsically unfair. A student who does his/her homework with friends (and who therefore, despite his/her own personal lack of understanding, gets good homework grades), but who takes his/her exams alone (and gets poor exam grades because of his/her lack of true understanding) could still garner an "A" or "B" in the course.

Perhaps a more pragmatic and immediate liability of the "organic" grading method is that, if a student complains about his/her course grade, then you have nothing to fall back on. You cannot show him/her your calculations, you cannot trot out the mean, the mode, and the standard deviation, and you cannot show where his/her score fits on a histogram. The trouble with an intuitive methodology is that you cannot explain it or defend it. Even though it sounds a trifle cold, you are much better off with an objective system of grading. In the end, everyone is more comfortable with a dispassionate approach. And, in the rare event that you have to defend yourself to the chair, or to an angry parent, or to the dean, or even to a colleague, you will be prepared.

One device that I have used in large calculus classes (see also Section 2.13 for additional thoughts on grading in large classes) is the following: I tell students that I determine their grade by weighting their midterms as 50% of the grade, the final as 30% of the grade, and the homework as 20% of the grade (for example). But the *caveat* that I throw in is that anyone who gets an "A" on the final gets an "A" in the course. This assumes, of course, that the final exam is comprehensive. Thus if a student comes to me during the term and is distraught about his/her homework grade or his/her midterm grade, then I can simply enjoin that student to do well

[15] I sometimes resort to the Stanislavsky method when handling a complaining student. I can usually see rather quickly whether I graded the problem correctly. If instead I screwed up, I put my hand to my forehead, look pained, and say, "Sometimes late at night my eyes get blurry on the fiftieth exam and I make mistakes. Let's have another look at this problem." The students just eat this up.

on the final. In fact not many students pull their grade up with the final exam (never more than 5%) and this simple device helps to keep morale high.[16]

As with many items in this book, I offer the last tentatively. I have had some of the brighter students complain about this policy: "Why should some jerk who didn't work all semester be able to pull off an 'A' at the last minute just by cramming?" I patiently explain that it is virtually impossible for such a "jerk" to pull off such a miracle, that the purpose of the policy is to help and to offer encouragement to students who have been struggling, or for whom this is the first difficult math class. It also sends the important message to students that what is important is that, by the end of the semester, they ultimately master the material. Some faculty have told me that the skewed value system that the just-described policy implies is sending a counterproductive message to the students. My primary motive in formulating the policy was to give the students hope, and to quell their misgivings and their fears; and not least I wanted to minimize their complaints. In a substantial course—say real analysis—in which it takes time for the students to internalize the ideas, this policy helps students to show (integrated over time) what they have ultimately learned.

Sometimes you must change a student's grade—either on an exam or in a course. Perhaps you made a clerical error in recording the grade, or you made an error in grading a problem, or you were the victim of any number of other human frailties. Do not be afraid to change a grade when it is merited. *However*: You do not want to develop the reputation among students as an instructor with whom grades can be negotiated. I've had this rep, and I don't know how I got it. But there was a time when, the day after an exam, 85% of my students lined up outside my office to take turns slugging it out with me—point by point and problem by problem—over their exam grade. I felt at times as though I should buy stock in the Kleenex company. This process is unpleasant and (can be) degrading both for you and for the student. Doing a careful job of grading in the first place, and posting carefully written solutions for students to see, can help to assuage much of student discomfort with grades.

Make the student read your (posted) solution before you agree to talk about the grading of a problem. Many times I, as an inexperienced instructor, have spent fifteen minutes haggling with a student over a problem only to realize that the student had not read the correct solution. Once he/she read it, his/her objections faded away.

No matter how fair and ethical and "right" your grading methods may be, the way you grade may run afoul of departmental policies. I know many a mathematician who has painstakingly prepared a grading curve for his/her calculus course and submitted his/her grades to the department, only to have the chair or the director of undergraduate studies call him/her on the carpet for not following the "approved departmental grading curve." The usual party line is, "In this department, we recommend that you give 20% 'A's, 30% 'B's, 40% 'C's, and 10% 'D's. We discourage 'F's." Of course these specific figures are manufactured, but the scenario occurs all the time—probably more often at large state institutions (subject to various public pressures) than at small private ones. This is no laughing matter. Many math

[16]It has become popular to tell students that you will drop their weakest midterm grade. Thus, if the course has three midterms, then you calculate the final grade based on only the highest two midterm scores. One can also drop the weakest homework grade(s).

instructors will just go along with the chair's wishes, simply because it is easier than bucking the tide. Others will stand on their right to autonomy in their own classroom.

This is a tough ethical call, and I cannot tell you what is right. You have to live in your department, with the colleagues and the policies that it has. Probably the best advice I can give is that you should find out what the departmental policies are *before* you begin to teach. If you think that you are going to have trouble living with the grading policies, then discuss them with departmental honchos and try to work out a position that everyone can live with.

It seems to be an increasingly common occurrence (see [**WIE**]) for a student to come to the instructor after the course is completely finished and say, "I got an 'F' in your course. Could we talk about how to raise my grade?" This is like buying a car, signing the papers, making the down payment, driving the car home, and then coming back a week later to see whether you can renegotiate the sale. It makes no sense.

A grade is supposed to be *an evaluation of the work that the student performed during the term.* When the grade is given, the work (or lack of it) is a done deal. The very notion that this is a point for haggling is a genuine travesty—it shows a true misunderstanding of the university's mission.

At a prominent university in St. Louis—not my own—there has developed a new process that is called "grieving." A student who receives a disappointing grade in a class will say "I'm going to grieve this grade." There is a Dean who is in charge of such matters, the student makes an appointment with that Dean, and then the young scholar puts on a dog and pony show to convince said Dean that the grade is unfair. Then the Dean changes the grade! Without consulting the instructor of the course!! Sadly, the practice of grieving amounts to an institutionalization of the wretched behavior that I described in the last two paragraphs. The institution in question is funded strictly according to its enrollment, and it takes great strides to see that its customers are happy. While I sympathize with the school's plight, I certainly do not endorse its practices.

2.11. The Syllabus (and the Course Diary)

Every mathematics course should have a syllabus. The teacher should give the course a little thought and planning before classes begin. What is the text? What material (i.e., what parts of the textbook) will be covered? What are the prerequisites? How many exams will there be? How will the grade be determined? What is the instructor's name, office number, phone number, email address, and office hour? What is the URL for the course Web page?

The syllabus should be in outline form—not paragraph form—and display essential information so that it is easy to find. A sample syllabus follows.

Fall, 2017 Math 411
Krantz Washington University

Syllabus for Math 411
Real Analysis

Course Description: This is a rigorous course on the foundations of mathematical analysis. Topics to be covered include set theory, logic, the real number system, sequences and series (both of scalars and of functions), compactness, topology of the reals, approximations, differentiation, integration. We will cover at least the first five chapters of the text.

Course Prerequisites: Calculus and linear algebra.

Instructor: Steven G. Krantz

Office: 103, Cupples I

Phone: (314) 935-6712

FAX: (314) 935-6839

Office Hours: To be announced. Consult the course Web page.

Course Web Page: `www.math.wustl.edu/~sk/math411.html`

Math Department Office: 100, Cupples I

Text: *Principles of Mathematical Analysis*, 3rd edition, by Walter Rudin.

Class Meeting Times: MWF 1-2

Classroom: 115, Cupples I

Exams: There will be two midterm exams and a final. Exams will be scheduled by the university. Consult the class Web page or the department office for information.

Quizzes: There will be weekly quizzes. Schedule and format to be announced.

Homework: There will be regular homework assignments and these will be graded. Late homework is not permitted. I will drop your two lowest homework grades to allow for missed assignments.

Grading: The components of the course are weighted as follows:

Homework:	10%
Quizzes:	15%
Midterm I:	20%
Midterm II:	20%
Final:	35%

This is a "bare bones" example of a course syllabus. It contains the critical information that should be a subset of any syllabus. Your syllabus may contain more information about the course topics, or about the text, or about scheduling, or about homework assignments, or about grading.

Some particularly nice examples of course syllabi, replete with loads of ancillary information for the students (about the math major, about how to take exams, about how to study, etc.) may be found at

`www.math.wustl.edu/~freiwald/131syllf10.html`

`www.math.wustl.edu/~freiwald/m309f11.html`

`www.math.wustl.edu/~freiwald/4171syllF13.html`

It is only courteous to provide students with the information that has been indicated in the simple sample syllabus provided above.[17] The syllabus is also a paper trail for your course. Try to stick to your syllabus as much as possible. If a student comes to you in the seventh week and says, "I didn't know that there would be two midterms." or "I didn't know that homework was such a big component of the course." then you can point out that these particulars were explained in the syllabus that you distributed on the first day of class and that has been posted all semester long. The syllabus serves as a sort of contract between you and the class. It keeps you honest and it keeps the students honest.

The syllabus should not be a *magnum opus*. In some large calculus classes, especially when there are several such classes being coordinated, instructors may find it useful to list the topic for every class period, relevant pages from the book, and the homework problems that are assigned for each day. That is fine in its place. For most courses, a syllabus of one or two pages is more than sufficient.

Ideally, the syllabus should be available in a stack outside your door, or perhaps in the undergraduate office, all semester long. If you set up a Web page for the course—and this is an excellent idea—then you should post the syllabus (together with all homework assignments, exam times, solution sets, etc.) there. This practice is just good business, like a restaurant posting its menu or a gas station posting its prices. There is admittedly a certain cachet to being completely disorganized and doing everything by the seat of your pants, but it doesn't pay. You end up wasting a lot of time covering your tracks, you create too many potential opportunities for aggravation, and it leaves students with a bad taste in their mouths.

For the same reason that you should have a course syllabus, I would also recommend that you keep a course diary. This device could be several sheets of paper that you tape inside the front of your copy of the text (and you should tape a copy of the syllabus in the same location), or it could be a separate little notebook that you keep together with your text and your grade book. Any time a student requests a makeup exam, or asks for an extension on an assignment, or whenever anything comes up in connection with the course—put it in the diary! When you and the class decide on a date for the midterm, put it in the diary. If you set up a review session, put it in the diary. If you are going to be out of town, and Professor Veeblefetzer has agreed to cover for you, put it in the diary. That is to say, make a dated entry with the item that you need to remember. It is even reasonable to put all appointments with your students in the course diary. That way you'll have

[17] In the old days I would distribute the syllabus in hard copy on the first day of class. Today I put the syllabus on the course Web page, and I do so a few weeks before the course begins. That way the students can begin to acquaint themselves with the course even before the formal *anfang*.

a record of all transactions pertaining to the course, and you also will be less likely to forget them.

2.12. Choosing a Textbook

Choosing a textbook sounds like a simple motor skill. It is not. As a mathematician, you are trained to be rather self-indulgent. You may tend to choose a text that pleases you. If there is a cute new proof of Stokes's theorem, or if there are four-color, computer-generated graphics, or if the book sales representative takes you to a particularly nice place for lunch, then you are liable to be influenced.

Remember that, while teaching the course, you will not spend nearly as much time reading the book as will the students. Try to step outside yourself as you look at a text. Ask yourself whether the students will be able to understand it, and whether they will be well motivated to try to do so. Are the definitions well formulated and the theorems clearly stated? Are important topics easy to find? Are the Table of Contents and the Index accurate and thorough? Are the examples well chosen, sufficient in number, and do they order from the simple to the complex (this device is known as "stepladdering")? Are the exercises suitable, and well-coordinated with the text, the lesson plans, and the examples?

There is a great temptation, especially in an advanced course, to choose a text that in your own mind is a "standard reference." But a *reference book* is not a *text*. Try to see things from the students' point of view. The text for your course should not be one of those books that is a pleasure to read *after* one has mastered the material. It should be a *text*, for someone who does not yet know the stuff.

If you choose a poor text, you will pay for it throughout the semester. Homework assignments will be difficult to design. They will be difficult to grade as well. Students will not perform well on exams if they do not have a good book to study. If the text is weak then the correspondence between it and your classes will be tenuous. This dilemma defeats class morale.

I will now give an example of a problem that exists in the current crop of calculus texts in this country. What does it mean to say that a function is discontinuous at a point $x = a$? Some calculus books state that a function is discontinuous at any point at which it is undefined. For example, the function $f(x) = 1/x$ is discontinuous at $x = 0$. But wait, it gets worse. Consider the function $g(x) = (x^2 - 1)/(x - 1)$. On the face of it, this function is undefined at $x = 1$ and hence discontinuous there. After division, however, the function becomes $x + 1$ which is both defined and continuous at $x = 1$. But are $(x^2 - 1)/(x - 1)$ and $x + 1$ the same function? Who knows? And beware: If you declare a convention different from that in the text, many students will become confused and some will be lost.

This issue of continuity is but one instance of the sort of mess that a textbook can get you into. Some people never use the first edition of a text because they reason that **(i)** all misprints will not be weeded out until the third edition and **(ii)** glitches like that described in the preceding paragraph will be mollified (in response to user protest) in later editions. In point of fact, the conundrum described in the last paragraph occurs in the third edition of a very successful calculus text. The (very famous and wealthy) author insists that this is the only way to handle continuity at a point where the function is undefined.

Let me expand on this last point. If a textbook uses notation or other conventions that you do not like, then don't use that book. You really are obliged to follow the notation and definitions and other paradigms in the text you have chosen. Otherwise all but the gifted students will be lost. If you repeatedly criticize the text as the course proceeds, then you will be sending a confusing message to the students: Why did you choose this book if it is obviously so full of flaws? Isn't it your job to select a text that you can teach from?

Now you can hardly be expected to read and digest every word of a text—especially one as cumbersome as a calculus book—before you use it in a class. The safest policy when selecting a text is to consult someone who has already used it. Don't be afraid to seek the advice of a more experienced faculty member when selecting a text—especially if it is for a class that you have never taught before.

Recently I used a text—in a multi-variable calculus class—that was a disaster. First, it was too formal. It stated results in a fashion that was far too abstruse. As an example, it defined multiple integrals on any spatial region whose boundary has "zero content". Do you know what that means? If you do then you must be a geometric measure theorist, for most people don't. Also, the exercises in the book had a great many small errors. Remember that young students read mathematics quite literally. They are unaccustomed, as are you and I, to making plausible adjustments to what they read. They do not readily generate error-correcting code. Perhaps the worst offense of the book is that I would encounter some strange piece of terminology on page 257, but could not find the page where it was originally defined. This irritation occurred repeatedly, because the book had a very poor index and because usually these obscure pieces of terminology were defined in some optional exercise. I was terribly embarrassed by the albatross that this book turned out to be for a very long fourteen-week semester. I should have asked one or more colleagues for advice on which text to choose. Or I should have read a sample chapter before I adopted the text. Instead I chose the book because I know one of the authors and he is a nice guy.

When you are selecting the text for an undergraduate course, then *do just that*. Pick one book—not more—and set it before the students. If we want our students to develop the scholarly frame of mind and the discipline of which we so often speak, then we must give them a hand. We owe it to our undergraduates to put a single book before them and to teach them to read it. To teach an undergraduate course by first dipping into this book and then dipping into that book may be charming and lots of fun for you, but it just confuses the students. Please don't do it. (*Caveat:* Philosophy courses and literature courses and history courses are different. Of course a "history of western philosophy" course will have many sources. Mathematics is a much more compact and directed subject. Therefore a single, well-chosen text is usually just what the doctor ordered.)

It is a revelation for a young student to become completely involved with a book, to realize that—when he/she wrestles with page 200—he/she can go back to page 100 (which he/she has presumably already mastered) for help. You assist this process by giving the students *a single book* and helping the students to plough through it. Choose that textbook carefully, for you will have to live with that book too—for an entire semester! You will have to fashion your lessons after the book, solve the problems, create the exams, answer student questions, and so forth. In choosing a textbook, you are defining your life for a non-trivial period of time. It is a bit like setting up housekeeping with the author—and you want to choose a partner who will not do you wrong. Choose wisely and well.

And be sure to select a text that was written for the students in your class, and for the sort of students that attend your university. You can teach calculus from the famous text of Courant and John [**COJ**], or from the amazing book of Spivak [**SPI**], but you had better have some pretty clever students if you are going to pull this off. Today there are calculus books at every imaginable level, ranging from *E. McSquared's Calculus Primer* [**SWJ**] (which is in fact a comic book about calculus) to very rigorous and advanced texts such as the two just mentioned. There is a similar, although perhaps less well-populated, range of texts for almost any undergraduate course. Give some careful forethought to what sort of students will be in your class, and whether they will be able to read the text that you choose.

Each textbook has its own *gestalt*, reflecting the author's view of the subject, and of the way that it ought to be learned. By the end of the semester you will have a pretty good idea what that *gestalt* is, but it is quite difficult to discern it during the brief perusal that will most likely inform your textbook choice. This is yet another reason why consulting an experienced colleague is a useful part of the textbook selection process. Let me discuss just one example to illustrate my meaning.

There is a noteworthy mathematical textbook series written by John Saxon. Saxon has actually appeared on the television show *Sixty Minutes* and wept openly because—if schools would only adopt his textbooks—the mathematics abilities of American students would be greatly enhanced. The hallmark of the Saxon books is that they have no chapters. The calculus book, for instance, has more than one hundred sections, each about three pages long. In any given section, the first page reviews what has come immediately before and the last page previews what is about to come. The lone middle page of the section contains a new little nugget of knowledge.

Can you see what might be wrong with a textbook of this sort? The answer—I think—is that the text is constantly holding the student's hand. The student has little need for retention, because the book is constantly reminding him/her of the old ideas, and only introducing new ones in minuscule quantities. The series, although well-thought-out and well-intentioned, has not been a success. As an experienced instructor, I can with little effort perform this sort of evaluation of a new textbook. If you are new at the game, then you will want to cultivate this sort of critical skill. While you develop it, consult experienced colleagues to help inform your textbook choices.

I once spent an evening fiddling with the CD-ROM version of a popular calculus text. It was a technological marvel. Using this disc the reader can, if he/she wishes, read each page (with black characters on a white background), much as though he/she were reading a hard copy text. Key words are hyperlinked—if the reader forgets what "local maximum" means, or what a "derivative" is, he/she can click on the highlighted term and a box will pop up reminding him/her of the main ideas. If that precis is not sufficient, then another click on an icon will move the user into a workbook. Another click will return the user to the place in the text at which the reading was interrupted. The reader can record marginal notes, and these are each marked (for future readings) with an icon that looks like a pencil on a notepad. Amazing. When doing exercises, the reader can click on any particular exercise and the solution will appear on the screen *instantly.*

I fell asleep that night with visions of sugar plums dancing in my head. I woke up in the morning convinced that I had seen the end of Western civilization as we know it. If a student can click on “derivative” any time he/she wishes—to remind him/her what the concept means—then I bet that he/she will *never* actually learn what a derivative is. He/she will never internalize it, and therefore never really be able to use it. The typical young student that I know, when “doing” exercises in this CD-ROM environment, will proceed in the following fashion: He/she will glance at an exercise, click to see the solution, say to himself/herself, “Oh yeah, I could have done that,” and then go on to the next one. In short, he/she won’t do the exercise and won’t learn anything. As things are at present, the student faces the small psychological block of having to turn 850 or so pages and find the answer in the back of the book. I would like to think that that barrier serves as an impetus for the student to try to do the problem himself/herself.

I have forty years of teaching experience that tell me that hard copy textbooks, more or less of the traditional form, work. New, untested teaching technologies make me nervous. Of course we have to try them, otherwise we will ossify and never learn anything new. But we should be aware of the pitfalls, and engage in careful self-evaluation, as we proceed.

And now a coda on cost. It is not impossible these days for a textbook to cost $200 or more. If the book that you choose is expensive, be prepared to defend your choice. Is there an equally good book that costs just half as much? (I’m not talking here about *your monograph* on *your special subject*. Rather, I’m considering something like a linear algebra text, for which you should have less emotional involvement.) Checking the cost of the text for your class is just the sort of courtesy that you would have expected your instructor to show when you were a student.

2.13. Large Lectures

At many large state universities, and some private ones, it is common to teach some or all lower-division courses in large lectures. The large lecture situation offers special teaching problems. How can you, as the instructor, make yourself heard? How do you fill the room with yourself? How do you field questions? Do you really want the students to ask questions? If so, how can you encourage the students to do so? What about exams? How do you organize your teaching assistants?

Most large lecture halls come equipped with a microphone for the lecturer. Unless you have a voice like William Jennings Bryan, use it. If you don’t use the mike, then either you will not be heard or you will be under such a visible strain that it will detract from what you are trying to accomplish. Like it or not, the instructor in a large lecture is putting on something of a performance. Of course you are not dancing the macarena, nor are you singing the blues, but you *are* trying to get through to an audience, and to engage them in the learning process. Obviously there will be some technique involved. If your face is beet red, and perspiration is popping out on your forehead, and your armpits are soaked with sweat, and you look like a nervous wreck, then you will not be a success at this job. Learning to use a microphone will help you to avoid these obvious liabilities.

You *can* learn to become relaxed with a microphone clipped to your collar. When you have done so, you will be able to speak in a normal voice and to be heard clearly. Your hands will be free and you can comfortably write on the board and gesticulate and point to students and act in a lively and engaging manner. You can then concentrate on getting your mathematics across to the students.

One way to develop student involvement in the classroom is to encourage questions (see Section 1.5 on the importance of student questions). The discourse that is built around student questions is a critical part of the learning process. But large lectures impose severe time constraints, and severe communications problems. You must learn to handle questions in your large lectures with care. Too many questions will bring the lecture to a grinding halt. But a good one can make everyone prick up their ears. You have seen Jay Leno and David Letterman walk out into the audience and engage in repartee with select individuals. Those in the audience who have been selected are usually happy to participate. Unlike our students, the participant in the Leno or Letterman show does not stare at the floor, nor hold his/her breath until his/her face turns blue. What is the trick?

It helps that Leno and Letterman are celebrities. Everyone wants to talk to a celebrity. If you think that's all there is to it then you are kidding yourself. Leno and Letterman know something that perhaps you do not. And that is how to handle all different types of people who behave in all sorts of ways. Imagine, for instance, that you have picked out a student and asked him/her a question and he/she plainly is embarrassed because he/she doesn't know the answer. The inept and counterproductive way to handle the situation is to stand there and persist and embarrass the student further. A more productive way to proceed is just as you would proceed in your home if one of your dinner party guests spilled his wine or dropped his potato on the floor:

- Make a joke of it, create a diversion, or pretend not to notice;
- Affect to get distracted and then ask someone else;
- Say, "I must not have formulated the question very clearly" and then try again on the other side of the room.

As any book of etiquette will tell you, the quintessence of politesse is to make everyone feel comfortable—under any circumstances. Never forget to use these same skills in your classroom! Most of the Hollywood movies that depict a college classroom invariably show the professor standing before his students denigrating their intelligence and abilities—and the students lap it up! This may be poetic license, but it is certainly not reality. Show your students the same courtesy that you would have wanted to be shown when you were an undergraduate.

