TOMMY DREYFUS

WHY JOHNNY CAN'T PROVE

(with apologies to Morris Kline)

ABSTRACT. The one sentence answer to the question in the title is that the ability to prove depends on forms of knowledge to which most students are rarely if ever exposed. The paper gives a more detailed analysis, drawing on research in mathematics education and classroom experiences.

1. INTRODUCTION

Recent changes in mathematics teaching at all levels include attempts to make learning experiences more cooperative, more conceptual and more connected. As a consequence, students are more and more frequently asked to explain their reasoning; for example, Silver (1994) suggested that written explanations should become a prevalent feature of school mathematics and predicted that 'unless and until solution explanations and interpretations become a regular item on the menu of instructional activities in mathematics classrooms, ... there can be little hope of substantially improving the poor mathematics performance of American students' (p. 315). On another continent and for a different student age group, the custom of scientific debate has been firmly established since 1984 as an opportunity for deep learning experiences in the framework of a large first year university mathematics course (Alibert and Thomas, 1991). Other examples will be referred to below.

Occasions for mathematics students to make their reasoning explicit may arise for a number of reasons: A student may want to convince a classmate of a guess or conjecture during a collaborative phase; another student may have asked for help; or the teacher may try to obtain clarification about students' thinking in order to help them, to assess their progress, or attempt to move them from a descriptive to a justificative mode of thinking about what they are doing (Margolinas, 1992). In these cases, the explanations students are asked to provide are thus arguments, possibly even proofs. This increased emphasis on explanation, argument and proof is consistent with the continued importance of proof in mathematics (Hanna, 1995; Dreyfus, in press).

Educational Studies in Mathematics **38:** 85–109, 1999.

The different situations mentioned above may require different kinds of explanations. This paper will focus mainly on written explanations given by college students as response to questions by a teacher or textbook, for example in homework assignments. Even within this limited framework, several questions immediately arise: As teachers and as educators, on what basis do we or do we not accept a student's explanation? Which arguments are acceptable to us under which circumstances, which are not, and why? What criteria do we use and what are these criteria based on? Do we expect a proof when we ask for a justification? And do we expect a proof when we ask students to 'explain why'? To what degree does an explanation need to convince? And if so, does it need to convince a mathematician, the teacher, fellow students?

These questions cannot be answered in general, outside of the framework of a particular curriculum or course. But often, the questions are not even asked; or if they are asked, they remain unanswered even where answers are feasible and essential. A first aim of this paper is therefore to provide appropriate background for analyzing these questions. The paper thus has a descriptive rather than a normative character. In this respect, it is different from Kline's (1973) book from which the title was adapted.

On the other hand, some changes in instruction may well be indicated: Sample explanations, even from the more successful students show that frequently their criteria for acceptable explanations appear to differ from their teachers' criteria. For example, students often provide chronological accounts of actions carried out rather than pointing out connections and implications. One might therefore add another set of questions: What do students consider a satisfactory explanation? What is the basis of their conception of a satisfactory explanation? And what is the origin of the differences between students' and teachers' conceptions? The second aim of this paper is to identify some of the reasons for students' limited conceptions of explanation and proof; in other words, I will attempt to clarify why students cannot be expected to have a mathematician's concept of proof, of its function and of its purpose.

In Section 2, some sample explanations will be exhibited; most have been given by students, and they are intended to show how difficult the task is, even for the more successful ones. This point will be strengthened in Section 3 by means of a review of research on proof and explanation at the college, high school and elementary school levels. In Section 4, the influence of typical high school and college mathematics experience on students' image of explanation and proof will be reviewed. In Section 5, the review of research will be continued with epistemological and cognitive work on the distinction between proof and explanation; the discussion will also be carried beyond mathematics education to the topic of proof in mathematics itself. And in the concluding Section 6, the material from all the previous sections will be pulled together into an appraisal of the difficulties students have with explanations and proofs.