The remarkable book *One L* by Scott Turow [**TUR**] describes the life of a first-year law student at Harvard. Central to the experience described in the book is the notion that all first year law classes at Harvard are conducted by the "Socratic method." Note that first-year law classes at Harvard typically have an enrollment of 140. In such a class, the Socratic method consists of the professor, in the first few moments of the class, getting in one student's face and humiliating him for (as much as) the rest of the hour. And the students take it! They hate it, but they take it. And the consensus seems to be that it makes them better lawyers. [The movie *Paper Chase* starring Timothy Bottoms also dramatically recreates the first year of Law School at Harvard.] The way that Scott Turow describes the first-year law experience, he makes it seem as though a class of 140 can be like a class of 10. The professor at Harvard Law plays the class like a harp, and sets the students against one another in draconian ways. At the end of the book, Turow vilifies the experience. But it is clear that it has made him stronger.

I certainly do not advocate that you read *One L* and teach your classes accordingly. Freshman calculus students have neither the proven abilities nor the determination of first year law students at Harvard. But the book shows that a large lecture can be taught both powerfully and incisively. There is a skill to it, and it is one that we can all learn. At Harvard, if *One L* is to be believed, the catalyst to success is fear. But my experience dictates that other catalysts work as well. Communication, intellectual inquiry, and discourse are some of these.

Of course you want to know that your class is alive—that it contains living, breathing people. One way to do this is to make students comfortable with asking questions. Once you have created an atmosphere in your class in which people feel natural asking questions, then you have a foundation that you can build on. If they are at ease asking questions, then they can move on to making statements, formulating conjectures, suggesting lines of reasoning, or pointing out errors in what you have written on the board. It is this atmosphere that I strive to create in my own classroom, and one that I enjoy immensely.

But there is a trap. Left to their own devices, students will lapse into asking questions of the rote form, "How do you do problem 6?" Such questions must be discouraged in any class, but especially in a large class, as they are boring and non-instructive. If you do get such a question (and if you want to consider and then to answer such a question), then don't simply turn to the blackboard and record the solution for all to see. Instead, engage the questioner in discourse. "Have you tried the problem? How far did you get? Where did you get stuck? Did any of the rest of you have any luck with this problem? Did you get stuck in the same place? Does anyone want to suggest a way out of this mess?" It is too easy for a question like "How do you do problem 6?" to degenerate into a private discussion between a single student and the instructor. It is also difficult for the other students to pay attention when you are addressing the needs of a particular individual. Endeavor to turn an individual's question into more than what it is. If you cannot use it as a catalyst for a useful classroom discussion, then tell the student to see you after class.

What you really want are questions like, "Why don't we define 'continuity' this way?" Or, "Why does the chain rule have this form rather than that form?" It is up to you to *prompt* the students for the questions that you want. In order that they not become rhetorical questions, you must put these issues to the class and then *wait for an answer.*[18] It is not enough to say, "Why does the product rule have this form? Well, here's why." If you want a reaction from your class, you must draw it forth.

When a student asks a question, *repeat it* so that you can be sure that the entire class has heard it (and that you have *understood* it). Write the question on the board. This is sound policy even in a small class. If you do not repeat the question, then the interchange between you and the student becomes a private conversation. The rest of the class is excluded. If other students (those who are sufficiently interested) are interrupting with "Huh? I can't hear. What's the question?" then you

[18]This classroom technique has been studied in detail by experts in the psychology of learning (see [**MOO**]). One such study recommends that, after you formulate a question to the class, you wait 30 seconds—but not more. Less than 30 seconds does not give students enough time to formulate an answer; more than 30 seconds is wasting time.

are wasting valuable class time and also losing control. Other private conversations will start up. Your class will go badly.

But there is another important consideration to repeating the student's question. If the question is not optimally formulated (or just plain wrong) then the repetition gives you an opportunity to clean it up or rephrase it. Then treat the issue raised with respect and answer it directly.

It is especially important in the large lecture situation that students be sure that you will not belittle them or make them look foolish in front of the other students. Be prepared to make even a dumb question look smart. Writing the question on the board and repeating it out loud gives you a chance to turn the question into something that you can answer, and that will make both you and the questioner feel witty and wise.

If the first question that you field gives rise to a second, then say something like, "Let's go back to the lecture for a bit. I think it will clarify this point for you." Or you can say, "This question session is getting a bit too specialized. Why don't you see me after class?"

There are virtually no data to support the contention that small classes are "better" than large classes for mathematics teaching. Statistics do not indicate that students in small classes perform better or retain more. The statistics *do* indicate that students feel better about themselves and the class, and enjoy the situation much more, if the class is small. Another way to say this is that large lectures are not a good tool for *engaging students in the learning process.* Just as nineteenth century social theorists noted that factory workers on an assembly line easily can become alienated from the work process, so students who are educated in a large class (that smacks of an assembly line) can become alienated, and therefore will not learn as well. An awareness of this problem, on the part of the instructor, is an important first step in dealing with it.

The role of intuition in our lives should not be minimized. And our intuition is that small classes are better. Why is this the case, and why is our intuition inconsistent with the statistics cited in the last paragraph? The answer (as Len Gillman has patiently pointed out to me) is that the advantages of small classes are intangibles: The friendly give-and-take between instructor and students (eminently possible in a small class but quite difficult in a large one) is a form of encouragement, and a way to make the subject seem fun and exciting. In a small class it is likely that you can cover more material more efficiently, thus you will have time to treat ancillary topics that will whet the students' appetites, and may cause some of them to decide to be math majors. The main point is that a small class is *personal*, while a large one is not. You would not want to have your annual physical exam in a large auditorium with 200 other people. You probably do not want to learn calculus in that fashion either.[19]

One device that works very well (when teaching a class either large or small), if you can manage it, is to make yourself available in the front of the classroom

[19] The main campus of Penn State has about 40,000 undergraduates. It used to be, when I taught there, that calculus was taught in *very* large lectures. Several hundred students per class. Use of a microphone was essential. Teaching such a course was an ordeal, to say the least. And the level of student complaint was considerable. Even the Dean's office got a fair dose of student complaints. So they decided to teach calculus in small classes of thirty—and they convinced the Dean to provide the resources to make this happen. And guess what? The level of compaints has fallen nearly to zero.

for fifteen minutes after class. Students feel much more comfortable talking to you when surrounded by their peers, and while their questions are fresh in their minds.

Never, ever get involved in a personal discussion of grades in front of a class of any size. If student T wants to know why problem n was given only p points, tell that student to see you privately.

It is imperative that the instructor for a large lecture course be extra well prepared. If you begin to get lost when doing an example or sidetracked with an incorrect explanation then you will quickly lose a large segment of the audience, a lot of talking will start to take place, and the room will soon be bedlam (see also Section 5.5 on discipline). Everyone has off days and makes mistakes, but you must take extra care in a large lecture to have the material down cold.

The teaching of a large lecture course offers complications of a special nature. Discipline and commanding attention are two of these that are treated elsewhere in this book. But there are others. You *must* have a syllabus for such a course. You *must* prepare your homework assignments and exams carefully—and well in advance. There are few things more unpleasant than facing down a hostile audience of 350 hungry freshmen right before lunch. Therefore do not give exams on which the problems don't work out; do not give homework assignments on which the problems don't work out; *do* plan ahead for *all* activities. Prepare and prepare and prepare some more. Have a fair and objective system of grading. If a student comes to you with questions about grades, then have a fair, consistent, and clear set of data to show the student.

If you are the professor in charge of a large lecture, then you will probably be in charge of a group of 2 to 10 or more Teaching Assistants (TAs)—refer to Section 2.16. You must exercise organizational skills with them as well. Meet with them once a week to be sure that they are covering the right material, are aware of upcoming assignments and exams, and to apprise them of any difficulties that have arisen. Make regular use of email to keep in touch with your TAs.

If the TAs are helping you to grade an exam, then you must tell them how you want the papers graded (see also Section 2.10 on grading). Grade exams *horizontally* rather than *vertically*. This means that you should put TA #1 in charge of problem #1 (on all exams), TA #2 in charge of problem #2, and so forth. This is the only way to insure some consistency. If you let each TA grade a stack of exams from start to finish, then you will have wild inconsistencies and many student complaints.

Horizontal grading is also a useful device even when you are grading a stack of just twenty exams all by yourself. It will discipline you, it will make your work more accurate, and it will tend to make the job go more quickly.

At a large university—with 30,000 or more students—there may be several large lectures of the same calculus or linear algebra course running at the same time. If you are in such a situation, you will often find it convenient to work cooperatively with the other lecturers (in some departments you may have no choice). In fact students in this situation seem to value a sense of overall fairness and uniform treatment more than they value the flair and pizzazz of the individual instructor's personal style. Thus you may find it useful to meet regularly with the other lecturers and to hammer out some uniform policies, and even uniform treatment of the topics in the course. You may wish to have common exams, and to take turns writing them.

The article [**MOO**] contains a sensible discussion of how the precepts of the management concept TQM (Total Quality Management) can be used to guarantee that the students are treated fairly in this type of learning environment.[20]

Here is a trick that some professors teaching a large lecture have used with success. The professor recognizes that students in a large lecture can develop a feeling of alienation. Such students are afraid to approach the professor, afraid to go to office hours, afraid to ask questions in class, afraid to lodge complaints. Therefore, at the beginning of the term, the professor asks for two or three students to volunteer to be the class *ombudsmen*. As you know, an ombudsman acts as a go-between: he/she fields questions from the constituency and presents them (with no mention of the source) to the professor. It is not difficult to see that this little device can serve to open lines of communication. Students are much more prepared to present their concerns to a peer (thus preserving their anonymity with the instructor) than they are to confront the professor one-on-one. Good ombudsmen will be able to answer trivial concerns without even getting the professor involved (because the ombudsmen will, as a matter of course, be meeting regularly with the professor). The ombudsmen will also consolidate and clean up student complaints and present them to the professor in a manner which the professor will find agreeable.

Faculty who have used student ombudsmen in the manner just described report that **(i)** it is a decisive means for bridging the communication gap between professor and students when a class is large, **(ii)** students respond readily and well to peer ombudsmen, **(iii)** the ombudsman device is a good way for the professor to develop a close relationship with at least some of the students in the class, and **(iv)** students will volunteer to be ombudsmen if they are made to understand that a professor can write an especially good letter of recommendation for a student whom he/she has come to know in this capacity.

It is tricky in a large lecture to help the students get to know you as a person, to cut through the barrier that always exists between the person in front of the room and the large audience in the back of the room, and to keep the lines of communication open. You know that successful performers can do these things. You know that preachers can do it. And you know that skilled teachers can do it. One of your career goals should be to develop this talent yourself.

2.14. Problem Sessions, Review Sessions, and Help Sessions

At many big universities, the large thrice weekly lectures in a lower-division math course are supplemented by once- or twice- weekly "problem sessions" or "help sessions." Usually the lectures are delivered by a professor or instructor while the help sessions are staffed by graduate student teaching assistants (TAs).

Imagine that you are the graduate student in charge of a problem session. It is easy to fall into the trap of not taking the work very seriously. After all, student attendance at these sessions is poor in general and spotty at best. Students seem to be inattentive and their questions are often puerile. But the quality of any class or help session is largely influenced by the attitudes and efforts of the person in

[20] I note that TQM—a management technique borrowed from industry—is an emotion-laden topic in the educational setting. I shall not try to treat it here. But see [**MOO**] for a quick, if not impartial, introduction to the idea.

front of the room. If your attitude is to treat the help session casually or carelessly, then you will get correspondingly disappointing results from the students. Consider giving weekly quizzes, sending students to the board, and other devices for livening up your problem session. I wish to concentrate here on more mundane matters.

It is arguably more difficult to conduct a good problem session than to give a good lecture or class lesson. For the problem session presents all the difficulties of a class period, and more. At least in a class lecture you are in complete control of the order of topics and can, if you wish, present them from prepared notes. In a problem session, if you really let the students ask what they wish, then you must be ready for anything. And you must be able to think quickly, on your feet, of the best way to present any given topic, give a hint on any problem, or handle any point of confusion. In a class or lecture you can always pull rank and say, "There is no time for questions now. See me in my office hour." (I don't recommend that you say this very often, but it is an option that is available). But help sessions are for questions.

If you are a novice, then it is probably safest to view the help session in the most naive way. Your role is to help students do their homework assignment for *that week*. Thus your preparation for a help session might consist of working all the homework problems for the week, or at least staring at them long enough to be sure that you know how to do them.

Be certain that the techniques that you present are consistent with those used in class and in the book. Some professors require their TAs to attend their classes, just to insure this consistency. Such a professor might even do a spot check of the grader's work, or drop in on help sessions to see how things are going.

I know of at least one professor who works closely with his grader and his TAs by attending, once per week, each problem session for his class *accompanied by the grader!* This requires some extra effort on everybody's part, but it shows real consideration for the student who has questions about the way that his/her homework was graded (or how the class, as a whole, is being conducted). It goes without saying that, in order to use this device to good effect, the professor will have to be well-coordinated with the grader on how he wants each homework assignment graded.

When you are helping with a homework problem that is to be handed in, don't give away the store. One reasonable answer to the dreary question, "How do you do number 14?" might be, "I'll do number 16 for you, which is similar." Another reasonable answer might be, "I'll get you started. You do the rest." A third is, "Here is an outline of the basic steps." The truly skillful instructor will turn this question-answer session into a team effort. Gently goading the students with his/her own prompts and questions, this instructor will resist simply doing the requested problem for the student. The trouble with just solving the problem—and nothing more—is that only the requestor and perhaps a few others will be paying attention. If instead the instructor can generate some repartee, and can get the students to want to pitch in, then there will be considerable student interest and a number of class members will learn from the experience.

Implicit in the last paragraph is the observation that the TA-led problem session can be an opportunity for active learning (see Section 3.7 for a discussion of this idea). The problem session is already a relatively small group, and the setting is informal. It is natural for the instructor (the TA) to pose a question and tell the students to discuss it with their neighbors. Or to get people to go to the board. Or to get students to pose problems themselves. Certainly, with a little imagination, it is possible to turn a problem session into a lively interchange rather than a boring list of "How do you do problem 16?"

There are subtle psychological forces at play in the scenario just described. If each student is worried about protecting his/her turf, and simply does not want to share what he/she knows, then you will have a hard time generating useful dialogue in your problem session. If instead the atmosphere is one of learning being a sharing activity, and of giving knowledge in expectation of receiving knowledge, then the problem session can be a worthwhile and nurturing experience for everyone. (We all know of mathematicians who collaborate easily and well, and of others who seem to be thoroughly incapable of collaboration. Perhaps these differences reflect attitudes similar to those being described here.) Of course you as the TA or instructor must set the example. If the signal you send is that *you* are not willing to help, that *you* are not willing to share, that instead you are like the oar master on an ancient galley, then you will get little in the way of cooperation and sharing from your students. If instead the example you set is one of patience and giving and caring, then you are likely to be the beneficiary of an enthusiastic response.

The advice to the TA (five paragraphs ago) to work all the homework problems the night before a problem session is one that I tender hesitantly. I never do this, but I've been teaching math for forty years. I am rarely surprised by any question in a calculus class or help session and, even if I am, I can usually slug my way through whatever new features are present. If I am at a review session for an exam and a student presents a really difficult question then I always have the option of saying, "That's an interesting question, but one that could never be put on the test. Let's discuss it privately."

In your first few years of teaching you will have to strike a balance between being thoroughly prepared (by working all problems in advance) and spending too much time on preparation (see also Section 1.2). Just remember that a large part of your job is **(i)** to show the students how to do the problems and **(ii)** to persuade the students that the problems are doable (by ordinary mortals). If you fumble around and act baffled by the problems, then you are presenting a poor role model and, more to the point, doing your job badly. Students find appealing the fact that I can do all the problems and that, moreover, I invariably know where the difficult spots are and can help them to chart their way through them. This ability can only come with experience. It is the model that you should strive to attain.

2.15. On Being a TA

Being a Teaching Assistant (TA) provides some experience in being a teacher. But it does not provide much, and the glimpse of teaching that it provides can be misleading.

When you are a graduate TA at a big state university, you are probably not your own boss. In most cases you work, alongside several other TAs, for some professor who is delivering lectures to a large class. On alternate days, the class will be broken up into smaller "quiz sections" or "problem sessions", and you will be asked to teach one or more of these (see Section 2.14). You will also be asked to help with grading, with other assigned activities, and (primarily) you will be asked to do what you are told.

Being told what to do lifts a great deal of responsibility from your shoulders. But this also means that a TA has never really taught. You've had some experience standing in front of a group, organizing your thoughts, answering questions, developing blackboard technique, and so forth. But you will have never made up an exam, written a syllabus, designed a course, given a course grade, or any of the dozens of other activities that figure significantly in the teaching process.

However, if you have never been a TA (either because in graduate school you were on a fellowship that had no formal duties attached to it, or perhaps because you were educated in another country), do not despair.[21] At least you are entering this profession with possibly fewer prejudices than are held by those who have stood as a TA before a hostile audience in this country. Perhaps reading this book will provide you with better information and a better outlook than having served as a TA under a professor who doesn't even care about good teaching.

Let me put an ameliatory note here. Some professors are well aware of the down side of being a TA and attempt to compensate for it. They give their TAs more responsibility. For instance, such a professor might write the first midterm exam for a class himself and then let the TAs write subsequent midterms (under close supervision). This is positive psychological reinforcement for the TAs, and good experience for them as well. Likewise, the TAs can be allowed to set the curve for grading (under supervision) and to perform the other ordinary functions of the instructor. The professor is not being lazy here. Rather, he probably has to expend more effort than if he were doing these tasks solo. But it provides awfully good experience for the graduate student TA.

At some schools, the TA is more autonomous. It is possible that the TA will be a free-standing teacher, creating his/her own exams and constructing his/her own grading system. If this description applies to you, then this section of the book does not. But the rest of the book does, and you may benefit from reading it.

For more information about the day-to-day duties of being a Teaching Assistant, see Section 2.14.

2.16. Tutors

A commonly asked piece of advice, usually from a student having trouble in your course, is, "Should I get a tutor?" I have a very simple answer to this question: "No." It is almost unavoidable that the student will treat the tutor as a crutch. The student figures that, by paying $35 per hour (or whatever is the going rate), he/she is *buying* knowledge. And now looms the specter that to my mind should be the benchmark for all educational issues. *All learning of significant knowledge requires considerable effort on the part of the learner.* This fact has not changed since Euclid told Ptolemy (over 2000 years ago) that, "There is no royal road to geometry." Instead of just slugging his/her way through a new idea, the student finds himself/herself thinking, "I don't get this. I'll have to ask the tutor."

I could go on about this point at length, but I will try to restrain myself. Go to any athletic facility and you will see young people spending hours perfecting their free throw or their skate board technique or their butterfly stroke. They don't hire tutors to achieve those goals. They also don't hire tutors for learning to build

[21] I must make the following caveat: These days, when you are applying for your first job, you had better have some teaching experience on your CV. If not then you will be passed over for sure.

model airplanes or learning to modify their cars. The reason is very simple. There is plenty of peer support for these activities. Young people are highly motivated to be proficient at them. An eighteen-year-old understands clearly when an athletic coach says, "No pain, no gain." However the same concept makes little sense to him/her in the context of mathematics or another deep academic subject.

It is a sad fact that many students—and their parents too—view the university situation in the same way that they view buying a car: You pay your money and you take your choice. The professor is expected to deliver an education (in exchange for the big bucks) in the same way that your local Ford dealer is expected to deliver a car when you fork over your down payment. The college instructor who says, "I am a scholar, and I set a standard, and I expect my students to rise to it." may find himself/herself in a very lonely place. I certainly do not think that such an instructor is misguided. Far from it, I agree with him/her wholeheartedly. But I also realize that, if I expect my students to rise to a certain standard, then I must teach them that this is a worthy goal, and then I must show them how to do it. It's not so hard, once you realize that that is what you must do.

A good math student must be self-motivated. In most instances, the hiring of a tutor is an attempt by the student to buy his/her way out of some work. I've been a tutor. It's a great way for a young person to make some extra money. But in most instances it is not a beneficial learning device. You might find it helpful to refer to Section 3.2 on math anxiety in connection with these issues.

Of course there are exceptions to what I am saying here. Some students are too timid, or too slow, or too far behind to catch up without help. Sometimes a student will have a legitimate and doctor-certified learning disability. If a student has been ill for several weeks, or has had a death in the family or some other personal crisis, then a tutor may be the only alternative.

It is a sobering thought to realize how different the students' point of view is from our own. There is at least one high quality large state university today where students routinely hire a tutor for each class that they take—*before they have even set foot in the classroom*. Clearly these students have convinced themselves that classroom instruction is inadequate, or that their own abilities are substandard, or that they do not know how to study and require a surfeit of hand holding. On days when you think that teaching is a straightforward process, stop and ponder this matter.

In any event, if you are a paid instructor at a college or university, then do not hire *yourself* out as a tutor for a student in a class that *you* are teaching. It is inappropriate, it is tawdry, it is a conflict of interest, and it might get you into trouble with your department. The safest policy is not to tutor students at your institution at all. The point is that you are already being paid a salary by your school to educate the students at that school. To further accept tutoring money from the students constitutes double dipping.

Even having to recommend tutors can put you in a position of conflict of interest. Most math departments maintain a list of qualified people (usually graduate students, but perhaps some undergraduates also) who can tutor for math courses. This is done as a service for the students, but it is also done as a service for the faculty. When a student asks you about tutors, send that student to the departmental office and the official list. It really is the best policy.

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CHAPTER 3

Spiritual Matters

3.0. Chapter Overview

This chapter addresses philosophical issues connected with teaching. How do students learn, how do they formulate questions, how should we answer those questions, what is the function of the mathematics teacher? An adequate instructor records the material accurately on the blackboard and then goes home. A truly dynamic instructor interacts with the students, excites their intellectual curiosity, and helps them to discover ideas for themselves. The material in this chapter should help you to pass from the first state to the second.

3.1. Breaking the Ice

The first day of class is simultaneously a day of happy anticipation and a day of stress. It is the first of these (assuming that you like to teach) because you are, after a restful summer, jumping into something that you enjoy and that you do well. It is stressful because you don't know what this new group of students is going to be like, or whether they will play ball with you, or whether you can get through to them.

I am a teacher of long experience. On days of exceptional hubris, I convince myself that I am rather a good teacher. Yet most semesters, especially in the fall, I meet a new class with new students and I have to demonstrate to these people that I'm a good guy. We begin as total strangers, and my goal is to turn us into a working group. Usually this takes a while—often several weeks.