2. SAMPLE EXPLANATIONS

Abstract characterizations of explanations will be discussed later (Section 5). Instead, we start with a number of sample explanations which will be used to raise some pertinent questions. Most of these examples have been collected, somewhat randomly, from students participating in introductory university level courses such as calculus or linear algebra. Moreover, the examples are answers to questions and problems on which the students demonstrated a certain proficiency and some understanding. This choice has been made in order to focus on the characteristics of the explanations rather than on misunderstandings of the questions or the concepts involved in answering them. My concern is thus not with the question 'Why were the students unable to give the correct answer?' but rather with the question 'Why were the students unable to give a decent explanation in spite of the fact that they seem to have a satisfactory understanding of the question and its answer (or of the problem and its solution)?' In this respect, the focus of the present paper is different from that of other recent work; for example, Ferrari (1997) analyzed similar questions answered by a similar student population but was interested mainly in the analysis of the students' wrong answers and the conceptual reasons for these wrong answers; Vinner (1997), on the other hand, analyzed ways in which students altogether avoid to conceptually deal with the questions which are presented to them, and exhibited reasons for such behavior.

Most of the following examples have been chosen from take-home assignments which first year undergraduate students at two universities in two different countries handed in after having been given about a week to prepare the assignment. Students were explicitly advised that explanations of what they did and why they did it were crucial and would account for a substantial part of the grade. I have chosen cases in which the answers led me to believe that the question was meaningful to the student, and the student showed a substantial understanding of the procedures and concepts needed to answer it. Apart from that, neither the examples nor the students are representative in any sense but have been chosen for illustrative purposes. Their choice, including the topics, the level of mathematics, and the level of students have been influenced by my own personal bias and experience.

Ferrari characterized his weakest students as ‘unable to use words to express even elementary mathematical ideas and relationships’ (loc. cit., p. 2–262). The use of language to express mathematical relationships is a crucial constituent of explanations. As a consequence, most of the following examples relate, in some way, to the use of language.

Example 1.
Determine whether the following statement is true or false, and explain:
If $\{ v_1, v_2, v_3, v_4 \}$ is linearly dependent, then $\{ v_1, v_2, v_3 \}$ is also linearly independent.

RP: True because taking down a vector does not help linear dependence.

The use of ‘taking down’ rather than, say, ‘omitting’ points to a lack of linguistic ability; this impression is compounded by the use of the word ‘help’ rather than a less vague but presumably more complex term. But ignoring the purely linguistic inadequacies of the explanation, we can try to speculate on its mathematical adequacy.

Maybe RP thought as follows: ‘I know that adding a vector to a given linearly independent set of vectors might produce a linearly dependent set; on the other hand, adding a vector to a linearly dependent set will not produce a linearly independent set. In other words, adding a vector to a set “helps” the linear dependence of the set. Thus, omitting a vector from a set does not “help” its linear dependence. Since I was instructed to state my reasoning concisely, I will only write down an abridged version of the last sentence.’

It should be noted that even this expanded (and invented) explanation has mathematical and logical problems. It is not sufficiently sharp, from the mathematical point of view. By means of the introduction of the term ‘help’ for ‘might produce’ a vagueness is introduced which can be interpreted as ‘in some cases adding a vector will produce a linearly dependent set’ or as ‘in all cases adding a vector will produce a linearly dependent set’. Such vagueness may be due to lack of conceptual clarity or to lack of linguistic ability, or to a combination. Next, the expanded version proceeds to take the converse of one part of the preceding sentence. Since I (rather than the student) invented the sentence preceding the converse, we cannot know on what the student’s claim was based. But we may go further and ask whether the expanded version, even after replacing the word ‘help’ by a less vague one would constitute a satisfactory explanation: Isn’t it almost tautological to state that the claim is true because adding a vector to a linearly dependent set will always produce a linearly dependent set? Shouldn’t the student have also explained, using the definition of linear dependence, why adding a vector to a linearly dependent set cannot produce a linearly independent set? How far back does an explanation have to go? How deep does it need to be in order to count as an explanation?

The above example is far from being a special case; in fact, it is rather typical. Similar analyses could be carried out for the following two examples:

Example 2.
Determine whether the following statement is true or false, and explain: If v_1, v_2, v_3, v_4 are in R^4 and it is known that $v_3 = 0$, then the set { v_1, v_2, v_3, v_4 } is linearly dependent.

RC: True, the nontrivial solution is possible because v_3 is equal to *0*.

Example 3.
Is the following statement true or false; justify your answer: A system of n equations in n unknowns has at most n solutions.

TA: False; if we will discover a consistent system with a free variable, there will be ∞ solutions.