Since I so enjoy a class once we have all become friends, I find the period of tooling up to that happy steady state generally too long and too painful. What usually happens is that there is a period of two to five weeks during which the students look at me as though I am from Mars. They don't laugh at my jokes, they don't answer my questions, they don't seem to take me very seriously. If the class is to be a success, then some magical thing must happen to change everyone's attitude.

You should consider ways to make yourself seem like a human being to your students. Being playful, or impish, or making fun of yourself, is certainly one technique for accomplishing that goal. If that doesn't work for you, or makes you feel uncomfortable, then try something else. Read them some history. Tell them of Bishop Berkeley and his doubts about calculus. Tell them about fractals, or dynamical systems, or wavelets, or why[1] mathematicians don't get the Nobel Prize.[2]

Find some way to open up to your students so that they will open up to you. Some instructors hide their unease behind regimen. They take roll, or put together a seating chart, or ask each student to introduce himself/herself. This routine is fine if you are comfortable with it. My view is that you should show students from day one that you are a person, and that you are going to spend the term doing your best to communicate with them. I don't think that taking roll is a good way to send that signal. It is better if you tell them what the course is about, or describe your grading policies, or give them some clues as to what *you* are like.

To repeat an important theme of this book: If your students are not talking to you it is probably because you are not talking to them. Set the tone on the first day. And never forget it.

3.2. Math Anxiety

About thirty-seven years ago the phenomenon of "Math Anxiety" was identified and described—by well-meaning people, educators endeavoring to explain why some people have more trouble learning or dealing with math than others (see [**TOB**] and [**KOW**] for both history and concept). We don't hear much about math anxiety in math departments because such departments are full of people who don't have it. Math anxiety is an inability by an otherwise intelligent person to cope with quantification and, more generally, with mathematics. Frequently the outward symptoms of math anxiety are physiological rather than psychological. When confronted by a math problem, the sufferer has sweaty palms, is nauseous, has heart palpitations, and experiences paralysis of thought. Oft-cited examples of math anxiety are the successful business person who cannot calculate a tip, or the brilliant musician who cannot balance a checkbook. This quick description does not begin to describe the torment that those suffering from math anxiety actually experience.

What sets mathematics apart from many other activities in life is that it is unforgiving. Most people are not talented speakers or conversationalists, but comfort themselves with the notion that at least they can get their ideas across. Many people cannot spell, but rationalize that the reader can figure out what was meant (or else they rely on a spell-checker). But if you are doing a math problem and it is not right then it is wrong. Period.

Learning elementary mathematics is about as difficult as learning to play *Malagueña* on the guitar. But there is terrific peer support for learning to play the guitar well. There is precious little such support (especially among college students) for learning mathematics. If the student also has a mathematics teacher who is a

[1]There are actually several versions of this story. The so-called French/American version is that a mathematician (Mittag-Leffler) ran off with Nobel's wife. The Swedish version is that Alfred Nobel was a practical man of the world who wasn't aware of, or did not care about, mathematics as a discipline. Thanks to the efforts of Lars Gärding and Lars Hörmander [**GAH**], most people now subscribe to the second theory.

[2]If you are tired of the standard Nobel Prize story, then tell the lesser known story of the Mittag-Leffler Prize. Mittag-Leffler set it up, of course, to spite Nobel. He mandated that the medal would be twice as large, and the award twice as grand in several notable aspects. It was only awarded twice because Mittag-Leffler invested the funds in the Italian railroad system and German World War I bonds.

dreary old nudnik and if the textbook is unreadable, then a comfortable cop-out is for the student to say that he/she has math anxiety. His/her friends won't challenge him/her on this assertion. In fact they may be empathetic. Thus the term "math anxiety" is sometimes misused. It can be applied carelessly to people who do not have it.

The literature—in psychology and education journals—on math anxiety is copious. The more scholarly articles are careful to separate math anxiety from general anxiety and from "math avoidance." Some people who claim to have math anxiety have been treated successfully with a combination of relaxation techniques and remedial mathematics review.

It would be heartless to say to a manic depressive, "Just cheer up," or to say to a drug addict, "Just say no." Likewise, it is heartless to tell a person who thinks he/she has math anxiety that in fact he/she is wrong—he/she is just a lazy bum. At the same time, mathematics instructors are not trained to treat math anxiety, any more than they are trained to treat nervous disorders or paranoia. If a student told you that he/she had dyslexia, then you certainly would not try to diagnose it or treat it yourself; nor would you tell the student that he/she just didn't have the right attitude, and should work harder. Likewise, if one of your students complains of math anxiety, you should take the matter seriously and realize that you are not qualified to handle it. Refer that student to a professional. Most every campus has one.

Never forget that you are a powerful figure in your students' lives. This fact carries with it a great deal of responsibility. Problems of the psyche can be severe and dangerous. If a student comes to you with psychological problems, then make sure that he/she gets help from someone who is qualified to administer that help.

Unfortunately, at some schools the "math anxiety" thing has gotten way out of hand. There are good universities where a student may be excused from a mathematics or statistics course (one that is *required* for the major) by simply declaring himself/herself to be math phobic, or possessed of math anxiety. It is a sad state of affairs, but there is nothing that we math teachers can do about it. Because you and I are, by nature, good at mathematics, and because we do not suffer from math anxiety, it is difficult for us to empathize with people who suffer from this malady. It is really best to let those who know the literature, know the symptoms, and know the treatments to handle students who have this form of stress. Do not hesitate to refer your students to the appropriate counselor when the situation so dictates.

3.3. Inductive vs. Deductive Method

It is of paramount importance, epistemologically speaking, for us as scholars to know that mathematics can be developed *deductively* from certain axioms. The axiomatic method of Euclid and Occam's Razor has long been the blueprint for the foundations of our subject. Russell and Whitehead's *Principia Mathematica* is a milestone in human thought, although one that is perhaps best left unread. Hilbert and Bourbaki, among others, also helped to lay the foundations that assure us that what we do is (for the most part) logically consistent.

However mathematics, as well as most other subjects, is not learned deductively; it is learned *inductively.* We learn by beginning with simple examples and working from them to general principles. Even when you give a colloquium lecture to seasoned mathematicians, you should motivate your ideas with good examples. The principle applies even more assuredly to classes of freshmen and sophomores.

Beresford Parlett recently said

> Only wimps do the general case. Real teachers tackle examples.

This simple idea should be a guiding force whenever you are preparing to explain a new idea to your students.

Take the fact that the mixed partial derivatives of a C^2 function in the plane commute. To state this theorem cold and prove it—before an audience of freshmen—is showing a complete lack of sensitivity to your listeners. Instead, you should work a couple of examples and then say, "Notice that it does not seem to matter in which order we calculate the derivatives. In fact there is a general principle at work here." Then you state the theorem.

Whether you actually give a proof is a matter of personal taste. With freshmen I would not. I'd tell them that, when they take a course in real analysis, they can worry about niceties like this. Other math instructors may have differing views about the question of proof.

And by the way—you know and I know that C^2 is too strong a hypothesis for the commutation of derivatives. But, really, isn't that good enough for freshmen? If a bright student raises this issue, offer to explain it after class. But do not fall into the trap of always stating the sharpest form of any given result. Great simplifications can result from the introduction of slightly stronger hypotheses, and you will reach a much broader cross section of your audience by using this device.

Ralph Boas had these thoughts about the inductive method:[3]

> I once heard Wiener admit that, although he had used the ergodic theorem, he had never gone through a proof of it. Later, of course, he did prove (and improve) it.
>
> ......
>
> I do not think my story about Wiener is very surprising. One can't always be going back to first principles.
>
> I quite agree that—at least for *some* people (I am one of them) calculation precedes understanding. I have probably said before that I knew how to calculate with logarithms long before I knew how they worked. The idea that proofs come first is, I think, a modern fallacy. Certainly—even in this calculator age—a child learns that $2 \times 2 = 4$ before understanding *why*. The trouble with "new math" was (in part) the fallacy of thinking that understanding needs to come first.

Ralph Boas was a great teacher, and there is wisdom in what he says. Don't put the cart before the horse when you teach. A young student is ill-motivated to learn the inner workings of a mathematical idea before he/she has understood what it is and how to use it.

Now suppose that you are teaching real analysis (from [**RUD**], for example). One of the neat results in such a course is that a conditionally convergent series can be rearranged to sum to any (extended) real limit. When I present this result, I first consider the series $\sum(-1)^j/j$ and run through the proof specifically for this

[3]Part of this quotation comes from a private communication between Boas and James Cargal, as quoted in [**CAR**].

example. The point is that, by specializing down to an example, I don't have to worry about proving first that the sum of the positive terms diverges and the sum of the negative terms diverges. That is self-evident in the example. Thus, on the first pass, I can concentrate on the main point of the proof and finesse the details. After doing the first example, and thereby instilling the students with confidence and understanding, I easily can go back to the general result and describe quickly how it works.

Go from the simple to the complex—not the other way. It's an obvious point, but it works. An example of this philosophy comes from the calculus. Many calculus books, when they formulate Green's theorem, go to great pains to introduce the notions of "x- simple domain" and "y- simple domain" (i.e. domains with either connected horizontal or connected vertical cross sections). This is because the authors are looking ahead to the proof, and want to state the theorem in precisely the form in which it will be proved. The entire approach is silly.

Why not state Green's theorem in complete generality? Then it is simple, sweet, and students can see what the principal idea is:

Theorem: *Let $\Omega \subseteq \mathbb{R}^2$ be a smoothly bounded domain. Let $\mathbf{F}(x,y) = u(x,y)\mathbf{i} + v(x,y)\mathbf{j}$ be a smooth vector field defined on Ω together with its boundary. Then*

$$\oint_{\partial\Omega} \mathbf{F} \cdot d\mathbf{r} = \iint_{\Omega} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dA.$$

When it is time for the proof, just say "to keep the proof simple, and to avoid technical details, we restrict attention to a special class of domains ..." This approach communicates exactly the points that you wish to convey, but cuts directly to the key ideas and will reach more of the students with less fuss. If a student asks you afterward what is lost by restricting attention to x-simple and y-simple domains, you can point out that more general (smoothly bounded) domains can be cut up into x-simple and y-simple subdomains. This is a *wonderful* answer, for the apt student will immediately begin to draw pictures and cut up domains, and will gain immense satisfaction from the process.

Here is a useful device—almost never seen in texts or discussed in teaching guides—that was suggested to me by Paul Halmos:

Suppose that you are teaching the fundamental theorem of algebra. It's a simple theorem. You could just state it cold and let the students think about it. But the point is that these are *students*, *not* mathematicians. It is your job to give them some help and motivation and context. First present to them the polynomial equation $x - 7 = 0$. Point out that it is easy to find all the roots and to say what they are. Next treat $2x - 7 = 0$. Follow this by $x^2 + 2x - 7$ (complete the square—imitating the proof of the quadratic formula). Give an argument that $x^3 + x^2 + 2x - 7$ has a root by using the intermediate value property. Work a little harder to prove that $x^4 + x^3 + x^2 + 2x - 7$ has a root. Then surprise them with the assertion that there is no formula, using only elementary algebraic operations, for solving polynomial equations of degree 5 or greater. (In the process, tell the students a little bit about Evariste Galois, and how he recorded his key ideas the night before the duel in which he was sure he would die—at the age of 21!) Finally, point out that the remarkable fundamental theorem of algebra, due to Gauss, guarantees in complete generality that any non-constant polynomial has a (complex) root.[4]

Notice how much depth and texture this simple discussion lends to the fundamental theorem. You have really given the students something to think about. Stating the theorem cold and then moving ahead, while *prima facie* logical and adequate, does not constitute teaching—that is, it does not contribute to understanding. As with many of the devices presented in the present book, this one becomes natural after some practice and experience. At the beginning it will require some effort. The easiest thing in the world for a mathematician to do is to state theorems and to prove them. It requires more effort to *teach.*

I once heard a splendid lecture by a distinguished mathematical physicist. He began by telling us that everyone thinks that someone else understands the Second Law of Thermodynamics. But in fact nobody really understands it. He went on to say that he had come up with a new formulation of the Second Law, and he could now say that he understood at least *this new formulation.* The rest of the talk concentrated on explaining his new idea. He began that portion of the discussion by saying, "Suppose you have a cup of coffee . . . " Throughout the talk, he illustrated sophisticated ideas from mathematical physics by way of the cup of coffee.

Think a minute about what this speaker did for his audience. First, he exhibited incredible humility by saying that, up until this moment, he had never understood the Second Law of Thermodynamics. Then he went on to say that he found a new way to think about it that at least *he* understood.[5] To illustrate his new ideas, he spoke of a cup of coffee. Who in the audience would not be at ease after an introduction like this? Who would not be dying to hear more? Who would not feel that the speaker was welcoming him into his world? Isn't this what you want to do when you teach?

One could go on at length about the philosophy being promulgated here. But the point has been made. Saki once said that, "A small inaccuracy can save hours of explanation." Mathematicians cannot afford to be inaccurate. But, for the students' sake, we can simplify. We can reach out to our audience, and find a meeting ground. We can communicate.

3.4. Who Is My Audience?

I think that, collectively, we academic mathematicians have made some tactical errors in the past fifty years. During that time, while our universities were undergoing explosive growth, and both student and faculty populations were increasing at a startling rate, we also saw a great deal of curricular development and change. And it happened, with some regularity, that other departments would approach the math department and say, "We think that our majors should know some mathematics. Can you help us?" Awkward discussions ensued. It came to pass that the "other departments" were unable to articulate their needs, and we mostly didn't care. So we would ultimately say, "Have them learn some calculus." The result is the huge, undifferentiated throng of students that populates our calculus classes

[4] I always take some delight at this point in telling my students that both of Gauss's sons settled in the St. Louis area.

[5] Bear in mind that this speaker is the holder of the Dirac Prize and many other high honors in the physics world.

these days. At some big state universities, as many as three or four thousand or more students enroll in calculus in the fall semester. But the hard truth is that we set up our service courses more to suit ourselves than to suit our customers.

Mathematicians of my generation were brought up to teach mathematics in general, and calculus in particular, in the French style. That is, we teach as though the classroom were full of little future mathematicians. We state theorems and we prove them. In the cold light of day, you know as well as I that we make little effort to speak to the business students in their own argot, or to the pre-med students in their own argot, or to the psychology students in their own argot. Schools with the requisite population and with adequate resources set up special sections of calculus for these different populations, and even use special textbooks. But the hard truth is that a calculus book for business students is little more than watered down calculus. Such a book is written just like a calculus book, except that most functions are polynomials, and the few exceptions are exponentials and logarithms. The business applications are usually just a tiresome appendage. A similar description applies to most calculus books written for the life sciences.

In fact the scenario that I've described above is a bit embarrassing for all of us. Probably a psychology major would get a lot more use out of a course that gave him/her some finite math and some statistics and maybe just a smattering of calculus. A business student needs to understand a statement like "the rate of increase of inflation is decreasing," but probably does not need to know the Weierstrassian $\tan(z/2)$ change of variable for integrals. A biology student needs to understand statements about average population density of mitochondria, but probably does not need to know how to analyze a hanging cable using hyperbolic trigonometric functions.

We have never given much thought to what our constituents really need, or really want. Recently at my own university the Dean of the Business School came to us with some uncertainties about our business calculus course. One upshot of the subsequent negotiations is that, now when business students take our business calculus course, they must concurrently enroll in a one-credit course—*given by the Business School*—in which a business instructor explains to the students what the calculus ideas mean *in their own language*. This is a wonderful idea, and one that I would like to see more widely adopted.

When we teach elementary mathematics courses, we must disabuse ourselves of the idea that our audience consists primarily of mathematics majors, or of future mathematicians. Even if you teach at M.I.T. this notion is most likely incorrect. Like a good writer, you should be aware of your audience. And this awareness should inform your teaching methodology.

At a prominent public university on the West coast, the most popular calculus course (measured by the number of students enrolled) is the one taught by the Forestry Department. No kidding. A moment's consideration will explain why this is so. Forestry majors at this institution are required to take calculus, and they are not the most mathematically gifted people around. They cannot handle the calculus course offered by the mathematics department, so the Forestry Department created a course of its own. Meanwhile, students in other majors—chemistry, biology, nursing, and so on—are required to take calculus. But the regulations do not say *which* calculus course. So they vote with their feet—they gravitate to the easiest calculus course they can find.

If you are like me, your reaction after reading the fourth sentence of the last paragraph might have been "Well, we don't want to teach those students anyway." But you probably did not anticipate the way in which the paragraph would end. It is important for a mathematics department to look after its courses and, yes, keep control over them. At a large Midwestern University, the Engineering School teaches its own Japanese courses. This tells me that the Japanese Department is not minding the store. We should not be guilty of such negligence. Teaching students is what pays the bills, and what maintains our credibility with the administration. Teaching calculus to forestry majors may not be your favorite activity, but it is important for the welfare of your department. You should do it, and you should do it well.

Now let me return to the subject of being sensitive to one's audience. When I taught at Penn State, I once participated in a College of Science "Open House." The Open House was the university's way of reaching out to the community, and helping citizens become acquainted with the institution. Groups of young people were bussed in from all over the state of Pennsylvania to participate in the activities. Unfortunately, there was poor coordination between the administration and the individual departments. Therefore it was difficult to suit the planned programs to the different groups. In one particular hour, we planned to show a film about "turning a sphere inside out." The kids that showed up for this piece of entertainment were gang members from inner city Philadelphia. You can imagine that they were not fascinated by four Frenchmen in black suits, chain smoking Gauloises and discussing the niceties of low dimensional topology. Being a quick thinker, I turned off the film and announced, "Let's talk about gambling. I'll show you some tricks with cards so that, when you gamble, you'll always win." One gang member fixed me with a grim look, pulled out a large knife, and said, "I don't need that %$!#&. I *already* always win."

As you can see, I thought I knew my audience, but I did not. The fact is that, if I wanted to entice inner city gang members to further their education, then I should have realized that mathematics—especially low-dimensional topology—may not be the medium that speaks to them. They might be more interested in law, or medicine, or engineering. As professors at universities, we take pleasure in being removed from the vicissitudes of daily life. But sometimes, to be good teachers, we must confront them.

3.5. Mentors and Neophytes

I've had many mathematical mentors, but never a teaching mentor. I've learned how to teach by making virtually every mistake that can possibly be made and then trying to learn from those mistakes.

Having a mentor—someone experienced in the craft of teaching—can save you a lot of travail. I encourage you to seek such a guide. Your mentor will not always give you the answers that you want, nor possibly any answer at all. But you can benefit from the mentor's experience.

I have frequently served as a teaching mentor, for faculty both young and old. This has come about partly because of my willingness to play the role of mentor, and partly because I have written this book. I have derived some satisfaction from the mentor role, and have often felt that I've done a little good. But not always. For example, when I visited Australia, the chairman of one mathematics department took me aside and told me of their special problem. The freshmen in their large lectures—especially the males—act like escapees from the Asylum at Charenton. They run through the aisles. They throw paper airplanes, and fruit, and books—both at the instructor and at each other. And they shout and scream and carry on at great length. I expressed some confusion and dismay at this description, and allowed that I knew of no American university with this problem. After some discussion, the chairman told me that their freshmen all live at home with their parents and are therefore still *extremely immature*. This information certainly helped to explain the situation, but gave no clue as to how to deal with it (my only thought was massive and regular doses of thorazine). In the end, we agreed that what they were already doing was probably the best that they could do. Instructors had to learn to be tough and harsh from day one, and to deal with miscreants both directly and severely. New instructors were given a crash course in this technique. Those who lasted the first year were survivors in the strong sense of the word.

Fortunately, most of the teaching problems that you will face in your first year or two at the job will be more prosaic. A mentor can show you the ropes when you are writing a syllabus or exam, help you to choose the right text, explain when you can skip a topic in a course, suggest how to handle disruptive students, or tell you what to do if you are running out of time in a course. A mentor can help you curve the results of a test. To ask for help is nothing to be ashamed of.

Some mathematics departments now assign a senior teaching mentor to each new young faculty member. If your department is one of these, then take advantage of your mentor's experience and perspective. You don't have to follow all of the advice that is tendered, but at least you will have food for thought. You will have somewhere to turn if you are confused or bewildered. If you are not assigned a mentor, then take the initiative and approach a senior faculty member. He/she will probably be quite pleased and flattered to be asked, and you will have initiated a useful relationship.

In American business schools, it is fashionable to speak of the *sempai-kohai* relationship. Borrowed from the Japanese, this terminology refers to the notion that one should revere older people, and heed their wisdom, just because they are old. If you had asked me to respond to this notion when I was twenty, then I would have quoted the Yardbirds:[6]

> The shapes of things before my eyes
> Just teach me to despise.
> Will time makes man more wise?

Now that I am sliding into my golden years, I have a slightly different view of the world. Nobody is *a priori* correct just because he/she is old. But an old person has fought the good fight for many years, and has survived. This individual will have stories to tell, and advice to give. Experience is a valuable commodity, and anyone who is willing to share the benefits of his/her experience is someone worth listening to. Find a mentor, ask him/her questions, and listen to the answers. And then use this input to craft your own style of teaching.

[6] The Yardbirds were a popular rock group in the 1960s—famous in part because guitarist Eric Clapton was in the group.

3.6. Teaching Reform

The teaching reform movement had its formal beginning in 1986, when Ronald Douglas of SUNY, Stony Brook and Stephen Maurer of the Sloan Foundation organized a small (twenty-five participant) meeting, held at Tulane University, to lament the sorry state of lower-division mathematics education in the United States—particularly calculus teaching. Some of the early reports pursuant to Douglas's meeting, and the ideas that sprang therefrom, appear in [**NCE**], [**DOU**] and [**STE1**]. For a more recent assessment, see [**STK**], [**LET**], [**TUC**], [**KLR**], [**BRE2**], or [**ROB**]. Other points of view are offered in [**WU1**], [**WU2**]. Some modern perspectives appear in [**GOT**], [**VDW**], [**CRW**], and [**QUI**].

Douglas is a great organizer, and he got people fired up with information that our attrition and dropout rate in calculus is embarrassingly high, that our failure rate is unacceptable, that our teaching is not optimal, and—yes—that the lecture is dead.

Douglas persuaded the federal government to be interested in the problem, and many different programs were set up at the national level to encourage mathematics faculty—at universities and colleges of all sorts—to rethink and re-invent their curricula. There are special programs sponsored by the National Science Foundation for work on

- calculus
- precalculus
- post-calculus
- high school curriculum
- instrumentation and laboratory improvement
- advanced technology education
- undergraduate faculty enhancement
- teacher preparation

and many others as well.