The most generous evaluation a teacher of a first year linear algebra class would presumably give to these answers is that they include important elements of the required explanation but are not substantial enough. Students at this stage of their education appear to find it extremely difficult to distinguish conciseness from lack of substance. I regularly have students who complain about my requirements arguing that they should not be required to write text because they are taking a mathematics class rather than a literature class. There are two extreme cases of not sufficiently substantial explanations. One is stating a tautology rather than an explanation by simply repeating the claim. The other is not giving an explanation at all but a computation, as in Example 4.

Example 4.
Are the columns of the matrix A linearly independent?

$$A = \begin{bmatrix} 3 & 4 & 9 \\ -2 & -7 & 7 \\ 1 & 2 & -2 \\ 0 & 2 & -6 \end{bmatrix}$$

$$AW : \begin{bmatrix} 3 & 4 & 9 \\ -2 & -7 & 7 \\ 1 & 2 & -2 \\ 0 & 2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 \\ 2 & -7 & 7 \\ 3 & 4 & 9 \\ 0 & 2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 13 \\ 0 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Although I have no specific information about this, it may well be that AW was one of the students complaining about my 'literacy' requirements. He reacted to the question by correctly carrying out a computation (reduction of the matrix) from whose result the answer to the question can be read off. He might even have read the answer off (the three column vectors are linearly independent) but he did not consider it necessary to leave a written record of this; nor did he consider it necessary to establish the connection between the computation and the question: Why and how can the linear independence of the column vectors be read off the reduced matrix? We don't know whether AW could have provided the why and how; we only know that he didn't – presumably because to him the computation constitutes the most important part of answering the question rather than the answer itself or the explanation justifying the procedure. It is not at all obvious that the same aspects of an answer (or solution) are considered important by the teachers and students of beginning university mathematics courses.

Above, explanations lacking in substance were considered. Occasionally, students exaggerate in the opposite direction and 'explain' by writing down whatever comes to their mind and might possibly be related to the question. This results in texts which include all the elements needed for the required explanation, and with redundant information added. The following is a relatively mild case.

Example 5.
Prove that the equation $x^3+9x^2+33x-8=0$ has exactly one real root.

AM: [Defines $f(x) = x^3+9x^2+33x-8$; differentiates f and shows that the derivative has no real roots. Then continues:] The fact that the derivative has no roots means that there are no critical points at which one has to check the behavior of the function.
$\lim_{x\to\infty} f(x) = \infty$. $\lim_{x\to-\infty} f(x) = -\infty$.
The derivative of the function is always positive. The function exists for all x. The function increases always and therefore, because it goes from $-\infty$ to ∞ there is only one real root.

By omitting the redundant part and reordering the rest, AM's argument can be made into:

AM': The function exists for all x. The derivative of the function is always positive. The function increases always; it goes from $-\infty$ ($\lim_{x\to-\infty} f(x) = -\infty$) to ∞ ($\lim_{x\to\infty} f(x) = \infty$). Therefore, there is only one real root.

This argument is quite acceptable, although far from perfect; for example, it states that the function exists for all x but omits to state that it is differentiable for all x; it also concludes that there is 'only one root', not making it completely clear that the intention is 'exactly one root'. The question arises why the argument AM is less acceptable than AM' as an explanation of the fact that the function has exactly one root. From the student's point of view, is the omitted part really redundant? After all, we did need the fact that there are no critical points, and critical points are points at which often the behavior of the function has to be checked; in the case at hand, if there was a critical point, it might be one at which the function is not differentiable, and AM has demonstrated consciousness of this possibility. Moreover, and again from the student's point of view, AM presents all arguments needed to draw the correct conclusion; why should it be so important that the arguments appear in one order rather than another? To the teacher, it might appear that AM was not quite able to correctly combine all the elements presented into a coherent proof; to AM, it might appear that she has provided the required proof and beyond, some elaboration which adds to the minimum which is necessary.

Uncertainty about how to handle redundancy might lead some students to add unnecessary, even superfluous elements as above; on the other hand, it might lead others to omit elements which are relevant and necessary for a complete argument. The next example is a case in point.

Example 6.
Show that if AB and BA are both defined, then AB and BA are square matrices.

DM: By definition, if A is an m$\times$r matrix and B is an r$\times$n matrix, then AB is an m$\times$n matrix. BA would then be an r$\times$r matrix which is a square.