When Persi Diaconis was asked about his MacArthur Prize (see [**MP1**]), he said, "...if somebody gives you a prize and says that you're terrific, well, that's nice. But if someone gives you a prize and says here is $200,000 to show you that we mean it, maybe they really mean it." So it is with teaching reform. Government funding has made the reform idea into a movement, and has made it necessary for the rest of us to take it seriously. It has enabled the investigators to buy equipment, to run teacher-training workshops, to hold conferences, to write books, and to get the word out. It has really had an effect. Like it or not, the Harvard calculus book (the first big reform book) had the greatest first year sales of any calculus book in modern history—by a factor of about four. It has really had an impact, and has caused many of us to take a new look at calculus. That effect alone may justify saying that the money was well spent.

And the new methods have much to offer. Some of the hallmarks of the new methodology are these:

- Lectures, used as an exclusive teaching device, are not sufficient to adequately educate, inspire, and motivate the college students of today.
- Traditional methods of teaching should be replaced by "cooperative learning."
- Students should discover mathematical facts for themselves.

- Students can help each other discover mathematical facts by working in small groups, with the instructor acting as a coach.
- Students can use computers to help them to visualize mathematical concepts.
- Students can use computers and calculators to take over the tedious aspects of mathematics (i.e., calculations and drill) so that they may concentrate on the conceptual aspects.
- Students can (and this is a significant new idea) use the computer to construct implementations of mathematical concepts in computer code, and thereby aid in the construction of those concepts in their own minds (see, for instance, [**DUB**], [**DDLZ**], [**DUF**], [**DUL**]).

Whether or not you agree with these precepts, they certainly merit your serious consideration.[7] I still use lectures to teach, but I do everything I can to let the students at least *think* that they are discovering the ideas. You certainly cannot stand in front of a group and *command* that they discover the fundamental theorem of calculus on their own (this is akin to a bald man commanding his hair to grow). But you can lead them through the woods, leaving a trail of corn that they can peck at and follow. You also cannot put students into a group and say, "Discuss conservative vector fields and see what you come up with." You must provide considerable guidance and context.

Let me not mince words about one key notion. The British educational system—especially as it is practiced at Cambridge and Oxford Universities—does a terrific job of teaching students the art of discourse. Each student at these extraordinary universities is assigned a tutor, with whom the student meets weekly, who literally drills the student in the arts of written and oral communication. The student's weekly assignment is to write an essay on a topic chosen by the tutor. The student is to use all the resources of the university—lectures, other faculty, the library—in preparing his/her piece. Then he/she must meet with the tutor and defend it. It is a wonderful regimen, and serves to keep language and dialogue and discourse alive.

We in the United States have not traditionally done as well at this sort of training. Instead, we take eighteen year old freshmen, many of whom have never articulated a thought more sophisticated than an order for a burger and fries shouted at the plastic effigy outside of the local Jack-in-the-Box,® and we suddenly expect them to formulate and deliver cogent sentences about the chain rule and linear independence. Talking mathematics is a high level skill. We should not simply *expect* our freshmen and sophomores to be able to do it. Instead, we should *plan to train them to do it.* We should develop techniques for *teaching* our students to talk to us. Many of the teaching reform projects—notably the Harvard project (see [**HAL**])—have led the way in teaching students the art of communication, and particularly how to write.

The Mathematics Association of America (MAA) had a project called CLUME (Cooperative Learning in Undergraduate Mathematics Education), the purpose of which was not simply to make faculty aware of new pedagogical techniques but also to help faculty to learn to use them. The project wrapped up in 1999, but its results are still useful.

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[7]The Web site `www.math.okstate.edu/archives/calcrefm.html` offers information about many of the reform resources that are available.

Information about this program may be obtained from the Web site: `www.maa.org/publications/maa-reviews/cooperative-learning-in-undergraduate-mathematics-issues-that-matter-and-strategies-that-work`

I say this elsewhere in the book but it is worth repeating here. If you simply tack new teaching techniques on to your existing lectures and problem sessions then those new techniques probably will not succeed. If you want to use the new techniques then—to some degree—you will have to learn to teach anew.

The hidebound among us, those who are content with traditional teaching methods and who have little patience for the reform practices, are fond of suggesting that "money talks and baloney walks." If I were a real cynic, I would recall the story about Willie Sutton—the man who staged the great robbery of the Bank of England. When asked how he got into robbing banks, Willie said, "Well, that's where the money is." I know many of the principals who are involved in calculus reform. Surely they like receiving summer salaries and other pecuniary support no less than the rest of us. But these are people who are sincerely committed to bettering undergraduate education in mathematics. And they probably work a lot harder at it than the rest of us. Their conclusions are certainly worth our careful examination.

Some proponents of the reform movement cite the Meyer/Briggs Type Indicator, commonly known as MBTI (see [**KRT1, KRT2**]), to substantiate their views about the way that students learn. The argument goes as follows. The Meyer/Briggs shows, objectively and quantitatively, that there are sixteen different personality types. These types are based on four dichotomies: extroverted/introverted, sensing/intuitive, thinking/feeling, and judging/perceiving. These various types have different values and aspirations and needs. Yet mathematicians have always taught basic mathematics to suit one personality type (i.e., themselves—people with a penchant for mathematics). So, argue these reformists, it is up to mathematicians to acknowledge that they need to develop and learn different ways to teach mathematics to suit the different types of people that there are.

This is an interesting line of reasoning. Certainly any successful teacher knows that there are students with mathematical ability and there are those without; there are students who are quick and students who are slow; there are students who think scientifically and there are those who do not. However, these are not the personality differences that Meyer/Briggs picks out. Rather, Meyer/Briggs focuses (as indicated in the preceding paragraph) on rather "personal" aspects of the personality. Should we adapt the teaching of calculus (for instance) to these different types of people?

My view is that one goes to college to get an education, and a substantial component of education is to learn different methods of discourse. The methodology and language and style of discourse of mathematics is tried and true—it has a noble history going back at least to Euclid. It conceptualizes certain types of problems in a precisely formulated language and it provides tools for solving them. We would certainly be doing our students—and ourselves—a disservice to abrogate that hard-won value system and methodology in an attempt to sociologize our subject. Our job is to teach logic and problem solving and the mathematical method. It is what we know and what we are good at. And it is what we can teach and teach well.

My thoughts in the last three paragraphs are not meant to be a slam at the overall value system of teaching reform. Instead, it is a discussion of one particular set of values promulgated by a special set of people who profess to subscribe to reform. Since I am a great believer in critical thinking and the art of discourse, I offer this discussion as grist for that particular mill. Frankly, I think that it is far beyond the ken of the typical mathematician to become involved in assessing various personality types and adapting teaching to those types. Most of us are not very good teachers to start. To make the job vastly more difficult—by endeavoring to adapt our teaching to a spectrum of personality types—strikes me as self-destructive.

In reading articles about reform, and listening to discussions about reform, I find that there is a confusion between *content* and *motivation*. To my mind, there is nothing essentially wrong with the content of any standard lower-division math course. However, the ways that we motivate the material, or endeavor to make it real for the students, or attempt to make the material speak to the students' interests, is hit-and-miss at best. Most of us have given very little thought to the matter, and do not have the necessary knowledge from other disciplines that would be necessary to really make mathematics come alive for a business student, or a biology student. I think that the reform movement can teach us a great deal about how to help the students to become *engaged* in the learning process. But we should not confuse all the good features that reform has to offer with the (I believe misguided) fear that parts of the existing curriculum may be lost in the process of reform. Traditionalists will be more willing to accept reform if they are not beset by fears that the curriculum that they know and believe in is going to be lost in the shuffle. Reformers may be more willing to talk to traditionalists if they can believe that traditionalists are willing to listen.

Debate about reform tends to be quite emotional. I believe that the vituperation is due partly to the fact that people are worried about defending their turf. Also, middle aged math professors probably don't want to discard the teaching techniques that they have been using half their lives and then retool. But I have also found (and this phenomenon often holds when people disagree vehemently and hopelessly) that different participants in the reform discussion actually have different concerns and goals.

- Some are concerned with teaching a well-defined body of mathematics to a given audience.
- Others are interested in maintaining and promoting the students' self-esteem, and in empowering certain groups.
- Still others wish to cut the attrition rate, so that more students can advance to upper-division mathematics courses—no matter what the loss in content and curriculum.

I am not judging the goals described in this bulleted list. But if one reviews the somewhat alarming debates about mathematics teaching that have been going on in California (see [**AND**], [**JAC1**], [**JAC2**], and [**ROSG**]), one sees in the end that frequently the warring factions are discussing entirely different matters. It is essential that we communicate, and find grounds for cooperation rather than for hostility (see particularly the article by Judith Roitman in [**GKM**]). In the end, we all want to learn how to teach our students so that they learn what they need to learn and go on to success in their later activities. See [**KLE**], [**GID**], [**JAS**], and [**RUDE**] for further thoughts about the math wars.

I am happy to say that I was an organizer of a conference held several years ago whose goal was to get reformers and traditionalists to talk to each other. We chose the participants carefully, avoiding certain well-known hotheads. And we were pleased to see how readily these different groups communicated, and how many shared values we discovered. The volume [**GKM**] is the proceedings of that effort.

Let us return now to the mainstream of the reform movement—the Harvard calculus project. I certainly do not agree with all of the tenets of the Harvard project [**HAL**], and I would probably have trouble using the Harvard book as a text. But at least the Harvard people have *done something*. They have formulated an approach to lower-division teaching, and they have written some books implementing that approach. They have run workshops to train people to use the materials that they have developed. Surely what they have done is much more constructive than the collective whining of which many of the rest of us are guilty.

Of course the Harvard book is not the final word on teaching calculus—nowhere close. As noted previously, one of the notable calculus books on the market is Stewart's "Reform" edition (see [**STEW**]), which professes to be an artful marriage of reform and traditional methodologies. (See also the newer book [**BLK**].) In fact I believe that the publication of Stewart's book marks the beginning of the "second generation" of reform.

Thus we might now see concerted dialogue between reform values and traditional values. I would expect and hope that other texts professing to be a miscegenation of tradition and reform will appear, and that we will thereby converge, as a collection of professional teachers, to a new platform of insights about calculus teaching.

As I have said, the Harvard group has dealt with teaching reform, and calculus reform in particular, by *doing*. There are others, notably Ed Dubinsky and his collaborators, who have studied the theory as well as the practice of mathematical teaching. Worth particular note are the references [**ASI**], [**BDDT**], [**DDLZ**], [**DUB**], [**CDNS**], [**ACDS**]. The first of these posits a theoretical framework for mathematical curriculum development. The second, and the third as well, concentrates on cognitive aspects of student learning of binary operations, especially in the context of elementary group theory. The fourth describes a computer programming language—`ISETL`—which Dubinsky has found particularly useful in helping students to build ideas in their own minds. The fifth considers how to help students understand what a limit is. All of these are questions that ought to be of interest to the mathematics instructor. Reading these articles, you may agree or disagree with their findings. But the articles are bound to be useful in helping you develop your own ideas about teaching.

One university that has been particularly successful with reform mathematics teaching is the University of Michigan. This is in part because the university has been willing to supply the necessary resources, and also because considerable effort is put into training the graduate students and postdocs who teach calculus. And this is a point that cannot be avoided. Many of us are comfortable with teaching by lecture because we understand it and we know how much time it will take to prepare and deliver a lecture. The reform methods take a good deal of time and effort and also retooling. Not all of us want to make such a commitment.

It is now generally held that many of the calculus reform methodologies do improve student performance, particularly with regard to grades, persistence in mathematics, and performance in future courses. Many of the reform techniques are particularly effective with underrepresented groups, including women, first-generation students, and at-risk students. But it must be noted that the reform techniques are *harder*. They require more time and effort on the part of the instructor, and they require fiduciary resources as well. Reform techniques do not necessarily travel well.

Franklin Roosevelt said that "Politics is the art of the possible." There is wisdom in these words. Nobody is prepared to discard wholesale the traditional teaching techniques that we have used for so long, and likewise no thinking person can ignore the new needs of our students, the new technologies that are available, and the new ideas about teaching that are being developed. Teaching is a lifelong passion, and we spend our entire lives developing and growing as educators. We continue to learn both from the reform movement and from re-thinking traditional methods of learning and teaching. Our ideas will continue to evolve.

3.7. STEM

The financial meltdown of 2008 and the election of President Barack Obama led to particular interest in the importance of education, and particularly in the STEM (Science, Technology, Engineering, and Mathematics) disciplines. Now there is a mandate to expand the technically trained workforce beyond white males, and also to improve education at all levels.

Two reports were published in 2012 that had a decisive impact on undergraduate STEM education. These were [**PCAST**] and [**NRC2**]. Both described the overwhelming evidence for the active learning paradigm and criticized the scientific community for being slow to employ it. Here active learning is a model of instruction that focuses the responsibility of learning on learners. It was popularized in the 1990s by its appearance in the Association for the Study of Higher Education (ASHE) report [**BOE**]. In this report they discuss a variety of methodologies for promoting "active learning." They cite literature which indicates that to learn, students must do more than just listen: They must read, write, discuss, or be engaged in solving problems. It relates to the three learning domains referred to as knowledge, skills and attitudes (KSA), and that this organization of learning behaviours can be thought of as "the goals of the learning process" [**BLO**]. In particular, students must engage in such higher-order thinking tasks as analysis, synthesis, and evaluation.

In the mathematical domain, Phillip Griffiths along with Uri Treisman, Eric Friedlander, Mark Green, Jim Gates, and Tara Holm established Transforming Post-Secondary Education in Mathematics (TPSE-Math) to promote and scale active learning methods at the top-tier universities in the country. The five principal math associations are now supporting this effort.

Project NExT, now sponsored by the MAA, is an important edifice for mentoring young mathematicians. It has played an active role in promoting active learning. The Educational Advancement Foundation (EAF) has also been prominent in this context.

The Association of American Universities (AAU) has launched a STEM Undergraduate Education Initiative. The Association of Public Land-Grant Universities

(APLU) has started the Science and Mathematics Teacher Imperative. Both of these intiatives strive to bring the use of active learning approaches up to a visible level at our flagship universities.

See [**BRE3**] for further evidence of the prominence and effectiveness of active learning.

3.8. How to Ask, How to Answer

If a pollster asked the average American voter, "What do you think of the upcoming election?" then the resulting answer would probably not be very enlightening. If you turn to your calculus class one day and say, "OK, now we've covered Chapters 3 and 4—any questions?" then you will get a bunch of blank looks. By the same token, if a textbook salesman hands a new calculus book to a math professor and says, "What do you think?", the professor will probably say, "I dunno; they all look the same to me." By the same token, students come to professors with questions such as, "Like, you know; I don't think I understand any of this stuff we're doing."

It is a strange facet of the human condition that most of us don't know consciously what we think about most things most of the time. A skilled questioner learns to ask *specific questions* in order to obtain meaningful answers. Rather than asking your class if there are questions about Chapters 3 and 4, ask them instead if they are comfortable with the chain rule, or if they can handle related rates questions, or falling body problems. The material in a person's memory is hung on hooks. You must reach for those hooks to get useful answers to your questions.

The same principle applies when you are holding a review session—for a midterm exam, let's say. If you are serious, if you *really* want to help the students, then it is simply not good enough for you to stand before the students and say, "Any questions?" They *do not know* what they want to ask. And, even if they think they know, they are timid about doing so. You must prompt them: "Do you understand integration by parts? Can you do partial fractions? What about the u-substitution? Is Section 7.5 confusing? Was the second homework assignment particularly difficult?" You, the instructor, must understand that your having said these things will **(i)** jog their memories, and **(ii)** make it OK for them to ask about these topics. I find that it breaks the ice for me to write a list of topics on the board. This is just one way to get the "review session ball" rolling. Remember: You must poke the students and prod them and, if necessary, embarrass them a little. Never forget the psychological aspects of teaching.

We implement these dicta naturally when writing an exam. You would never set an exam question for freshmen that said, "Tell everything you know about differential calculus." Instead you ask very specific questions. You want to train yourself to do the same when talking or lecturing to students. More, you want to train yourself to do the same in reverse when you are trying to elicit questions from students.

There is a gentle art of getting your students to pose questions. And I don't mean questions like, "Will this be on the test?" I mean the kind of meaty, well-thought-out questions that we all live for. Perhaps the most common complaint that I hear from disillusioned mathematics instructors is that they cannot develop any participation from their lower-division classes. The matter of garnering good questions is a non-trivial issue, and one to which an entire separate book could be devoted. You are going to have to find methods that suit your personality, and your teaching style, and that work for you. (See the Appendix to this section for some specific suggestions on how to increase student participation and inject some life into your class.)

The devices that you use can be quite simple. For example, giving a good quiz once or twice per week is a device for focusing student attention on some *particular issues.* The quiz is a little bit like a traffic officer pulling you over and threatening you with a citation. We are all aware—in a general sort of way—of the traffic safety laws. But if a cop gets in your face and starts telling you things that you are doing wrong then suddenly the penny drops.

The devices that you use can also be complex. You could have each student develop an ongoing, long-term project. Such a project might have the property that it must include material based on the ideas from each week of the course. And each student must be prepared to report to you at any time on the status of his/her project.

You may very well think that quizzes are too trite and semester-long projects are too massive for you to consider. Fine. I use quizzes frequently in my own course, and I'm frankly too lazy to do semester-long projects. Finding a way to get students to participate is something that you must do for yourself. Consider wheedling, threatening, cajoling, joking, challenging, priming. You can get through to your students by making them like you, or by scaring the hell out of them, or by conning them, or by being gruff with them. I am not necessarily recommending any of these. But if you want to be an effective teacher then you must find something that works for you.

As you experiment with ways to liven up your class, bear in mind the nature of the enemy. One enemy is that young adults, for the most part, are quite unsure of themselves. Unlike an experienced mathematician, who in effect makes a career out of asking (often stupid) questions, the student is deathly afraid of looking silly in front of his/her peers. He/she is not intellectually mature, and not experienced. He/she is not expert in the art of discourse (see also Section 3.9).

This last point is worth developing. If you have survived in the academic game, then you have learned to ask questions. You would never go up to a member of the National Academy of Sciences and say "Duh. I was trying to prove an interior regularity theorem for the Laplacian, and I just cannot seem to do it. I tried integrating by parts, but I couldn't decide what to do with the boundary term." Your friend the National Academy member would—justifiably—probably conclude that you were an idiot. A safer way to pose the question would be: "I've been thinking about interior regularity for the Laplacian. I know the classical ideas, but what is the modern approach? What would be a general context in which to fit this type of question?"

If you know something about elliptic partial differential equations, then you are probably not sent into paroxysms of ecstasy by the second question either. But it certainly sounds more intelligent than the first. And it gives the questioner some room to maneuver. Students simply don't have this skill at discourse, so they resort to the obvious subterfuge—they clam up. Part of your job as teacher is to help your students learn to engage in scholarly discourse. Help them to ask questions. If a student asks a weak question, help him/her to turn it into a better one. Try to create an atmosphere in which you and the students are co-explorers. Convey that you will sometimes make false starts, and so can they. It's a knack, but you can learn it.

Another enemy, besides the observed fact that students are uncertain and don't want to talk, is that mathematics *can be* (it is not by nature) a dry, forbidding subject. Part of your job as teacher is to make the subject come alive and to motivate the students to want to learn the material. This book supplies a variety of techniques for achieving that goal.

APPENDIX to 3.8: Suggestions for Encouraging Class Participation

This appendix contains several techniques, drawn from the literature or from conversation or from my own experience, for bringing your class to life. Take them for what they are worth. Some may appeal to you, and some may not. But reading about them may give you ideas of your own. Note that the activities discussed here are designed for classes of manageable size. They do not lend themselves well to a large lecture of 350; see Section 2.13 for a consideration of techniques suitable for that environment.

In lower-division political science courses, it is common for the instructor to begin a class by saying, "Today we are going to be a medieval village. Who wants to be the mayor? Who wants to be the executioner?" And so forth. It is quite natural for a mathematician to react to that type of classroom activity with derision, to observe that it appears to be childish and non-productive. Perhaps, but such devices are a wonderful way to get students involved with the subject matter. What can we do in our math classes that will **(i)** teach the students something of value and **(ii)** get them involved with the subject matter? Here are some possible answers.

1) Get students to go to the blackboard. I have noted in Section 2.14 that this is not necessarily the most efficient use of time. But it *is* a way to get the students to participate. If you wish, and if it is feasible in your learning environment, you could record problems on the board before students come into the classroom. Those who wish can go to the board—even before class begins—and work problems. To avoid having the same old students monopolize this activity, you could institute a rule that no student may work a problem at the board twice in one week. Of course the *entire class* should discuss the various solutions that are so recorded.

2) Have students prepare oral reports or mini-lectures. This will really only work with a relatively small class. The activity is usually best reserved for the last part of the semester, when everyone is tired and students are receptive to a change of pace. Since most of the students will be inexperienced in activities of this nature, I recommend that you assign students to each give a fifteen minute lecture on a very specific topic.

3) Have students take turns writing and grading quizzes. It might be appropriate to assign a team of three students to each quiz. Not only will this activity cause the students to think critically about the material that they are studying, but it will also imbue them with an appreciation for the sorts of things that you, the instructor, must do.

4) If a student *cannot* do a problem, and brings this fact up in class, then have him go to the blackboard and explain what he/she tried and where he/she got stuck. It is certainly true that some students will be too shy to pull this off, but most students will be secretly thrilled to be treated like fellow scholars. You can orchestrate a similar activity for a student who *does* know how to do a problem.

5) Use "Minute Notes". These work in the following manner. Once every week or so, ask students to jot down on a slip of paper anything that is bothering them—problems that they cannot do or concepts that they cannot understand or anything else that pertains to the class. You give them just one minute for this task (hence the name). Do it at the beginning of the class hour, and collect the notes right away. Read them on the spot. You will suddenly have a much clearer picture of what is going on in the class, what concerns the students have, where you should go from there.

Perhaps more importantly, you will have given the students a feeling of empowerment. You will have helped them to understand that their input is a constructive part of the class. After a few weeks of Minute Notes, you will generally find that students are much more willing to raise their hands in class and make meaningful contributions to the learning experience.

6) If you are truly daring, then you can design your course so that it is more like a literature course. That is, you give the students regular reading assignments and homework assignments, but you do not lecture directly on a linearly ordered sequence of topics. Instead you come to class each time with an air of, "Well, what shall we talk about today? Who would like to begin?" The idea is that your classroom is a marketplace of ideas. You need to really know your stuff, and have an engaging manner, to pull this off. But it is bound to be great fun. See also Section 4.5 on flipped classrooms.

7) Have guest instructors. To use this tool well, you must work closely with the guests to be sure that they will talk about material that is salient to the class, and will present it at an appropriate level. If you think of the fourteen weeks (give or take) of your course in the same way that I have discussed single lectures or classes, then having guest instructors is a way to prevent your course from being an "uninflected monotone". You can also consider roles that graduate students, teaching assistants, and "teacher's aids" (i.e., teachers in training doing their practicum in your class) might play in livening up the atmosphere.