As in the earlier examples, several different interpretations of DM's explanation could be given, ranging from a rather severe critique of the explanation's inadequacy as a mathematical proof to a rather positive evaluation of the correctness and the relevance of the elements which are presented, and related to each other. While it is not difficult to point out how the explanation could be improved, we do not have the means to find out why the student did not give a better explanation. The characteristics of this as well as earlier student explanations show that giving an argument or explanation is a very difficult undertaking for beginning undergraduates from at least two points of view: In most cases, they still lack the conceptual clarity to actively use the relevant concepts in a mathematical argument; and, more generally, they have had little opportunity to learn what are the characteristics of a mathematical explanation.

The two final examples show, that not only students have such problems.

Example 7.
The series $\Sigma \frac{1}{k(k-1)}$ converges because S(2) = 1/2, S(3) = 2/3, S(4) = 3/4, S(10) = 9/10, S(100) = 99/100, and thus S(n) = (n-1)/n. For infinitely large values of n, the partial sums S(n), S(n+1), ... differ from the limit S=1 and consequently among themselves by an infinitely small quantity.

Did the student compute S(100) or guess it inductively? What exactly does the word ‘thus’ mean. And what notion of convergence was used? While this example looks typical for a modern calculus student, its treatment of convergence is historically based on Cauchy’s (1821) definition: ‘When the values successively attributed to the same variable approach indefinitely a fixed value, eventually differing from it by as little as one could wish, that fixed value is called the limit of all the others.’ Seen from today’s vantage point, Cauchy’s definition lacks conceptual clarity (Lakatos, 1978) and Cauchy was able to use it to prove the (incorrect) result that the limit of any sequence of continuous functions is continuous.

Example 8.

The following exchange took place within a six hour period in June 1997 on an electronic list whose topic is post-calculus mathematics teaching and whose participants are university level mathematics teachers and researchers:

DE: Here is a problem from Arnold (1991) I’ve just managed to solve. Let f(x,y) be a polynomial with real coefficients, such that $f(x,y) > 0$ for all (x,y) in the plane. Does f necessarily achieve its minimum?

AD: It is too easy. It can’t be a polynomial of odd degree, otherwise it will take both signs. So it’s even degree, and bounded below by 0. It must have a positive minimum at a critical point, which will be its absolute minimum, so, yes, it attains its minimum. Am I missing something here?

RR: No; but one does not even need critical point theorems. Once you know P takes positive values only, knowing it to be a polynomial tells you the leading coefficient is positive, because the term of highest (combined) degree outclasses all the others combined as $|x|+|y| \longrightarrow$ infinity. Let P(a,b)=M, then choose a box outside of which $P(x,y) > M$, the box also big enough to contain the point (a,b). (Here the positivity of the leading term and the triangle inequality turn up.) Then by continuity alone, P has a minimum somewhere in the closed box, and since its value is $< M$ there it is an absolute minimum for the whole plane.

SL: I don’t find RR’s proof convincing because the crucial step is unclear. There is no one term of highest degree. Obviously, if one can show that the homogeneous part of highest degree is positive definite, the rest follows, just as RR says. AD’s proof is also not convincing. Why does even degree imply (in a two-variable polynomial) the existence of a critical point?

DS: Does f necessarily achieve its minimum? No. Take $f(x,y) = x^2 + (xy-1)^2$. It is clear that $f(a,b) > 0$, and $f(a,b) = 0$ implies $a = 0$, $ab = 1$, which is impossible. However, $f(a,1/a) = a^2$ can be as small a positive number as desired.

How are we to evaluate the (mistaken) explanations by AD and RR? They use relevant and correct input, they are convincing, and they support what many list readers, including this author, clearly expected to be the correct answer. On the other hand, the problem cannot be considered very difficult. Many calculus teachers use examples like $g(x,y) = xy^2/(x^2+y^4)$ in order to show that the two variable case is much more complex than the one-variable case; the function g converges to zero along every straight line through the origin but does not converge to zero at the origin. This is fairly similar in spirit to DS's function which has a minimum on every line through the origin but not in the plane. How could professional mathematicians go so wrong? Are explanations so difficult to give and judge?

To conclude this section, the questions which were raised are collected for further reference:

- Which aspects of an answer (or solution) are considered most important: computation, statement of the answer, relationship between computation and answer, procedural or conceptual? Are the same aspects considered important by the teachers and by the students of beginning university mathematics courses? How are the students supposed to know what the teacher considers important?
- How difficult is it to give and judge mathematical explanations? Should we see our students' imperfect explanations in a rather forgiving light?
- How deep does an explanation have to be? Does it always have to go all the way back to the relevant definitions? Does it have to live up to the same stringent criteria as a proof, namely to use only definitions and previously proved statements?
- How important is the order of the reasons from which a conclusion is to be drawn?
- Is redundancy wrong? Explanations given in teaching situations are often redundant; lecturers tend to repeat statements many times over, giving different points of view, and connections to various related concepts. Why then should a student's explanation be free of such redundancies?
- How much accuracy is required from an explanation? Under what circumstances may students use vague terms such as 'does not help' (example 1)?
- What counts as tautological? What might sound tautological to the teacher might constitute a considerable conceptual step for the student because (s)he is less well versed in the subject matter.
- Can we tell whether a student's problem is linguistic rather than conceptual? How should we deal with linguistic problems?

The examples and questions of this section have been collected here somewhat informally because we lack a better research base on student explanations in undergraduate mathematics. There is a shortage of research data, and it is one aim of this paper to point out the need for such research. A first conclusion – as informal as the data on which it is based – is that the task of explaining is extremely difficult, even for reasonably proficient students who were accepted into a university and exhibit some understanding of the topic.

3. RESEARCH ON STUDENTS' CONCEPTIONS OF PROOF

Student answers were selected for presentation in Section 2 only if they showed some features which can be considered as justificative; moreover, some well known cases like proof by example were not even illustrated. Nevertheless, the variety of the answers is great and may show different stages of development. With this in mind, mathematics educators have attempted to classify students' developing notions of proof. Balacheff (1987), for example, distinguishes pragmatic proofs and intellectual proofs, subdividing each into several subclasses; and Harel and Sowder (1998) propose a large set of schemes intended to make a classification of college students' proof-like productions possible.

Our aim here is not to classify such productions, or follow their development but rather to identify some of the reasons for the fact that many students appear to have a very limited conception of proof. Indeed, research results on students' conceptions of proof are amazingly uniform; they show that most high school and college students don't know what a proof is nor what it is supposed to achieve. Even by the time they graduate from high school, most students have not been enculturated into the practice of proving, or even justifying the mathematical processes they use.

Fischbein (1982), for example, provided close to 400 high school students with a proof of the statement 'For every integer n, the number $E = n^3 - n$ is divisible by 6'. Although over 80% of the students affirmed that they had checked the proof and found it to be correct, less than 70% agreed that $E = n^3 - n$ will always be divisible by 6, less than 40% concluded that a purported counterexample must contain a mistake, and less than 30% agreed that there was no need for additional checks in order to decide on the truth or falsity of the statement. Fischbein concluded that less than 15% of the students really understood what a mathematical proof meant.

Coe and Ruthven (1994) found that when proof contexts are data-driven, and students are expected to form conjectures by generalization or counter example, then students' proof strategies are primarily empirical. It seems that in such a context students are willing to replace deductive argument by a sufficiently diverse set of instances.

Similarly, Finlow-Bates, Lerman and Morgan (1993) found that many first year undergraduates had difficulties following chains of reasoning, and judged mathematical arguments according to empirical or aesthetic rather than logical criteria.

Martin and Harel (1989) provided preservice elementary teachers with correct deductive, incorrect deductive, and inductive arguments for the same statements. Every inductive argument they presented was accepted as a valid mathematical proof by more than half of their students; the acceptance rates for the deductive arguments were not much higher than those for the inductive ones; and the false deductive proofs were accepted by close to half of the students.

Finally, Moore (1994) found that even apparently trivial proofs are often major challenges for undergraduate mathematics majors.

Is it, then, a fact that students cannot argue mathematically at all? This would be an unwarranted conclusion. In fact, all examples provided in Section 2, while being rather far from constituting rigorous proofs, contain clear seeds of mathematical argumentation and justification. Moreover, several studies carried out at the upper elementary level show that in suitable environments some students develop promising abilities.

Maher and Martino (1996) report a sequence of eleven events in the development of one elementary student's justificative arguments over a five year period. While most tasks given to the student required the classification and organization of data, the student progressively developed not only her ability to classify systematically but more significantly, her ability to accompany the classification by verbal argumentation showing, for example, that the classification is indeed complete. The authors conclude that the student's interest in justifying arose out of her idea that mathematics should make sense.