8) If you have the resources, and the breadth of acquaintance, or if your department has the contacts, you could bring in guest speakers from industry or government or business. Imagine a calculus class in which you bring in someone working on the NASA space station project to talk about how calculus is used to design the work platform for the engineers in space (I'm not making this up; there really is such a project—and I have written research papers about it. In fact one of my papers has a picture of the space shuttle.). Students would really wake up and smell the coffee when confronted with such a class experience.

9) This technique was devised by Jean Pedersen. She asserts that it works extraordinarily well for her. It is called the method of "mathematical POST-IT® notes."

We all know that POST-IT notes are those little squares of colored paper that easily can be affixed or un-affixed to a document for the purpose of making remarks or memos. The idea for the application of these devices in a math class is that the professor comes to class with a tablet or two of these notes, each having the professor's name (or some other identifiable epithet) stamped on it. Whenever a student asks a good question (not "Will this be on the test?" or "What is this stuff good for?" or some pseudo-question that the student just cooked up), then he/she is rewarded with a POST-IT note. "So what?" you ask.

When the next exam comes around, the students are instructed to bring their POST-IT notes along. They are to affix them to the front of the exam that they hand in. The student then receives two extra points (or some number to be predetermined) for each POST-IT note.

Reports are that, when this policy is announced in class, it is as though a jolt of electricity has run through the room. Suddenly hands are waving in the air, and previously uninterested students become the life of the party.

Now let me be the first to admit that this teaching device, like any other, is not perfect. Some students who are already alienated will become more alienated if they are unable to garner any POST-IT notes. Other students may object that they are being treated like children. Think carefully before you try this, or any, new technique.

10) I have saved the most frivolous suggestion for last. Although you probably will not choose to use it yourself, it may suggest analogous techniques that more naturally suit you and your classroom. And, although the technique is a bit silly, it is currently in use by at least one successful math teacher.

On the first day of class the instructor announces that he is very embarrassed to report that he/she simply cannot spell. Students should feel free to correct his/her dreadful spelling. Then he/she begins to lecture, spelling "line" as "lien" and "book" as "buk". Students are so delighted confidently to be able to correct the professor's spelling that participating constructively in the mathematics portion of the course becomes very natural.

I find this last technique of deliberate misspelling to be a bit dishonest, but it's hard to argue with success. In my own classes, I endeavor to create the feeling that we are all creating the lesson together. I do this with a constant line of patter, much like that used by a magician or an illusionist. With this technique, I have the students talking all the time as well. If a mistake is made in class, then it is *our mistake*, and we fix it together. If a problem is solved correctly, then that is our shared triumph.

The key to bringing your class to life is to become involved with the students and to make learning a shared activity. Perhaps this is one of the great lessons of the reform movement. It is not an ideal learning environment to have the teacher as stick-man preaching before an audience of sponges. Learning should be done symbiotically, and it is up to the instructor to structure his/her class accordingly.

3.9. Inquiry-Based Learning

The Soviet launch of the Sputnik satellite in 1957 caused a virtual panic in this country regarding science education. One of the upshots of all this educational activity is an interest in inquiry-based learning. The premise of this type of learning is that subject matter should not be presented as “received knowledge.” Instead, the teacher or facilitator should pose questions, problems, or scenarios to the students. Then the students learn by doing.

Historically speaking, the basis for inquiry-based learning comes from ideas of Piaget, Dewey, Vigotsky, and Freire. In the 1960s, Joseph Schwab studied the methodology and gave it some structure. He in fact divided inquiry into four distinct levels. Later on, Marshall Herron, Heather Banchi, and Randy Bell developed these ideas further. In a modern formulation, the four levels of inquiry are:

(a) **Confirmation Inquiry:** Here students are supplied the question and solution procedure for soving the problem, and *the results are known in advance*. Confirmation of the result is the goal of the inquiry.

(b) **Structured Inquiry:** Here students are provided with the question and the solution procedure, but now the task is to generate an explanation that is supported by evidence collected in the procedure.

(c) **Guided Inquiry:** Now students are provided only with the research question. The task is to design the solution procedure and then to test the question and the resulting explanations. This method is more successful when the students have had several previous opportunities to learn and practice different solution techniques.

(d) **Open Inquiry:** Here the students formulate the questions themselves, design solution procedures, and communicate the results.

Joseph Schwab believed that science did not need to be a process for identifying stable truths about the world around us. Instead science should be a flexible and multi-directional inquiry-driven process of thinking and learning. Science in the classroom should more closely reflect the work and methods of practicing scientists. Clearly the inquiry-based learning technique is an outgrowth of his ideas.

The Educational Advancement Foundation has developed a modified version of the Moore method that incorporates inquiry-based learning. It continues to support and promote efforts to develop inquiry-based learning.

3.10. The Art of Discourse

Ask yourself this question: If a student has a successful and fulfilling college education, then what does he/she take away with him/her? Twenty years after graduation, what does that student still retain? What intellectual framework does he/she have to build on?

Comic Don Novello, in his role as Father Guido Sarducci (on the television show *Saturday Night Live*), gave the following answer. “If you majored in Economics, all you remember twenty years later is ‘Supply and Demand.’ If you majored in French, all you remember is ‘Parlez vous Français?’ If you majored in Physics, all you remember twenty years later is ‘Every action has an equal and opposite reaction.’ ” (He might have added, “If you majored in math, all you remember is

'Take the exponent and put it in front.' ") So the good Father Sarducci proposed that people not spend four years and $200,000 on a traditional university education. If this is all you are going to retain, argued the good cleric, Father Sarducci will teach it to you in five minutes—and charge you much less. He called his solution "Father Guido Sarducci's Five Minute University."

We who devote our lives to university teaching hope fervently that there is considerably more to higher education than Father Sarducci's droll diatribe would suggest. In this section I am going to endeavor to say what that "more" is.

The naive answer to the question "What does a student get from his education?" is that the student receives career training. Certainly career training has significant value, and should not be dismissed lightly. But, if we take the long view, then we can see a larger picture. We can see depth and texture. What a student ought to take away from college is **(i)** critical thinking skills and **(ii)** knowledge of and experience with discourse (see also the discussion in Sections 3.7. 3.9). These two aspects of education are essential, and they are not disjoint.

In college, a student declares a major. And that is the area in which the student obtains advanced training. But most of the student's courses are *not* in the major. In those other courses, the student is learning philosophical discourse, humanistic discourse, the discourse of social thought, and scientific discourse. The student is learning *different modes* of critical thought.

For example, Renaissance philosophers considered the questions, "What is the world we see and what is the world we experience and what is the world that is *actually out there*? Are they one and the same world? If not, then how do they differ? And how can we tell?" Renaissance mathematicians studied algebraic equations. Renaissance musicians studied the lute. All of these are valuable avenues of inquiry, and they are all quite different. An important part of gaining an education is learning about these different modes of thought.

When we teach undergraduates—especially lower-division students—we are primarily teaching non-majors. It can certainly inform our teaching, and remind us of what we are about, to be cognizant of our goals when we teach. When you teach calculus to a pre-medical student, or finite math to a business student, you are endeavoring to acquaint him/her with modes of mathematical thinking, with our special method of reasoning and analysis.

In fact, when I teach my undergraduates, I have in mind a much larger and more ambitious goal. I want to teach my students that the world need not be a place in which they are passive observers. They need not spend their lives "letting things happen." Put in other words, we do not—at least should not—live in a world in which some nebulous *other people* generate ideas, and hold office, and make decisions. In fact it is *we* who are to become educated, to assume the positions of leadership, and to make the important decisions. To my mind, this is the role in society of an educated person. Perhaps we instructors have, in our own lives, realized this truth. But we should determine to pass it on to our students.

Surely it is more constructive, and more fulfilling for everyone involved, to bear these thoughts in mind as we lecture these unformed lumps of clay. Do not view the teaching process as a sorry labor—akin to shoveling out the Aegean stables. You are *not* trying to turn these eighteen and nineteen year olds into little mathematicians. Instead, you are trying to *educate* them, to stretch their minds, to teach them to analyze and to think critically.

3.11. Do I Have to Teach Calculus Again?

Yes, you do. Especially if you teach at a large state school, chances are that you are going to spend a non-trivial amount of your time teaching calculus and statistics and linear algebra and sophomore ODEs (ordinary differential equations). It's all in the numbers. Just count the feet and noses and divide by three. Most students who take a math course—if they are not taking *pre*-calculus—take calculus or statistics or linear algebra or ODE's.

Of course you can, if you wish, treat this task as a dreary duty. And hardly anyone would blame you. But the harsh reality, at the dawn of our new century, is that your department's reputation around campus hinges on how well you and your colleagues teach these four magic courses. And the Dean will know what that reputation is and will act accordingly. There was a time when we could thumb our collective noses at the students and the administration and say, "We prove good theorems and we all have research grants. Go away and leave us alone." I'm afraid that, at most schools, this attitude will no longer fly. Administrators and parents and students—*and society*—have new expectations for the denizens of the universities. Like it or not, we live in the real world.

Doing a creditable job with lower-division teaching is not a lot of trouble. You may neither desire nor seek the reputation of "teaching hero," but everyone—including your colleagues but especially including your chair and your dean—will appreciate it if you pull your weight in the teaching game. And if you don't create trouble or cause extra work for others.

Teaching is really a team effort. While it is certainly true that each instructor is completely in charge of his/her one or two or three (or more!) classes per semester, it is also true that the aggregate of professors is responsible for the entire curriculum and for educating all the students. You are part of a faculty that is offering a curriculum of mathematics. It is a great help, and a boost to morale, to think of your teaching assignment as part of this group effort. If you do a good job, it will make your life easier. But it will also make your chair's and your colleagues' lives easier. I hope that reading this book will help you in that task.

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CHAPTER 4

The Electronic World

4.0. Chapter Overview

The Internet has had a massive impact on all aspects of our lives, and particularly on teaching. Thought of as a tool for communication, the Internet offers a vast variety of means for making course material available to students. It also presents a great panorama of methods for drilling students and testing them.

The Internet also offers ways to reach (and teach) many thousands of students at once. Sometimes at no cost. Many times with results that are difficult to measure.

The Internet also raises critical issues of academic integrity. How can we tell whose work is being submitted, and under what name? How can we measure what any individual student actually knows?

Thus, as with many aspects of life, the Internet is a sword that cuts in many different directions. In the present chapter we explore some of the byways and crossroads that we may encounter with the Internet.

4.1. Teaching with the Internet

The Internet is a marvelous tool for making information available to a large body of people quickly, and at no charge. It reaches people who do not have the means to go to school in the traditional manner, or who lack quality preparation, or who are burdened with a family or an outside job.

The Internet is a new teaching device in the strongest sense of the word. This chapter will explore the idea that the Internet can actually replace the traditional classroom with new insights, new experiences, and new information streams. But let us begin with very simple uses of the Internet in a teaching context.

Create a Web page for your class. Put the course syllabus on the Web page. You could have a page about prerequisites for the course, or ancillary reading, or ways to prepare for exams. Post homework assignments and due dates on your class Web page. Put information about upcoming exams there. If you need to write up a correction to something from class, or disseminate a list of errata to the text, or post homework solutions or exam solutions, then the Internet is just the ticket.

Some instructors go further, and type up their course notes for each class and put them on the class Web page. This is an awful lot of work, but it is a great resource for the students. And it will probably decrease the number of students who pester you with trivial questions at office hour.

You can use the Internet as the nerve center of your class, to keep everyone informed of up-to-the-minute information and last-minute changes, to post new homework assignments, to post grades, to change your office hours, to give last minute room or seating assignments for the upcoming exam, and so forth. The concept of fielding questions over the Internet, or with email, is a fascinating one. [The one obvious impediment is that most students don't know how to enter mathematics using the keyboard.][1] This certainly is more efficient than trying to remember to photocopy the information and bring it to class, it avoids the class time wasted when you distribute handouts, and it is more permanent (that is, the material can always be found right there on the Web for the duration of the term).

I believe that the full picture of the value of the Internet as a teaching device is yet to be determined. But I caution you against thinking that it can be a substitute for classroom learning.

4.2. Online Learning vs. Classroom Learning

I once read a proposal for an "Internet Mathematics Curriculum." The premise was that, at certain universities with a great many part-time and commuter students, absenteeism is a problem. Students have families and jobs and cannot always make it to class. In the electronic age, modes of communication are changing—so why not take advantage? The proposal was that the professor would still give his/her lectures, and those who could attend would do so. But there would be assigned note-takers who would post official notes on the Internet. The Internet could also be used to cut through the problem that students will not—or are too shy to—participate in class. The math class would have its own electronic bulletin board(s), and students could post their queries there—anonymously or not. Other students, or the professor or the TA, could answer the queries as they saw fit. Since many students have the same questions, this use of a bulletin board would allow the professor to use his/her time more efficiently. [As you will see below, many of these ideas have today been taken up by the `MOOC` movement.]

The proposal that I just described was not funded. In fact it didn't even make the first cut. But I think there is real merit to some of the ideas just described. I also, however, think that the concept of an Internet University abrogates much of what the learning process is all about. Classes are held for a good *reason*, and it is this: Many things that we do in life have a ceremonial/substantive aspect. We hold funerals to come to grips with someone's passing, and to create a sense of closure; we have graduation ceremonies to pause to think about an important moment in a young person's life; we select people for prizes (the Nobel Prize, the Cannes Film Festival Award, etc.) in part to recognize talented individuals and in part to ponder the human condition and what we are trying to achieve. Just so, we hold classes so that the students will take an hour, go to a special place, sit in a controlled environment, and think in a focused manner about a particular subject under the guidance of an expert. Three thousand years of experience with this technique suggests that it may be effective. If this were not as important as you and I know it to be, we would not do it.

At the risk of being repetitious, let me explicitly note these reasons why the traditional, physical classroom is important:

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[1]The software product *NetTutor* by Link-Systems is designed to cut through this problem. It presents the student with a white board on which to write his/her query by hand. Or else the student can click on icons to pull down mathematical symbols. The student can submit a question anonymously or not. The professor can answer questions in real time or at his/her convenience—and he/she can do so publicly or privately. The professor also can, with little effort, create a database of frequently asked questions that he/she can allow the students to access.

- Going to class gets students out of their normal daily routine so that they can dedicate some quality time to the topic of study;
- Being physically present in a classroom separates students from their regular responsibilities, thus allowing them to focus their thinking on a specific topic with some depth;
- The classroom brings people together who likely would not otherwise meet;
- The classroom promotes bonding and networking among the students—they can meet their counterparts and establish personal and study relationships.

If you decide to conduct an Online class, you might consider using the software `D2L` (short for `Desire2Learn`). This gives a rather structured environment for creating a course on the Web. You divide the content up into separate HTML pages (called "topics" or "modules"). The students can download papers and homework assignments. Then upload their solutions to repositories (called "dropboxes"). There are facilities for discussion blogs, grade spreadsheets, and online chats. D2L is similar to other "learning management systems" like `Blackboard` or `Moodle`.

Internet classes, while they may have their place, eliminate what is powerful about attending a class. Glancing at prepared lecture notes for your calculus class on your computer screen is (for the student) a bit too casual, and too much like turning on the radio. The student attempting to learn in this manner could be interrupted by the telephone, the doorbell, a pot boiling over, a baby crying, or any number of other exigencies (again, this is why traditional classes are a good thing). A mature and disciplined person with suitable scholarly training might be able to learn successfully from an Internet class (see [**MCR**]). I'm not so sure about inexperienced eighteen-year-old students.

4.3. MOOCs

A `MOOC` is a "Massive Open Online Course." Typically a `MOOC` is offered at no cost, with open enrollment. Many `MOOC` courses have literally hundreds of thousands of students. The course is entirely electronic—all lectures, all coursework, all quizzes, all homework, and all exams are conducted Online. Homework and exams are graded electronically.

The first `MOOC`s emerged from the open educational resources (OER) movement. The term `MOOC` was coined in 2008 by Dave Cormier of the University of Prince Edward Island and Senior Research Fellow Bryan Alexander of the National Institute for Technology in Liberal Education. Their work was in response to a course called "Connectivism and Connective Knowledge" (also known as CCK08). CCK08 was led by George Siemens of Athabasca University and Stephen Downes of the National Research Council. The enrollment consisted of 25 tuition-paying students in Extended Education at the University of Manitoba, as well as over 2200 online students from the general public who paid nothing.

There are many hundreds of `MOOC`s out there for you to sample. And many of them are created by quite distinguished scholars—Nobel Laureates, Professors at the Wharton School, Nevanlinna Prize winners at Stanford, and others. Students flock to `MOOC`s because they have an opportunity to sit at the feet of the world's great scholars—at no expense—and to be exposed to classical ideas as well as to learn current skills and technologies.

The MOOC movement, which is extensive, is part of a broader endeavor called the "Open Education Resources Movement." Open educational resources are freely accessible, openly licensed documents and media that are useful for teaching, learning, and assessing as well as for research purposes. Although some people consider the use of an open file format to be an essential characteristic of OER, this is not a universally acknowledged requirement.

The development and promotion of open educational resources is often motivated by a desire to curb the commodification of knowledge and provide an alternate or enhanced educational paradigm.

The idea of open educational resources (OER) has numerous working definitions. The term was first coined at UNESCO's 2002 Forum on Open Courseware and designates teaching, learning and research materials in any medium, digital or otherwise, that reside in the public domain or have been released under an open license that permits no-cost access, use, adaptation and redistribution by others with no or limited restrictions. Open licensing is built within the existing framework of intellectual property rights as defined by relevant international conventions and respects the authorship of the work. Often cited is the William and Flora Hewlett Foundation term which defines OER as: "teaching, learning, and research resources that reside in the public domain or have been released under an intellectual property license that permits their free use and re-purposing by others. Open educational resources include full courses, course materials, modules, textbooks, streaming videos, tests, software, and any other tools, materials, or techniques used to support access to knowledge." The Organization for Economic Co-operation and Development (OECD) defines OER as: "digitised materials offered freely and openly for educators, students, and self-learners to use and reuse for teaching, learning, and research. OER includes learning content, software tools to develop, use, and distribute content, and implementation resources such as open licences."

Now let us return to MOOCs. People criticize MOOCs because

- The vast majority of students who sign up for a MOOC typically do not attempt even the first homework assignment.
- The rate of completion of MOOCs is relatively low.
- It is difficult to measure the success of a MOOC—with regard to how much learning has taken place, and by whom.

A recent MOOC offered by Coursera[2] reports that 130,000 students enrolled, 15,000 completed at least one exam, and 9,000 completed the course. That is an astonishing attrition rate, and causes one to wonder what is going on.

To be fair, MOOCs are set up so that the student must enroll before he/she can see what the course has to offer. So a great many students will decide on that basis not to continue.

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[2]Coursera is a for-profit educational technology company founded by computer science professors Andrew Ng and Daphne Koller from Stanford University. It offers massive open online courses. Coursera works with universities to make some of their courses available online, and offers courses in physics, engineering, humanities, medicine, biology, social sciences, mathematics, business, computer science, and other subjects. Coursera has an official mobile app for iPhone and Android. As of April 2014, Coursera has 7.1 million users in 641 courses from 108 institutions. All Coursera courses are available for free. Other companies that develop and promote MOOCs are edX and Udacity. Major universities are involved in all these projects.

There are many advantages of MOOCs, especially for more mature students who have families and jobs. For one, you can tune into your MOOC any time and any place that you have an Internet connection. Secondly, you can go at your own pace. Third, you can ask questions and have them answered—through various user forums—by the other students, by TAs, and sometimes by professors. Fourth, on completing the MOOC course you are awarded with a very attractive certificate.

The Internet has been with us now for nearly twenty years. It was not long, after Mosaic and Netscape and other Internet pioneers were put in place, that Online learning systems evolved. In fact, for a few years, the pages of the *Chronicle of Higher Education* were packed with ads for various Online learning systems.

But, at some point, some of the big corporations in America announced that they did not want to hire people educated on the Internet. They wanted folks educated in the traditional fashion. And that was the end of that. The ads in the *Chronicle* disappeared, and things went back to normal.

But now that, in the form of MOOCs, Online education has taken on a new, more grandiose form, people are ready to embrace it again. And damn the consequences. You can search the Web and find articles about MOOCs. Two particularly incisive articles are [**MCR**] and [**GHR**]. They all discuss how useful a MOOC is for reaching out to a large audience, and what a feel-good device it is. But few of the articles discuss how to assess what is being learned. And *absolutely none* of the article considers whether graduate schools want to admit students who are educated with MOOCs, or whether the big employers (Microsoft, General Motors, Coca Cola) want to hire MOOC-educated students.

On the positive side, MOOCs can present the material in new ways. Some professors create cartoon characters who ask incisive questions (on behalf of the students). Sometimes these characters inject levity into the presentation, thus turning an otherwise dry lesson into a more entertaining one. Another potentially positive aspect of MOOCs is that the students themselves are encouraged to go to the class bulletin board and answer questions posed by other students. This is clearly quite different from what goes on in a traditional classroom. Even in a class of 100, I (the professor) can manage to answer virtually all student questions. But I could not dream of doing so for a class of 100,000. Obviously I would need some help. And the sociology of the construct makes it possible for others to play a role in this process. Students who enroll in MOOCs are encouraged to view the lessons together, to formulate questions and have discussions, and to work the ideas out among themselves.

Everyone seems to agree that the big plus of a MOOC is that it extends quality educational knowledge to a very broad audience, and for free. There are no prerequisites, so a MOOC is available to anyone. While production values of MOOCs are often low, one can imagine that this will improve when MOOCs become more accepted and more prevalent.

As with all new learning devices, I think that we should be aware of MOOCs and should consider, cautiously, learning how to use them. But we should also endeavor to assess how effective the MOOC learning experience is, and whether it is a valid substitute for traditional classroom learning.

4.4. The Khan Academy

The Khan Academy is a non-profit educational Web site created in 2006 by Salman Khan. His goal is to provide "a free, world-class education for anyone, anywhere."

The Khan Academy has enjoyed notable success. It claims to reach 10 million students per month, and that it has delivered 300 million lessons in total.

The Khan Web site showcases thousands of educational resources, including a personal learning dashboard, over 100,000 exercises, and over 5,000 micro lectures. Many admirers of Khan note that it is not a full-fledged curriculum. Attending the Khan Academy is not the same as attending Harvard University. But it is a valuable supplement to a standard curriculum.

The Khan Academy is funded by donations from the Bill & Melinda Gates Foundation and other charitable organizations. Bill Gates is an admirer of Khan, and claims that he uses Khan with his own kids.

Khan consciously avoids the format of having a person standing in front of a whiteboard. Instead it presents concepts as though they are "popping out of the universe" and into the user's mind.

The Khan Academy provides a Web-based exercise system that generates problems for students based on their skill level and performance. The Khan software, according to Salman Khan, can create tests, grade assignments, highlight challenges for certain students, and encourage students who are doing well to help struggling classmates. One of the attractive features of Khan, as has also been noted for `MOOC`s, is that one can pause a Khan tutorial in order to do some sideline thinking or investigating. Obviously this is not possible with a traditional lecture.

On the negative side, Khan himself is not an educator, and he is not a mathematician. In fact most of the Khan materials are created by computer scientists. This feature tends to make math faculty nervous.

Most of the Khan materials are at the high school level. They have ambitions to cover the college curriculum as well, but they are not there yet.

The TED presentation by Salman Khan, at the Web site

`https://www.youtube.com/watch?v=gM95HHI4gLk` ,

is quite compelling. He makes a case that the Khan videos can be used as the basis for "flipped classrooms," which we discuss in the next section. This is a profound rethinking of how classroom learning should work, and is well worth considering. Whereas many traditionally trained faculty (like myself) are uncomfortable with the ideas and techniques of teaching reform, the flipped classroom concept is one that almost anyone could embrace and use effectively.

4.5. The Flipped Classroom

One of the more innovative math instructors today is Ole Hald at U. C. Berkeley. His idea is simple but effective. It goes like this.

A typical math instructor will walk into the room, explain how the chain rule works, and then do five examples. And that is the lesson for today. This is pretty much how I teach, and how most of my colleagues teach. The trouble with that tried and true method is that the kids go home thinking that everything is crystal clear; but when they sit down a couple of days later to do it themselves they find that they cannot. They have lost the thread.

What Hald does instead is that he does a couple of examples, and then he has the kids do an example right there on the spot. In fact he has an army of assistants who hand out quiz sheets to the students, and they get right to work. Then Hald explains how the example should be done. Next, he does one more example. And he closes the class with the students doing another example.

What is the advantage of Ole Hald's technique? It is simply this: He *empowers* the students. When they walk out of class, they *know* that they have in hand a new technique. One that they can *actually do.* My dean once enjoined me grimly (when I was chair of my department) that we should teach our classes so that students leave the room excited by the ideas, talking among themselves about the wonderful new thing that they have just learned. Ole Hald's methods make this injunction a reality.

The flipped classroom takes this idea a step further. The idea here is that students view Online videos (such as the Khan Academy videos—see the preceding section) at home. This is their exposure to lectures. This is where they see what Salman Khan calls the "one size fits all" presentation of the material. But the advantage of doing it this way is that they can do it at their own pace. They can repeat passages. They can sit with friends while watching and ask questions and have discussions.

To put this matter in perspective: some instructors find the Khan videos too lightweight to serve the purpose described in the last paragraph. In their version of the flipped classroom, they create their *own* videos for the students to view. Of course this is a lot more work for the teacher, but it also gives that teacher a lot more control over the learning process.

Then, and this is the main point, when students go to class they do *not* have lectures. Instead they do problems and ask questions and interact with each other and with the instructor. And they get immediate feedback on their work. In short, they are participating in a learning system that empowers them, and makes them confident that they are actually learning something.

Eric Mazur of Harvard created several years ago what has become known as the *Harvard method of teaching physics.* With his technique, the teacher puts a pithy question up on the overhead projector screen. A sample question (sometimes attributed to Richard Feynman) is

> A spinning garden sprinkler is attached to the end of a hose. The sprinkler is placed at the bottom of a swimming pool. Then a pump is attached to the hose and water is pumped *out* of the pool through the sprinkler. In which direction does the sprinkler spin?

Then he tells the students to talk to their neighbors on the left and on the right, and to come up with an answer. Finally, the teacher calls on some students to present their answer, and he uses that to generate a discussion.

It would be easy to criticize Mazur's method as not using time well. One could also argue that mathematics does not lend itself well to pithy questions of the sort that we discussed in the last paragraph. But the upside of the Mazur method is that it helps the students to feel that they are learning, and that they are internalizing the ideas. Again, it empowers them.

Certainly Mazur's technique is very much in the spirit of the flipped classroom. Accompanied by Khan Academy videos that the student views at home, it can be a powerful method for teaching basic science.

I will conclude this section with a note of caution. The idea of the flipped classroom is attractive—*if* it really works. But consider these caveats:

- What do you do about students who don't bother viewing the videos beforehand, but who simply come to class hoping to get help with the homework?
- How do you manage the group interactions in the classroom? Will the smart kid who has carefully viewed the videos and mastered the homework problems dominate all the discussions? Will he/she answer all the questions?
- How do you manage students of varying abilities?
- Will the really bright students be turned off by the the fact of having to "carry" the weaker students?
- What do you do about students who have nothing to contribute during the classroom discussions?

There is real technique, and considerable patience, needed in order to manage student groups properly. In a group of five students, it is almost always the case that one or two will emerge as the leaders. What you do *not* want is for the leaders to do all the work and the others to vegetate. You must develop techniques for drawing out the *other* students, and getting them to carry some of the weight. When it is time for a group to give a report on its work, do not have one of the leaders give the report. One of the quieter students should do it. Every student in the group should be required to prepare a written report on the work. And, since students have almost no experience writing such reports, you should criticize the first draft of the report and then demand a rewrite.

A good department that believes in group work can arrange for training of faculty that want to use that didactic technique. And, in turn, the instructor should spend some time telling his/her students how group work should be implemented. It can be a rewarding experience if it is done right.

4.6. Computer Labs

Computers can calculate quickly. They can draw beautiful graphs. They can do simulations and numerical experimentation. They can answer "what if" queries. But they cannot think. They cannot answer questions. They cannot get a sense of what the students are learning and what not. They cannot engage in meaningful interaction. So, when we think about computer labs, we should keep these points in mind.

Math departments today have computer labs for several reasons:

- Adminstrators understand physical facilities. Traditionally they have had trouble funding the math department because they didn't know what they were paying for. But paying for a computer lab makes sense.
- All the other sciences have labs, so why shouldn't we have labs?
- We seek some tactile activity that will help the students learn. The lab seems to play this role.
- Going to the computer lab once or twice per week gives the students some discipline and some regimented activity.

Many mathematics departments now have a computer lab for instructional purposes. A number of elementary courses have custom-designed computer activities to augment the material that is presented in class. What might this consist of?

For any class, one could create `Mathematica` notebooks. These are self-contained, interactive pieces of software that require no knowledge of syntax or of `Mathematica`. The notebook presents ideas, asks questions, and evaluates the answers. Thus one could use the `Mathematica` notebook to drill students in basic concepts, to present animations of key geometric ideas, to walk the student through how a Riemann sum works, to illustrate the righthand rule (for the cross product), to do tricky graphing in three dimensions, and so forth.

In my own department we have used custom-made software to teach the students how to find numerical solutions to ordinary differential equations. Obviously the computer is a terrific tool for doing phase plane analysis, implementing the Runge-Kutta method, implementing Newton's method, comparing Euler's method with the improved Euler method, and so forth.

For several years in my department we had a big grant with the biology department. We used our funds to set up a calculus lab that involved the students actually doing a physical experiment (such as analyzing a bouncing ball) and then doing a computer analysis of the experiment. This was a useful device for really getting the students involved with the material, and for helping them to mentally internalize the ideas. And the faculty found that the lab activities really augmented their lectures effectively. Further details about these labs may be found in [**BKT**].

Today, in the technical world, it is fashionable to speak of "data scientists." These are people who sit on the cusp of mathematics, computer science, and domain knowledge. People who work at some of the national labs, like the National Security Agency, think of themselves in these terms. How does one train to be a data scientist? Computer labs could be one way to go. But these would have to be computer labs in which the students really had to do something proactive—like write in code, or develop a mathematical model, or design a computer graphic.

It is a fact that, for many math departments today, the computer lab is a part of life. Every TA and every faculty member is assigned certain hours each week in the computer lab. The computer lab is used to augment every lower division course, and many upper division courses as well. This is not so different from chemists having a lab or biologists having a lab. We just need some time to get used to it.

Having said all this, it may be noted that the use of computers in teaching blossomed in the 1990s and then declined in the early twenty-first century. See [**BRE2**] for some of the details. The reason is in part because sustaining a computer lab is labor-intensive and expensive, and the results were not always satisfying.

4.7. Clickers

Clickers are an interactive technology that lets the instructor pose questions to students and then immediately collect and view the responses of the entire class. A typical application of clicker technology is this:

(a) The instructor presents a multiple-choice question to the class.
(b) Students click in their answers using remote transmitters called *clickers* (which are available at the campus bookstore).

(c) The system instantly collects and tabulates the results, which the instructor can view, save, and display (anonymously) for the entire class to see.

Examples of how clickers have been used by instructors are as follows:

(1) A physics instructor checks student comprehension of the material being presented at several points in the lecture by having students click in answers to set questions. The system immediately displays a graphic representation of the student answers, and the instructor can then determine whether he should slow down, repeat, clarify, provide an alternative example, or pick up the pace.

(2) An engineering instructor delivers a mini-lecture and then poses a conceptual question for clicker response. She can quickly scan the resulting data display. If she observes that a number of students are answering incorrectly, then she asks them to discuss their answers with neighboring classmates. Students then have the chance to modify their answers based on their discussions. After students have clicked in their final answers, then she discusses the results with the class.

(3) A philosophy instructor has redesigned his course around what is called "question-driven instruction." He begins the class by posing a meaty philosophical question. Then he breaks the students up into small groups to discuss the question. Then he asks each group to click in its answer. Afterward, the entire class discusses the result. The instructor follows all this with a short lecture.

(4) In a `YouTube` video from McGill University, a professor poses to his class the question

> You want to put three distinct balls into two indistinguishable boxes. In how many different ways can you do it?

He has the students discuss the matter with their neighbors. Then, at the propitious moment, they submit their answers using clickers. He assesses how many students got it right, offers some guidance as to how to get closer to the right answer, then gives students a little time to re-figure and then resubmit their answers. Then he sums everything up and calls it a lesson.

It is easy to see that many math instructors would claim that this approach to teaching does not mesh well with an idea-driven subject like mathematics. And it is also an inefficient use of time. Traditionally, mathematics is a discipline that has been practiced by fanatics of the single-combat-warrior genre. It is not in our nature to want to poll the class to see what they think of what has been going on. The subject speaks for itself.

I have been to cross-disciplinary educational conferences where clickers were heavily promoted. My informal impression is that they are a more natural fit for subjects other than mathematics. There is evidence that instructors like clickers more than students. But this is a new tool that we can learn to use as best we see fit.

4.8. Homework Solutions on the Internet

One major concern of high school and college teachers these days is that students can find solutions to homework problems—especially problems taken from the most popular textbooks—on the Internet.

To make matters worse, there are resources available such as Wolfram `Alpha` that can, at least in principle, solve virtually any homework problem on the spot. I can tell you that all of my students know about Wolfram `Alpha`, and they use it.

And in fact computers present an even more profound imbroglio. There are now available—*for free* on the Internet!—computer systems that can solve most elementary calculus and statistics and linear algebra problems in real time—right on the spot!! [There are also commercial products that one must actually purchase, such as `Mathematica`,®`Maple`,®and `MatLab`.®Some of these are rather pricey, and they all have specialized syntaxes that one must learn.] Three free systems of note are `WolframAlpha`,®`Sage`,®and `Maxima`.®Let us discuss the first of these freeware products.

Go to the URL

`www.wolframalpha.com/`

You are greeted by a window asking you to type in your computational request. Remember that `WolframAlpha` has no specialized syntax. If you type in

`Integrate x^3 - x`

then the machine almost instantly spits out the answer

$$\int (x^3 - x)\, dx = \frac{x^4}{4} - \frac{x^2}{2} + \text{constant}\,.$$

But there is more! The screen also displays a graph of the function. It calculates alternative forms of the integral. And it calculates some definite integrals for you too, and relates them to area.

But it gets worse. If you pay $65 per year to subscribe to Wolfram `Alpha` Pro, then you get the following benefit. In the upper righthand corner of the display screen is a button labeled <`show steps`>. If you click on that button, you will see that `WolframAlpha` shows every step of the calculation in horrendous detail. [The free version of Wolfram `Alpha` gives a pared-down version of "show steps."] The software might be of limited utility if all it did was supply the answer. But, since it provides a complete solution, the student need look no further.

One might argue that Wolfram `Alpha` is a handy device that enables students to get homework help, or to check their answers, at 1:00am when other help is not available. Unfortunately it is also a handy device to enable students to dodge the learning game.

Now let us challenge `WolframAlpha` by asking it to calculate

$$\int e^{-x^2}\, dx\,.$$

We all know that the antiderivative of this integrand cannot be written down in closed form (see or [**ROS**] for a complete analysis of how to tell when an integral does or does not have this property). But `WolframAlpha` is no fool. It has to take a few moments this time to think things over, and then it yields the answer

$$\int e^{-x^2}\, dx = \left\{ \frac{1}{2}\sqrt{\pi}\,\mathrm{erf}(x) \right\}\,.$$

One may argue that this is an unsatisfactory answer, since erf is *defined* in terms of the integral we are trying to evaluate. But we gave `WolframAlpha` an essentially impossible task and it came up with a correct answer. A user can find a book of tables and look up values of erf. Or else he/she could go to the remarkable

NIST Digital Library of Mathematical Functions (DLMF) (`dlmf.nist.gov/`) for the needed numerical information.

In principle the student should be accustomed to the idea that some functions cannot be calculated by hand—numerical techniques, or at least a hand calculator, are needed to get reasonably accurate values. The functions sine, cosine, logarithm, and exponential are common examples of this phenomenon. Perhaps erf should be added to that collection. But erf will be much less familiar to the student.

Let us try to twist `WolframAlpha`'s arm to see whether we can get some more concrete information about this nasty integral. Type in

```
integrate e^{-x^2} from - 1 to + 1
```

Now `WolframAlpha` thinks for a while and ultimately gives the answer

$$\left\{\sqrt{\pi}\,\mathrm{erf}(1)\right\} \approx 1.49365\,.$$

You can see the nasty erf function rearing its ugly head again. It is of coures helpful to have an approximate numerical answer. If you click the "more digits" button then Wolfram `Alpha` will give you the answer to 100 digits.

It is clear that, to make the best possible use of `WolframAlpha`, one needs some mathematical maturity (see [**KRA2**]). A student probably would not know what to think if he/she were confronted with an answer involving erf. I note also that, when integrating `e^{-x^2}`, `WolframAlpha` does not offer a <show steps> button.

The Online software packages `Sage` and `Maxima` also offer mathematical calculation environments similar to `WolframAlpha`. `Maxima` is particularly interesting because it is a direct descendant of the early computer algebra system `MACSYMA` that was developed at M.I.T. in the 1960s. We shall not provide any further details here.

There are yet other Online systems that endeavor to provide answers to questions. `Yahoo Answers`, for example, is pretty good at math, but not nearly as smart or as accurate as Wolfram `Alpha`.

Clearly the math instructor who is aware of the issues raised here must give the matter some careful thought. How do you give proper credit to students who do the homework on their own (without the aid of the InterNet), and how do you reign in the students who endeavor to cheat? Can you still give takehome tests? An obvious conclusion to draw is that the only fair measure of what students have learned is a traditional, written, in-class exam in which each student does his/her own work *alone*. Without the aid of a calculator or computer. But there may be other answers.

4.9. Online Software

A big industry these days is Online software that performs some or all of the following functions:

(a) Generates problem sets/homework assignments.
(b) Generates exams.
(c) Grades problem sets.
(d) Grades exams.
(e) Provides Online tutorials.
(f) Tracks student progress in the course material.

There are many products that address some or all of these issues. Some of them, like `WeBWorK` from the University of Rochester and `WebAssign` from North

Carolina State University are free or nearly free. Others, like `ALEKS` from McGraw-Hill and `WileyPLUS` from John Wiley & Sons and `CengageBRAIN` from Cengage and `MathXL` from Pearson are commercial products from publishing houses. In order to access and use the first two mentioned products, you just go on the Web (either to `webwork.maa.org` or `www.webassign.net`) and sign up. For the products from commercial publishers, you must be using one of their textbooks. In some arrangements, when the school has a big adoption, then the publisher will provide the Online product for free. In other arrangements there is a charge per student. The URLs for these products are `www.aleks.com`, `www.wileyplus.com`, `www.cengagebrain.com`, `www.mathxl.com`.

`ALEKS`, which professes to be developed from artificial intelligence research at the University of California at Irvine, *guarantees* that a student's grade will go up one notch if he/she uses `ALEKS`. It is difficult to discern how the guarantor can say what the student's grade would have been without `ALEKS`. But there you have it. One of the things that `ALEKS` is famous for is its branching processes (no doubt a product of that AI research). If a student gets a problem wrong, then `ALEKS` points out the error and suggests an auxiliary problem that the student should try. It may also point out a passage in the text that the student should read.

Of course you should not misunderstand what these products can and cannot do. You simply *cannot* write an exam in TEX, print it out on a piece of paper, wave it in the air in front of your computer, and expect `WebAssign` to grade it. You instead have to write the exam *in* `WebAssign`, using the syntax of `WebAssign`, and the answers must be entered in `WebAssign`, in order for the software to be able to grade it.

We use `WeBWorK` at my university. It is a product that was created and developed by the Mathematics Department at the University of Rochester after the Dean there shut down their graduate program. In an effort to show that they were good citizens of the university, they put in the time and resources to develop this software. We use `WeBWorK` in large courses, where there are not resources available to hand-grade homework. Students like and appreciate `WeBWorK`, because it gives them immediate feedback on their work, and a true sense of how they are doing in the course. In some courses `WeBWorK` counts towards the grade; in others it is used only to provide the students extra practice.

We must bear in mind that the products described here are only tools to satisfy a short-term goal (namely, to get homework graded). Using `WeBWorK` does *not* teach students how to think about mathematics, and it certainly does not teach them how to write mathematics. That is another set of issues, to be dealt with elsewhere.

4.10. The Course Web Page

It is quite common these days for each math course to have its own Web page. I start each course with a Web page that displays the course syllabus (in fact I no longer distribute hard copies of the syllabus in class). As the course progresses, I add to the Web page. These additions include

- Homework assignments.
- Solutions to homework assignments.
- Announcements of exams.
- Solutions to exams.
- Other course information.

For the solutions, I write out the solutions by hand on $8'' \times 11.5''$ paper, scan it in, and post the *.pdf file. Those among us who are more energetic might like to write out the solutions in TEX, generate a `*.pdf` file, and post *that* file.

A particularly good course Web page (for calculus) appears at

`www.math.wustl.edu/~freiwald/131syllf10.html`

One sees here that, in addition to some of the obvious items described in Section 2.11, the instructor includes

- A discussion of background needed for the course;
- A separate link for homework assignments;
- A link to a study group page;
- A link for the discussion sections;
- A link regarding academic integrity;
- A link entitled *Advice for Success*;
- A link with suggestions for doing the homework;
- A link that allows the student to give anonymous feedback to the TA;
- A link on the use of calculators in class;
- A link to old exams;
- A link about the math major;
- A link about the undergraduate math program;
- A link about taking calculus in summer school.

and there is much more.

I also like to take advantage of the fact that our system is set up so that I can send an email to the entire class (without having to create my own alias). I can use that for announcements, and also to offer any small emendations to the recent lecture that may be necessary. I have always found it awkward and counterproductive to distribute paper corrections to my lectures (one hopes that there will not be too many of these). I find it much easier and more comfortable to send around emails with corrections.

Our Undergraduate Director requests that we leave our course Web page up long after the course is complete. This is a good idea, as it leaves a "paper trail" for the course, and is a good reference.

It is easy to imagine people who are more adept at the Web posting additional material for the course, extra problems, even animations that illustrate the course ideas. The course Web page is a powerful and convenient teaching tool, and I encourage you to develop the habit. Just be careful of copyright. You should not post portions of books without permission from the copyright holder.

4.11. Social Media

`FaceBook` is an Online social networking utility that allows people to keep in touch, make new contacts, share events in their lives, share photos, and do other types of personal bonding. It has one billion users. `FaceBook` adds half a petabyte of data to its Web site every day. It has been immensely successful. Other social media utilities are `Twitter` and `Instagram`.

It actually occurred a few years ago that a mathematician used `FaceBook` to prove a theorem. How can this be? One of the interesting features of `Facebook` is that it makes it very easy to contact, or communicate with, a collection of friends. This mathematician was stuck on a step in his proof, so he used `FaceBook` to put a message on his FB wall to ask for a hint. Within a day he got the hint he needed and his problem was solved.

An issue that comes up with teaching today is if and how to use `FaceBook` with your class. In `FaceBook` you arrange for certain people to be your "Friends," and these are, in effect, your contacts. Should you make some of your students your Friends? Or all of your students your Friends? If you do, how should you make use of FaceBook to augment your teaching?

My view is that it is a mistake to make only some of your students your Friends. That could be construed as favoritism. I also think that, as a professor, you must make some effort to maintain both the fact and appearance of objectivity. Having some or all of your students as Friends on `FaceBook` just smacks too much like meeting your students for a drink in a bar—something you should never do. People routinely use `FaceBook` to exhibit and discuss rather personal features of their lives. This is not necessarily something that you want to share with your students.[3]

Some teachers are great adherents of `FaceBook` and use it routinely in their classes. Some of the attractive applications are

- Creating a Timeline or `FaceBook` Group to support the teaching of any curriculum subject;
- Creating a space and platform for homework and revision resources;
- Peer tutoring and support;

The book `FaceBook` *Guide for Educators* at the URL

`www.ednfoundation.org/wp-content/uploads/Facebookguideforeducators.pdf`

contains a number of suggestions for how to use `FaceBook` as a teaching tool.

An interesting and not terribly well known social networking utility is `Piazza`. It differs from some of the others because the user gets to control who has access to his/her site. Also one can use TEX commands for mathematics. Some users can be identified as instructors and have corresponding privileges. Students can choose from three levels of anonymity. A `Wiki`-style format encourages collaboration. You can explore this tool at `https://piazza.com`.

My advice is to proceed with caution as you explore the idea of using social media with your class. The class Web page and the class email utility should be all the means you need to maintain contact with your class, and to keep the lines of communication open. My advice is to stay away from `FaceBook` and `Twitter` and `Instagram`.

4.12. SmartBoards

The SmartBoard is an interactive whiteboard produced by the company Smart Technologies. The Smart Technologies Company was founded in 1987, but began producing SmartBoards several years later.

The SmartBoard allows the user to prepare a lecture at home on the computer, bring it to the lecture room on a flash drive, and then plug it into the SmartBoard to get a high-tech presentation. The screens scroll and display very much like `PowerPoint`.[4]

[3]You could, at least in principle, have a personal `FaceBook` page and a professional `FaceBook` page. And of course you would restrict your professional page to strictly business matters. Although you could put fun things, like cartoons and math news stories, on the page.

[4]Of course one can do something similar with a notebook computer and a projector.

Conversely, one can write on the SmartBoard with a marker and record the lecture as a binary file on a flash drive. Thus it is saved for future use.

The SmartBoard is very smart. It responds to the special pens on the whiteboard, but it also responds to the touch of a finger or other solid object, such as a pointer or stylus.

SmartBoard has its own proprietary sofware that works with projectors and digital cameras to make the SmartBoard a very versatile and powerful technology.

The user of the SmartBoard can copy, cut, paste, and otherwise manipulate portions of the presentation—very much as he/she would do on a computer screen.

SmartBoard interacts with standard Microsoft software such as PowerPoint, Excel, Word, and AutoCAD.

It is clear to me that preparing a SmartBoard presentation may take more time than preparing a traditional chalkboard presentation. But, once you have the presentation fixed up, then you have it for repeated future use. So it may very well represent time well spent. My personal preference is for a traditional chalkboard, because I can use it skillfully to make the mathematics come alive for the students: They actually get to see *me* doing mathematics step-by-step, just as they will be doing it. It is worth thinking about the merits of the traditional approach.

4.13. Reference URLs

We use this concluding section of the "electronic" chapter to collect useful Web sites that have been referenced and described in the text.

- The Appendices, written by ten different mathematicians, to the second edition of this book:

  `www.math.wustl.edu/~sk/teachapps.pdf`

- The Web site `Rate My Professors`, which provides evaluations of faculty teaching at all colleges and universities in the country:

  `www.ratemyprofessors.com`

- Some particularly nice class Web pages by Ron Freiwald:

  `www.math.wustl.edu/~freiwald/131syllf10.html`

  `www.math.wustl.edu/~freiwald/m309f11.html`

  `www.math.wustl.edu/~freiwald/4171syllF13.html`

- A Web site about calculus reform resources:

  `www.math.okstate.edu/archives/calcrefm.html`

- A Web site regarding CLUME (Cooperative Learning in Undergraduate Mathematics Education), which has materials to help faculty to become aware of, and to learn how to use, new pedagogical techniques:
  `www.maa.org/publications/maa-reviews/cooperative-learning`
  `-in-undergraduate-mathematics-issues-that-matter-and-strategies`
  `-that-work`

- A video by Salman Khan about the Khan Academy:

  `https://www.youtube.com/watch?v=gM95HHI4gLk`

- The URL for the calculational software Wolfram Alpha:

  `www.wolframalpha.com`

- The reference work on functions and tables published by the National Institute of Standards and Technology:
  dlmf.nist.gov
- The Web site for the Online homework utility WeBWorK:
  webwork.maa.org
- The Web site for the Online homework utility WebAssign:
  www.webassign.net
- The Web site for the McGraw-Hill math learning utility Aleks:
  www.aleks.com
- The Web site for the Wiley math learning utility WileyPLUS:
  www.wileyplus.com
- The Web site for the Cengage math learning utility CengageBRAIN:
  www.cengagebrain.com
- The Web site for the Pearson math learning utility MathXL:
  www.mathxl.com
- A discussion of uses of FaceBook in teaching:
  www.ednfoundation.org/wp-content/uploads/
  Facebookguideforeducationrs.pdf
- A social networking utility called Piazza is specifically designed for teaching mathematics:
  https://piazza.com
- A definition of and discussion of plagiarism:
  en.wikipedia.org/wiki/Plagiarism

http://dx.doi.org/10.1090/mbk/089/05

CHAPTER 5

Difficult Matters

5.0. Chapter Overview

Any activity that involves people interacting with people is going to generate occasional friction. In teaching, this friction could result from student cheating, or from late work, or from excessive mistakes in the lecture, or from ill behavior (by the student *or* the teacher). An experienced instructor knows how to dispatch these situations both decisively and efficiently. The inexperienced instructor is liable to bungle the matter and make the situation worse. Such errors can annoy everyone, and make the instructor's life unnecessarily miserable.

This chapter offers advice on how to handle many of the most common "problem situations" that arise in teaching.

5.1. Non-Native English Speakers

Many new instructors in this country are not native speakers of the English language. Besides worrying about all the usual stumbling blocks that dog a new teacher, such a person needs to worry about **(i)** lack of fluency with the language and **(ii)** lack of familiarity with classroom techniques in the United States.

If you are not a native speaker of English, then fluency is entirely your responsibility. Every university offers a short crash course on English as a second language. Take the course. Watch television and movies. Read books in English. Converse with American colleagues. One friend of mine, from Vietnam, perfected his English by listening every morning to the wartime speeches of Winston Churchill—and regaling us with quotations every day at lunch!

If you are a non-native trying to learn English, then don't seek out and spend all your free time exclusively with people of your own ethnic and national background. It is of course natural for us to seek the company of people like ourselves. We all should do it from time to time, if only to relax. But if you want to learn English then you must first learn some discipline, and you must force yourself to talk to native speakers. It is not enough to know the words and the syntax of the language. You must learn both to speak and to understand it in its natural speed and rhythm. If you are going to succeed as a teacher in this country, then you must speak the language well enough to **(a)** be understood and **(b)** be able to field questions. It is point **(b)** that causes more trouble than **(a)**. Most students can get used to a teacher who has less than perfect proficiency with the mother tongue, or who speaks with an accent, or both. But if you as instructor cannot understand their questions then you will be a complete failure in and out of the classroom.

Some students are prone to complain, and will use any excuse to justify a complaint. I speak and articulate accentless English. Yet, when I taught at Penn State, some students complained that I did not speak with the local accent. Some students complain about teachers with British accents and teachers with Australian accents. There is nothing that you can do about such complaints, so you should not worry about them.

If you are organized, if you speak up, if you treat students with the respect that *you* would desire from an instructor, and if you show some enthusiasm for what you are doing, then students will forgive a lot. Your foreign accent will fade into the background. Nobody will hear it any longer. And you will be a successful teacher.

Remember that there is a big difference between **(i)** speaking good English with an accent and **(ii)** not knowing how to speak English. Many people who fall into the second category rationalize their lack of ability by saying to themselves that they actually fall into the first category. This is a trap, and you should avoid it.

Many professors who received their training in other countries are (justifiably) impatient with our students. American students do not specialize as early in their education as do, say, European students. Even in a sophomore differential equations class in America, there is a broad cross section of students that includes pre-medical students and others from outside the mainstream of mathematical science.

I once taught a junior/senior level real analysis course. One student's primary interest was chemistry, but he was studying for an advanced degree in statistics, and this in turn required that he take real analysis. Fine. He was a bright and hard-working student, and I couldn't help but like him. One day I gave a rigorous definition of "continuous function" and he raised his hand and said, "That's not what I think of as a continuous function." A part of me was incredulous. But he was coming from a different world, and he had posed a serious comment that demanded a serious answer. I was really on the spot. I had to *defend* my definition. I certainly learned from this dialogue. And I have, as a result of this experience, become more open to such questions. I would encourage you to do the same.

My message is this: Learn to be patient. Students will ask you to repeat terms. Students will ask you "non-mathematical" questions. Students might seem less able, or less well prepared, than those in your country. But they are bright and they are willing. You must learn to work with them. *After* you have learned how the American education system works, and what the students are like, you will find that your colleagues are receptive to your thoughts about its shortcomings. Before you have made this acquaintance, you are working in a vacuum and you should keep your own counsel.

In some countries it is the style of the university professor to stand at a lectern in the front of the room and to read the textbook to the class. Questions are considered to be a rude Americanism. An extreme example of a teaching style that is virtually orthogonal to what we Americans know is one that has been attributed to the celebrated Hungarian analyst F. Riesz. He would come to class accompanied by an Assistant Professor and an Associate Professor. The Associate Professor would read Riesz's famous text aloud to the class. The Assistant Professor would write the words on the blackboard. Riesz would stand front and center with his hands clasped behind his back and nod sagely.

My point is that, in the United States, for better or for worse, we have our own way of doing things. The style here is to indulge in discourse with the class. Some professors make the discourse largely unilateral. That is, they lecture. Other professors encourage more interchange between the students and the teacher. Reading this book will help you to become acquainted with the traditional methods, and some of the newer methods, of teaching in this country.

5.2. Late Work

Late work is a nagging problem. The easiest solution to the "Can I hand just this one assignment in late?" dilemma is to "Just say 'no'." But what of the student who has a *really good excuse*? What if there has been a death in the family or some other crisis that the student cannot avoid?

The trouble with making one exception is that it tends to snowball at an exponential rate to N exceptions. In a large class this can be catastrophic. One possible solution is to tell the students that, when you calculate their cumulative homework grade, you will drop their two worst grades. That means that any student can miss one or two homework assignments with essentially no penalty. It's a remarkably simple solution to an otherwise difficult problem.

There are a number of other possible answers to the late homework problem. You can downgrade late assignments, or you can assign extra work. You can just forget the missing assignment and base the student's course grade on the remaining course work. The point is that you should think about this matter in advance, and formulate a policy that you will use consistently. A choice of incorrect policy toward late work could lead to a lot of extra effort and/or aggravation for you. Don't be afraid to ask a more experienced colleague for help in this matter.

5.3. Cheating

Cheating is a big, and probably unsolvable, problem. Academic dishonesty is demoralizing for the teacher and for the non-cheating students. Honest students react to cheating with emotions that range from outrage to pity to melancholy. What is the point of studying so hard if cheaters can get good grades through skulduggery? And the cheaters' inflated grades affect the grading curve, which in turn affects everyone. On the whole, cheating is a moral outrage—for both instructor and student alike.

You will find it difficult to deal with the sort of students who cheat, for they may be dishonest with themselves and with others in a number of aspects of their lives. You want to be firm and fair and just all at the same time. But you *must* deal with them, and you must do so directly and firmly.[1] As with late work and other difficulties, you must have a clear and consistent policy to apply to cheaters. It should be noted (see below) that the university may have already formulated such a policy for you.

You may wish to set a moral tone against cheating by making an announcement on the first day of class. For large lectures, this may be especially important. Declare that you consider cheating to be an egregious offense—against yourself, against the other members of the class, and against the university. While you admit to the class that you may not be able to catch all cheaters, you assure the students that anyone caught cheating will be punished to the full extent of the law—*including expulsion from the university when appropriate.*

[1] To be fair and compassionate, I must say that some forms of cheating are like venial sins and some are like mortal sins. The former should probably be dealt with lightly—at least for a first offense—while the latter should be dealt with more sternly.

Be forewarned: Most American universities have set policies about handling cheaters. You are not free to act as you please when you catch a miscreant. In particular, there are due process procedures set up (to protect the rights of the accused cheater) that you must follow if you wish to punish a cheater. You do not necessarily have the right to tear up the student's exam, to give the student an "F," or to mete out other retribution. Check with the Director of Undergraduate Studies in your department to determine the proper course of action when handling a suspected cheater.

At one Ivy League university, entering students are required to sign an oath that they will adhere to the university's Honor Code. Part of the Honor Code is that, at the start of any exam, the professor will record on the blackboard the statement "I pledge my honor that I have neither given nor received information during this exam." Each student is to copy the pledge verbatim onto his/her exam sheet and then sign it. The instructor is then required to leave the room for the entire duration of the exam. The professor may, if he/she wishes, return briefly in the middle of the exam to answer questions. The critical part of the Honor Code, that the student signs at the outset of his/her education, is that he/she pledges not to cheat and he/she also pledges to turn in any other student whom he/she observes cheating. Since the instructor must leave the room at exam time, we see that the entire onus of catching cheaters is placed on the students themselves!

An interesting policy, and one that would not work at every institution. A notable feature of making each student copy and sign the pledge on his/her exam is the following. If he/she is planning to cheat, then the university is forcing him/her to lie as well. Having served on university committees that adjudicate cheating, my experience with students is that they are disinclined to rat out their peers. Most people want to be told what to do most of the time, and the students whom I have known prefer there to be an authority figure who will identify and deal with cheaters. This means you, so you had better figure out how to do it.

The best defense against cheaters is offense. Give your exams in a large room. Space the students far apart. Check picture IDs to make sure that students have not sent in ringers (substitutes) to take the exam for them. Patrol the room. Avoid turning the exam into a power trip situation. Just maintain control.

Another aspect of cheating is plagiarism. Plagiarism is not as likely to arise in a mathematics class as in, say, a history class.[2] But you should be aware of what it is and how to deal with it. Plagiarism is the appropriation of another person's words or ideas. It is too large to treat in any detail here, but see [**MLA**] and the Web site

`en.wikipedia.org/wiki/Plagiarism`

One advantage from your point of view is that you do not have to handle plagiarism in real time. You have the plagiarist's work, together with the putative source material, in front of you. You may consider it carefully, show it to colleagues, ask your Undergraduate Director how to proceed. The best policy is not to attemp to act alone.

[2]Although you might catch a student copying from `Wikipedia`, for instance. Some students will solve their homework problems using Wolfram `Alpha` or the like. I don't know whether we want to label the latter as cheating or as something else.

One could easily write another book about techniques to catch cheaters. In some departments, exams are photocopied (or at least a sample of them is photocopied) before they are returned to students. This is to dissuade a student from altering a graded exam and then coming back to the instructor to request more points. Some departments (such as my own) use elaborate statistical procedures to detect unnatural correlations among students' answers on multiple choice exams. (A student caught by means of such a mathematical technique finds it quite difficult to defend himself/herself!) Many other devices are available.

The point is that it is worth spending a few moments thinking about how you will handle cheaters. There are many pitfalls to be avoided—in particular, you must respect the accused cheater's rights as specified in your university's code of conduct. There is nothing very pretty about a situation involving cheating. Just remember that you are not free to act on your own. Become acquainted with your university's procedures. Learn to be consistent and fair.

5.4. Incompletes

The profession of teaching, while certainly a stimulating and rewarding one, is littered with nasty little details. One of these is the "incomplete." The theoretical purpose of an incomplete is to provide a vehicle for handling certain problem situations. Perhaps a student *has* completed a substantial amount of material in the course, but has been ill or has suffered a death in the family or some other setback. He/she needs to defer completion of the course work until the next term. The professor fills out an "Incomplete Form," and records the student's term grade as an "I" or "Incomplete," to formalize the understanding that the student will complete the work at some pre-specified future time. Many universities find it convenient to let professors administer incompletes as they see fit. As a result, there is much inconsistency and abuse.

Frankly, I've given a lot of incompletes in my life and very few of them were ever completed. Students get busy with the next semester's work, and never get around to things past. In fact I did not complete the only incomplete that I ever took as a student. It is also unfortunately the case that certain students will simply blow off a course and then ask for an incomplete at the end of the term. Often it is easier for you as the instructor to just grant the incomplete, given that an otherwise undisciplined student is not likely to complete it (the grade then usually, but not always, reverts to an "F"). You may very well wonder what is the point of engaging in a long interview with such a student to determine whether the incomplete is merited.

All this having been said, it is probably best, as with all matters in teaching that impinge on fairness, to have a uniform policy for handling incompletes. But think this through. Are you going to require that the student provide *proof* of his/her excuse? This sounds reasonable, but what if the student says, "My mother is dying of cancer." or "My grandmother just died and I cannot concentrate on my work." I know professors who will demand a letter from the physician or the undertaker, but this strikes me as a bit extreme. It could also prove to be uncomfortable or embarrassing for all concerned.

One convenient way to handle the request for an incomplete is to instruct the student to approach a professor teaching the same course the following term. The student should ask whether he/she can audit the course, having his/her work graded. The new professor of course will *not* submit an official grade for this student (after all, the student is not registered in his/her class). Instead, he/she will transmit the resulting grade to you. You then fill out a form to remove the student's Incomplete grade and replace it with that letter grade. This is clean and simple, and it works. You certainly don't want to have to re-teach some or all of the course for the benefit of just one student.

You are the academic analogue of a middle management executive in the business world. Executives exist, presumably, because they are smart enough to handle exceptional circumstances. Teaching is loaded with all of the sorts of exceptions that are connected with dealing with *people*. I have used the "incomplete" here as but one example of the problems and potential enigmas that can arise. Your department probably has set policies, or at least guidelines, for handling incompletes. Become acquainted with the routine procedures before you give your first "I."

5.5. Discipline

One hopes that, in a college environment, discipline will not be a big problem. But there are difficulties that can arise.

In a big class, with two hundred or more people, student talking can get out of hand. Many students read the newspaper, or knit, or eat their lunches, or write letters to their friends. Some students, when the lights are turned low, engage in romantic activities. Students come in late and leave early. Students sleep.

I see no point in making a spectacle over a student who is not causing a disturbance. If a student is quietly eating lunch, then that is no problem for me. If a student comes in late or needs to leave early and does so in an orderly and non-disruptive fashion, then I let that student alone. (A truly courteous student will tell me in advance that he/she needs to leave early or come in late, and I am suitably appreciative. If you find this custom attractive, then tell your class that you want to be notified in advance of such temporal irregularities.) To make a scene will only alienate the whole class.

How should you handle a student who causes a disruption? First try something gentle like, "OK, let's quiet down." Another technique is to simply stop talking until you have everyone's attention. If one or two applications of "nice" is ineffectual, then come down on the offender quickly and sternly. Example: "Mr. Trump, if you want to talk then please leave the room." or "Ms. Lewinsky, everyone else is here to learn. Please keep quiet." If you deliver these injunctions firmly and with confidence, then they will have a chilling effect.

Often you can arrange for the other (non-offending) students to be the bad guys. If you simply sit down and wait for silence and cooperation, then the other students will "shush" the offenders. You can just provide a modicum of stern looks.

William James was both a father of twentieth-century psychology and also a renowned teacher. When he felt that his class was not cooperating—either inattentive, or talkative, or simply dull—he would fold his arms and lie down on the floor. After a while, the class would fall into puzzled silence. Then he would announce, "Education goes both ways. You have to participate too!" And he would try again. As you can imagine, a clever teacher could make something like this device into a productive game with his class. I have a special place on the wall of my classroom where I ceremonially bang my head when my students are being slow. You should create your own tricks—ones that suit your personality and your style.

On those absolutely rare occasions when a class is beyond control, you might throw down your chalk and say something like, “This is hopeless. I’m through for today. We’ll try again on Friday.” I have never used this last device, and I fervently hope that I shall never have to resort to it. But somehow it gives me strength to know that it is at my disposal. If you do take this extreme measure, you had better let your chair know what you have done.

I have seen large mathematics classes (of about 400 or more students) which looked like a cross between a rock concert and a Hieronymous Bosch painting (see also Section 2.13). Private conversations and mini-dramas were taking place all over the room while groups of students roamed the aisles. The professor stood at the front of the room, bellowing away on his/her microphone, while a small percentage of the students attempted to learn something. Such a situation is plainly unacceptable. Nobody can ask a question, nor can there be any interchange of ideas, in such an atmosphere. Certainly the question of learning the art of discourse is all but absurd in this context. Generally speaking, a situation like this comes about *gradually* over the course of the semester. It happens because someone (most likely the professor) lets it happen.

Some professors prefer to deal directly, and in advance, with the discipline problems connected with large lectures. On the first day of class in a large lecture, these instructors tell the class that large classes present special organizational problems. In order to make the experience as beneficial as possible for everyone, the instructor goes on to prescribe certain rules of behavior in the classroom. These include no eating, no talking, no reading of the newspaper, no coming in late, no leaving early, and so forth. Remember that the first few lectures of your class are your chance to set the tone. You may wish to take the opportunity (gently) to “lay down the law.”

Of course if you are going to use the stern system described in the last paragraph—and it is a perfectly reasonable one—then you must follow through on it. If a student breaks one of the rules you have laid down, then you must call him/her on it: “Mr. Goerring, I said no eating in class.” or “Ms. Flowers, if you need to leave class early then you shouldn’t come at all.”

Because you are an authority in your field, because you give out the grades, and because you hold sway over students’ lives, you have both moral and *de facto* authority in the classroom. As a result, if you comport yourself like a concerned, dedicated professional, then you should have relatively few disciplinary problems. If you nip behavioral problems in the bud, and handle them with dispatch, then they will not get out of hand and your classes will go smoothly. But your antennae should be out for trouble. You will figure out quickly who the wise guys and troublemakers are. When they start to rattle, you start to roll.

Some instructors, in extreme problem situations where there are continuous interruptions and talking during a large lecture class, enlist a confederate. The confederate can sit in the back of class (unidentified to the students) and take notes on student behavior when the (official) instructor’s back is turned. That way, after consultation with the confederate, the instructor can be absolutely sure who is doing the talking, throwing the spit wads, and causing the trouble. And he/she can act accordingly. I was always impressed by my grade school teachers who had eyes in the backs of their heads. They knew when I was going to misbehave even before I started thinking about it. Most of us have not developed such a skill, and may find it (on rare occasions) useful to invoke the device described here.

Not long ago I had a student come to me and tell me that he'd skipped the previous two weeks of class. But he was now returning and would do his best to catch up. I said "fine"—if he needed some guidance, then he should let me know. Indeed, he showed up in class that day. I began the lecture by saying, "OK, let's have another look at Stokes's theorem." My prodigal friend, who had just been in to see me, said, "Could you give a quick review of this concept?" (I had been discussing Stokes's theorem for most of his two week absence.) I am ashamed to say that I lost it. I said, "No. Not for someone who hasn't been to class for two weeks." The other students supported me in this. I could tell by looking in their eyes. But I felt like a rat. When I conducted my personal debriefing after class, I wondered whether I had done the right thing. It all came out well, because a few minutes later he came to my office and apologized to me, I apologized to him, and everything was hunky dory. In retrospect, I think I should have said, "We've been studying Stokes's theorem for two weeks, and it doesn't lend itself to a quick summary. See me after class if you want more help."

Always remember that you have the power to command respect, but you cannot *demand* it. If you present the image of an organized, knowledgeable scholar who is trying to do a good job of teaching, then most students will play ball with you. If instead you are bumbling and unprepared, and look like you don't care about the class, then you can expect like reactions from the students.

I used to have a colleague who handled late arrivals to a large lecture in the following fashion. He would cease to lecture and make a show of timing how long it took the student to get seated. Then he would say, "There are 150 students in this class. It took you 90 seconds to get seated. Thus you wasted 3.75 hours of their time. Now each student here has paid N dollars to take this course. Let us next calculate how many of their dollars you have wasted." And so forth.

Let us consider the effect of this practice. Certainly any student who planned to come in late in the future would wear a bag over his/her head. But I cannot help but think that this sort of arrogant behavior on the part of the instructor suggests a serious attitude problem. Students will lose respect for an instructor who behaves in this fashion.

As the instructor in a classroom, you are in charge. You have every right to demand a certain type of behavior from students, and to enforce discipline. But you must not, in the course of disciplining a student, diminish his/her self-respect. Students are young adults, and should be treated as such.

We have all seen parents who cannot control their children. We have also seen 95 pound fathers who hold tremendous sway over their 250 pound linebacker sons. Parents of the latter sort understand the difference between *demanding* respect and *commanding* respect. The first is easy, is a convenient way to vent your spleen, and often doesn't work. The second is more gentle. It is an art that you need to cultivate. The techniques suggested in this book should help you in that task.

Remember this: You should be conscious of maintaining discipline in class from day one. This is not to say that you should be an unbending authoritarian—far from it. But if you let a class slide out of control for six weeks and then try to use the techniques suggested here (or other techniques) to take back the reins of power, then you will have an extremely difficult and unpleasant time.

The famous mathematics teacher R. L. Moore (see Section 1.8) is said to have once brought a Colt 45 to an unruly math class, set it conspicuously on the table, and then proceeded into his lesson in a room so quiet that one could have heard the sun rise. This technique may have been suitable in Texas fifty years ago. These days, however, I would recommend the use of more civilized techniques for keeping order.

5.6. Mistakes in Class

The most important rule to follow before giving a class is to prepare (Section 1.2). How much you prepare will depend on you—on your experience, your confidence, your training, and so forth. Being fully prepared gives you the flexibility to deal creatively with the unexpected.

But nobody is perfect. No matter how well prepared you are, or how careful, you will occasionally slip up. In the middle of a calculation, a plus sign can become a minus sign. An x may become a y. You will say one thing, think a second, write a third, and mean a fourth. It is best if you can handle these slips with a flair, and particularly without sending the class into a tailspin.

I endeavor in my classes to create an atmosphere in which students are comfortable to shout out, "Hey, Krantz, you forgot a minus sign." Or, "Is that a capital F or a lower case f?" This is a form of participation, and it can be a very constructive one. If you handle these situations badly, then students will be less inclined to ask questions or to approach you on other, more important, matters.

If mistakes are small, and occur in isolation, then they will not damage the learning process. But if they are frequent or, worse, if they snowball, then you will lose almost everyone, give a strong impression of carelessness, set a bad example, and (to oversimplify) turn off the class.

You may endeavor to bail out of an example that you are lousing up by saying, "Well, this isn't working out. Let's start another example." It won't work. This is in the vein of two "wrongs" not making a "right." The only solution here is not to make mistakes and to handle those that you make anyway with a certain amount of finesse.

However: If you can see that the example you are working on is getting out of control, if you *know* that it is going from bad to worse, that you are so bolloxed up that you will be unable to bail out of it, then what do you do? Do not spend the rest of the hour trying to slug it out. Doing so is uncomfortable, counterproductive, and will not teach anyone anything.[3] Instead apologize, say that you will write up the solution and hand it out next time (or put it on the class Web page!), and move on. My advice here may seem to fly in the face of Section 2.7, and to contradict the last paragraph, but it is only meant for extreme situations. Making mistakes is one of the surest ways to lose control of a class. It is the mathematical analogue of an equestrian letting go of the reins. Strive not to do it.

Besides preparing well, there are technical devices for minimizing the number of errors that you make. When I am working an example in a lower-division class, I pause *frequently* to say, "Let's make sure this is right" or, "Let's double check this step." I often pick out a student (who I know will respond well) and ask him/her whether that last step was done correctly. This procedure provides a good

[3]What I am describing here is a good argument for completely writing out in advance any example that you intend to present in class. That is what I do.

paradigm for the students. It also allows note takers to catch up and allows the bright students to strut their stuff in a harmless manner.

One of the most common ways that students make mistakes in their work is by trying to do too much in their heads. Therefore you should set a good example. Write out all calculations. Point out *explicitly* that you have had many years of experience with this material yet you still use lots of parentheses and write out every step.

Put in other words, one of the biggest mistakes that even experienced instructors make when teaching math is to skip steps, or to do steps in their heads. *Never, ever* do this. Write out everything. Every litle detail. Not doing so will cause a notable percentage of the students to get lost. Anyway, it is good discipline for you, and it will slow down your pace. All good things.

5.7. Advice and Consent

If your students take a shine to you, and many of them will, then they will view you as looming larger than just "the math teacher." They will come to you for advice on all sorts of things, from the purchase of a computer or computer algebra software, to advice on the purchase of a car, to advice on how to handle their parents, or advice on very private matters.

A good rule of thumb for you as teacher is to stick to things that you know. You are probably well qualified to give guidance about math books, which section of calculus to sign up for next semester, which computer to buy for which purposes, or whether `MACSYMA` is preferable to `Mathematica`. If you are an auto buff you could give advice about wheels. But being in a position of authority and being asked for advice by a semi-worshipful student is heady stuff, and you had better be careful.

When you are advising students as to which math class to take, it is easy to fall into the trap of unintentionally (or, more is the pity, intentionally) criticizing your colleagues. The practice of this indiscretion is unfortunately rather common. Please do not fall into it.

The students are your clients, but in some sense you work for their parents (since they probably pay the freight). You are almost certainly out of place to advise your students on how to behave toward their mother and father. Do so at your own risk.

When a student starts asking you about private matters then you are in dangerous territory. It is often difficult to discern the difference between **(i)** a student asking how to deal with a significant other and **(ii)** a student making a come-on. Unfortunately, sexual harassment and political correctness, are a part of life these days. Defending yourself against an allegation of either is one of the loneliest and most miserable battles that you may ever have to fight. It can threaten your self-respect, your career, and your marriage. A word to the wise should be sufficient in this matter. Section 5.8 treats sexism, sexual harassment, and related topics in more detail. In any event, if a student has personal problems then he/sheuld be sent to a counselor who is qualified to deal with these matters.

Many undergraduates enjoy having a faculty member as a friend. If you are open to it, you can have two or three students hanging about your office at just about any time of the day (or night). It's a boost to the ego to have these attentions, young people are often quite refreshing, and this device provides a convenient way to rationalize wasting a heck of a lot of time. You will have to decide for yourself how you want to handle this trap.

Consider the matter a bit differently. If you make yourself available all day long to help your students with their math (never mind getting involved in their personal lives), you will indeed attract customers. Make yourself available to other professors' students and you will have even more customers. You can provide tutorials, make up extra homework assignments for students, and find innumerable other ways to while away the day. But the set of activities that contribute to a successful academic career and the set of activities that I have just described have a rather small intersection. That sounds rather Machiavellian, so let me be more gentle. Almost all learning is ultimately accomplished by the individual. I've engaged in nearly all the activities described here. They have produced few lasting results and, in the end, truly have helped very few students.

Most of those who end up surviving in the academic game are people who decide that, no matter how much they love their students, they love themselves a bit more. Your students won't like you any the less for saying, "I have to do some work now. Let's talk at another time." You will figure this out for yourself eventually, but you heard it here first.

Finally, all of the advice in the last three paragraphs must be filtered through the value system of the institution at which you teach. There are certainly colleges—whose mission is primarily teaching—at which professors are *expected* to have an open door all day long. Such a policy is consistent with what such a school is trying to accomplish, and how it expects its faculty to spend its time, and what sorts of academic activities its faculty might pursue. Most research universities have a different policy toward faculty office hours. A common policy is that you should have 1.5 office hours each week for every course that you are teaching. Apart from those designated hours, your time is yours (although the presumption is that you are doing something scholarly during the other work hours).

Swarthmore and M.I.T. take different approaches to education—and both are excellent. Do be sensitive to what is expected of faculty at your institution.

5.8. Sexism, Racism, Misogyny, and Related Problems

Nobody, certainly no educated person, thinks of himself/herself as a sexist, or a racist, or a misogynist. That is what is so insidious about these misdoings when they come up in a university environment. People often are not aware that they are being offensive.

I don't want to preach about any of these topics. Rather I would like to be pragmatic and to mention some pitfalls. Common complaints from students are these: **(1)** The professor calls on male students more than on female students, **(2)** The professor will not answer questions from female students, **(3)** The professor shows favoritism toward female students in his grading policies, **(4)** The professor talks down to minority students, **(5)** The professor seems to believe that women are less able than men, **(6)** The professor seems to believe that Caucasians are more able than minorities, **(7)** The professor demands more from Asian students, **(8)** The professor makes suggestive cracks in class, **(9)** The professor uses vulgar language.

None of these complaints are cooked up. They have all been tendered, very seriously, by genuinely outraged students. It goes without saying that, in all instances, "male" may be switched with "female," "Caucasian" may be switched with "minority," and so on. The issue here, and I cannot emphasize this point too strongly, is not that any particular class of people is persecuting any other particular class of people. Rather, the point is that every instructor has his/her foibles and shortcomings and biases and these will often be perceived by students through the filter of whatever issues are currently in the air.

Matters regarding the issues being discussed here are rarely clear. Sometimes a student with one of the complaints described above is doing poorly in the course and is looking for an excuse or a scapegoat. Sometimes the professor just doesn't realize that he/she is behaving in a manner that some students find offensive. In short, there is plenty of room for misunderstanding.

You should never physically touch your students. An arm around the shoulder or even a prolonged and enthusiastic handshake easily can be misinterpreted.[4] But many people, especially those new in the teaching profession, are not aware of the subtleties involved in the legal definitions of sexism, racism, and so forth. Let me surprise you with some other aspects of harassment that you may not know.

- If you are in the habit of saying, "I don't want to blow smoke up your dress." or "This is a pregnant idea." or "This problem is a bitch." or "Bulls — !" during your classes then, by a strict interpretation of the statute, you may be guilty of sexual harassment. This is true regardless of your sex or the sex of the members of your audience. In fact, if you even tolerate this language from others who are in the classroom, then you also may be guilty! The spirit of the law is that if someone *feels* offended then they are offended.
- If you have displayed in your office a *Playboy* centerfold, or a (closed, unread) copy of *Playboy* sitting around where others can see it, or a *Frederick's of Hollywood* catalog, or a Chippendale's poster, or in fact a great many of the other posters that can be seen in popular stores (even in the campus bookstore), then you may be guilty of sexual harassment.
- Many items that you may think of as art—paintings or sketches or sculptures or frescos—may be seen by others as suggestive or offensive. Look at the effect that the work of Robert Mapplethorpe has had on support for the National Endowment for the Arts.
- Signs or posters that have religious, political, emotional, or sexual content may be deemed offensive.

You may feel that being forced to monitor your language or other behavior this closely is an abridgment of your First Amendment rights, and you may be correct in this feeling. My view, much as I hate censorship, is that there is no point in going through life wearing a "kick me" sign. You can use this language, or display these artifacts, all you like when alone or in private circumstances. As a teacher you are something of a public figure and must suffer certain restrictions.

A number of universities in this country have distributed detailed guidelines to their faculties about the issues being discussed here. Some have gone further, and indicated specific words or phrases that ought not to be used. For instance, you should not say "snowman" but should use a suitably laundered asexual alternative.

[4] Several years ago a major US bank instituted new policy strictly forbidding all back-patting, arms around the shoulder, and prolonged handshakes. The new rule came about just because there had been a large number of sexual harassment suits.

Various standard English phrases are suggested to be off-limits and substitutes are recommended. The point is that we are dealing with very delicate and emotion-charged issues of legality and morality here and one has no choice but to take them seriously.

It recently came about that an offer that had been tendered by a big state university to a job candidate was rescinded because the new employer learned that the candidate had been found guilty of sexual harassment at his previous place of employment. Note that the offer had been tendered *in writing*, and note that it had been accepted *in writing*. Furthermore, the job candidate was never found guilty in any court of law. Rather, he was sanctioned in a private university proceeding. Yet his new putative employer felt that it was too great a risk to put this man in a position of *in loco parentis* before eighteen-year-old young women. Of course the university risked a lawsuit from the spurned job candidate, but it felt that that was a lower risk than what might transpire if he were allowed to teach in its math department.

The book *The Lecherous Professor* [**DZW**], a serious but inflammatory study of the sexual peccadillos of male faculty at American universities, played a key role in the sanctioning of the instructor described in the last paragraph. A glance at the present section of this book will show you that emotions run high over questions of sexual misconduct. I am not suggesting that sexual misconduct is not a serious crime—it most certainly is. But we must all be aware that the gun could be pointed at any of us—even if we are innocent. Most accusations of sexual harassment or misconduct are cases of one person's word against another's. There are rarely any witnesses. It is just as easy for an allegation to be made against you as it is difficult for you to thereafter clear your name. Hence forewarned is forearmed.

It is no fun having to deal with issues of racism or sexism or sexual harassment or misogyny. A complaint lodged against you is, in effect, an attack on your character and your integrity. So be aware that these areas are a potential problem for all of us. Behave accordingly. If you are called on the carpet for any of these matters, then do not become defensive. Show respect for the complainant. Get help from your department chair or from your dean of human resources. This is serious business.

5.9. Begging and Pleading

Some students will come to you with unreasonable requests. They will tell you, after doing poorly on an exam, that if they do not pass this test—or this course—with a certain grade then they cannot continue in the pre-medical program, or the microbiology program. Of course you, as professor, can verify rather quickly whether this claim is true. But it does not matter. If the test was so important to the student then the student should have studied harder. The student should have come to your office hours for help before the test. If the student's homework was weak then the student should have seen the writing on the wall. Be cautious of these pleas. While you do not want to be heartless and unsympathetic, you also do not want to find yourself gradually being drawn into an ever more complicated morass of tricky moral dilemmas (see [**WIE**] for a rather hard-nosed view of this sort of student negotiation).

Students are sometimes *unable* to see the writing on the wall. A student will come to me and ask why he/she got a grade of "D" in a certain course. I will look at his/her record and say, "Well, you got a 'D' on the first midterm, a 'D' on the second midterm, a 'D' on the homework, and a 'D' on the final. So a grade of 'D' seemed to be in order." Amazingly, this line of reasoning never occurred to the student (no, I am not making this up). So I have to be patient and explain how the world works.

A desperate student will offer you all sorts of inducements to change grades. Discretion prevents me from enumerating what some of these may be, but they range from the pecuniary to the personal. *You must brush off these attempted bribes with the disdain that they deserve.* If you act as though you are considering and then rejecting them, then you are looking for trouble.

It really is true that if you look and/or act like a student then students will find you more approachable. They will more readily come to you with propositions that they wouldn't consider broaching with a more wizened (or older) faculty member. In short, younger faculty are more vulnerable. This is one reason for dressing differently from students and maintaining a slight distance. Again, this may sound cold. But I speak here from hard personal experience.

As has been mentioned in Section 5.8 and elsewhere, you must be sensitive to sexual harassment issues. Sexual harassment is not a pretty subject, but acting receptive—even mildly so—to any proffered inducements is only courting disaster.

Sometimes a student—who has done poorly on an exam—will ask to be allowed to take the test again. You simply cannot allow this type of favoritism. For one thing, it is unfair to the other students. Second, if the others find out then they will become angry—and justifiably so. You *can* give the student a second try off the record and go over the test afterwards with the student. This artifice can be a device for giving the student some encouragement. You might tell the student, "I can see from this unofficial exam that you know the material better than your official exam suggests. If you do well on the final then you can still probably get a grade of 'B'." However you must engage in this charity sparingly, if for no other reason than it can use up large chunks of your time. Also, it is too easily misinterpreted.

A favorite student response to a poor test grade is, "I did very well on the homework but my test grade does not reflect what I know." Of course some students may "choke," or panic, on a test. As a student, I have done so myself. But unfortunately many students do their homework by copying examples from the text, or the lecture—merely changing the appropriate numbers. This leads to a minimum of understanding. You must stress to your students that, when they study for a test, they should reach proficiency *without* recourse to the book or notes. If the book and notes are necessary, then the technique has not been mastered. For interest's sake, refer to Tom Banchoff's technique, described in Section 2.9, for handling students who choke on a test.

One of the most common student remarks is, "I really understand the material but I cannot do the problems." A variant is, "I can do the problems on the homework but I cannot do the problems on the test." Consider if you will these analogous statements: "I really understand how to swim but every time I get in the water I drown." and "Playing the piano sure looks easy when Arthur Rubinstein does it. I wonder why I cannot do it?"

If you are a good teacher, then you will make the material look easy, or at least straightforward. Thus you can lull students into a false sense of security. You must continually warn them of the importance of mastering the material *themselves*—and of *practicing.* And this point leads to the second ludicrous statement recorded above. The fact that a student can do the homework problems is meaningless *unless* the student can do them cold, and with the book closed.

This last observation seems so obvious that it hardly bears mention. But recall that this is a book about the obvious, and this point bears not only mention but repeated mention. Many students view the learning process as a passive one—something like getting a massage. You must constantly remind them that this attitude will not do. You can remind them by just telling them. Or you can remind them by giving pop quizzes. Or you can remind them by giving an exam and watching them flunk. But, one way or the other, you must attempt to break through this psychological impasse.

In a related vein, many students think that

(i) studying

and

(ii) just sitting in front of the book

are one and the same thing. We all know that studying requires discipline, tenacity, and hard work. It is not something that just happens to you. It is instead something that you *make happen.* Helping your students to understand this point begins with being conscious of the problem yourself.

It is an observed fact that most students—but especially freshmen and sophomores—have no idea how to study. David Bressoud recently conducted a survey in which he asked students in large calculus lectures how they studied. What he learned is that the student conception of "studying" is to attempt to do the assigned homework problems by emulating a cognate example from either the text or the class hour. This is certainly a meaningful activity, but it is only a minuscule portion of what true studying is. It is worthwhile to consider how you, as a mathematics instructor, can force the students to read the text and to turn the ideas over in their own minds. Some instructors have required their students to keep journals recording the development of ideas in the course. Lab activities—really tightly constructed ones that force the students to reinvent the ideas for themselves—are another way to accomplish this goal. Group discussion is a third method.

In my opinion, most students want to be told what to do. Your job as teacher is to tell them. Don't make them guess what are the important topics in your class. Tell them. Don't make them guess what they will be tested on. Tell them. Don't make them guess how to study for an exam. Tell them. Don't make them guess what are the pitfalls in studying. Tell them. Is there any reason not to do this? Would you rather deal with the begging and the pleading?

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CHAPTER 6

A New Beginning

6.0. Chapter Overview

At this point in time, American academe is at a crossroads. Both the university and society as a whole are making new demands on the professoriate. One of these is that we be directly accountable for our teaching. The purpose of this book is to point out that such demands are not antithetical to our scholarly pursuits. In fact your teaching activities can complement your research activities rather nicely.

I have discussed both philosophical issues and pragmatic issues in this text. Certainly good teaching is the sum of many particular skills, but it is also the product of attitude and purpose. I hope that reading this book has sharpened your focus on teaching.

6.1. The Role of the University Professor

A distinguished mathematician—well known to us all—joined the University of Chicago Mathematics Department, as an assistant professor, in the early 1960s. As he was settling into his office, the Chair came by and chatted him up for a few minutes. When the chair departed, he waggled his finger at the new faculty member and said, "Remember: Our job is proving theorems."

In retrospect, one wonders why the chair felt moved to make such a statement. Chicago is and was one of the pre-eminent mathematics departments in the country. In the early 1960s, the teaching reform movement was still a twinkle in somebody's eye. Teaching evaluations had not yet been invented. Everyone agreed with Paul Halmos that proving theorems was not just the main thing—it was the only thing.[1]

If we were to make a sequel to this movie, filmed in 2015, then the scene (at least at many universities, and especially public institutions) would be a bit different. The chair would still drop by to chat up the new faculty member. He would remind the newcomer that he was hired for his ability with mathematics, and for his achievements in research. (Proving theorems, and learning new mathematics, is the highest and finest thing that we do. This fact has not changed, and I hope it never will.) But as the chair departs, he will now waggle his finger and say, "But don't forget: It's teaching that pays the bills around here. Undergraduates come here expecting to be taught. And parents pay tuition because they want their children to be educated. I expect you to do a creditable job with your teaching. And I don't want to hear any complaints from students or parents. If I do, you will be making my job more difficult, and I in turn will make your *life* more difficult. A word to the wise should be sufficient."

Again, I am not trying to sound sappy. And I am also not endeavoring to be draconian. If you are new to the mathematics profession, then you may as well

[1]Personal communication.

know what sort of world we now inhabit. You have a choice: You can prove the Riemann hypothesis or you can learn how to teach.

If you have been in the profession for a while, and have never given any thought to teaching, then perhaps it is time you had better do so. You no doubt have your own ideas about the subject, but perhaps viewing the ideas presented here will give you food for thought.

My own experience is that my teaching meshes rather nicely with my research. I've had good ideas (for a research problem) while preparing a calculus class, and I've gotten inspiration for my calculus class from serious mathematics that I was working on. I suppose that this is the way it is supposed to be, and I believe that a part of the reason that these different facets of my professional life interact so well is that I am open to such interaction. I encourage you to foster this symbiosis in your own life.

6.2. Closing Thoughts

Sometimes the easiest way out, when we are faced with some difficult or distasteful task to perform, is to resort to cowardice. We are all guilty of this sort of avoidance. At one time or another we have all lied or engaged in subterfuge to avoid unpleasantries.

Our students, of course, suffer from their own shortcomings. One of my colleagues had a student knock on his door and ask for some help with calculus. The professor said, "I haven't seen you in class for three weeks. Why do you come to me now?" The student replied that he didn't need to go to class—he had the book. "Do you read the book?" intoned the impatient professor. The student replied, "Well, I could."

What are you going to do? I would tell the student that when he wanted to have a serious conversation he should phone me up for an appointment. Until then, he should not darken my doorstep.

I do not wish to dwell here on human frailties. But I think that the method of teaching that many of us use—and I have been guilty of this to a degree with certain classes that I really did not want to be teaching—is a form of cowardice. We just skulk into the room, write the words on the board, and convey with body language and voice and attitude that we are not interested in questions or in much of anything else connected with this class. Then we turn tail and skulk out of the room. I was once told (tongue-in-cheek, I think) that the secret to success in undergraduate teaching is, "Never let a student get between you and the door." Not an admirable attitude, but one that many of us have held from time to time.

How to Teach Mathematics has been an effort to fight this form of cowardice, both in myself and in others. Teaching can be rewarding, useful, and fun. To make it so does not require an enormous investment of time or effort. But it does require that you have a proper attitude and that you be conscious of the pitfalls. It does require being sufficiently well prepared in class so that you can concentrate on the *act* of teaching, rather than on the epsilons. And it requires a commitment.

We must believe that being a good teacher is something worth achieving. We must provide some peer support to each other to bring about this necessary positive attitude toward teaching. The last thing I want is for mathematicians to spend all day in the coffee room debating the latest pedagogical techniques being promulgated by some well-meaning educational theorist. I want to see mathematicians learning and creating mathematics and sharing it with others. But those others should include undergraduates. That is what teaching is about.

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Index

This third edition is a lively and provocative tract on how to teach mathematics in today's new world of online learning tools and innovative teaching devices. The author guides the reader through the joys and pitfalls of interacting with modern undergraduates—telling you very explicitly what to do and what *not* to do. This third edition has been streamlined from the second edition, but still includes the nuts and bolts of good teaching, discussing material related to new developments in teaching methodology and technique, as well as adding an entire new chapter on online teaching methods.

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