Zack (1997) analyzed the work of a team of fifth grade students who considered patterns in a counting problem. They used what they knew of the patterns to refute arguments by other teams. Zack found evidence of conjecture, refutation, generalization, and aspects of proving.

In teaching a fifth grade classroom, Lampert (1990) consciously and systematically initiated and supported social interactions appropriate to making mathematical arguments. As a result, her students began 'to make assertions that were based on their inductive observation of patterns and to move back and forth between these observations and deductive arguments about why the patterns would continue, even beyond the numbers they had tested' (p. 49). She concluded that classrooms can be led in such a way that 'in [students'] talk about mathematics, reasoning and mathematical argument – not the teacher or the textbook – are the primary source of an idea's legitimacy' (p. 34).

These reports appear to show that, in terms of deductive argumentation, fifth graders may show as much ability as college students. One has to keep in mind, though, that the elementary school children were observed in classes carefully planned and taught so as to support mathematical reasoning, argument and justification. Therefore, the studies only show that the transition to deductive reasoning is possible, not that it normally happens. And the studies at the high school and college level show that it often does not happen. Much of the remainder of this paper, in particular the next section, will discuss reasons why it does not.

At the most general level, the reason is obviously that most students never learned what counts as a mathematical argument. Although this sounds trivial, it isn't: Yackel and Cobb (1996) have coined the term 'sociomathematical norms' in order to discuss how environmental influences (teachers, classroom activity, . . .) determine students' mathematical beliefs and activity in the framework of a class or course; for example, '. . . what counts as an acceptable mathematical explanation and justification is a sociomathematical norm' (p. 461). Yackel and Cobb show how, in a second grade classroom, teacher and students interactively and consciously constitute sociomathematical norms regulating mathematical argumentation.

College textbook authors and teachers are rarely conscious of the need to establish sociomathematical norms, and their actions are often more apt to confuse rather than help students. College students do not usually read mathematics research papers, or see research mathematicians in action. But they do listen to lectures and participate in exercise sessions; they see and experience the talk and actions by their teachers; they read textbooks; they hand in assignments and tests, and they consider the grader's remarks when they receive them back; their mathematical behavior is shaped, consciously or subconsciously, by these influences. In the next section, I will present a number of examples which I consider symptomatic if not typical, and which might contribute to students' difficulties with explanation and proof. No systematic analysis of textbooks has been carried out; but it is at least conceivable that the given examples are the norm rather than isolated cases.

4. TEXTBOOKS AND CLASSROOM TEACHING

The examples which follow should by no means be seen as a critique of the experiences students are subjected to, but simply as a description. They illustrate introductory college and university courses, including service courses. They are likely to be inappropriate for advanced mathematics courses and for transition courses which have been instituted in some places specifically to help beginning mathematics majors make the transition to formal argument (Hillel and Alvarez, 1996).

In many textbooks used at the level under consideration, more or less formal arguments are used, together with visual or intuitive justifications, generic examples, and naive induction. Even the formal arguments are often only formal in appearance. But more importantly, students are rarely if ever given any indications whether mathematics distinguishes between these forms of argumentation or whether they are all equally acceptable.

For example, what does a textbook author expect when he asks students in one exercise to 'show that', and in the next one to 'show by example that' (Anton, 1994). What could and should the student conclude about the expectations author and teacher have in tasks such as Example 6 of Section 2? Indeed, a considerable number of students answered that problem by writing down a specific 2 by 3 matrix for A, a specific 3 by 2 matrix for B, computing AB and BA and, possibly, adding: 'You see!'

A very thoughtfully written recent linear algebra textbook is the one by Lay (1994). It stresses connections between the various concepts and methods usually taught in elementary linear algebra courses wherever possible. But it does not help the student distinguish different forms of justificative argument. For example, after computing the determinant of a 5 by 5 matrix with only two non-zero elements below the diagonal, the author states 'The matrix in Example 3 was nearly triangular. The method in that example is easily adapted to prove the following theorem. THEOREM 2. If A is a triangular matrix, then det A is the product of the entries in the main diagonal of A.' (p. 165) Isn't the student invited to conclude that a computational method, carried out for specific examples, counts as a proof?

Generic examples are used liberally in textbooks; but how often is their role clearly identified, and how often is the range of their genericity and thus their validity discussed?

Experimental and visual arguments are, of course, common in calculus textbooks. An example is Fraleigh (1990) who intuitively introduces the slope of the tangent as limit of the slope of the secant, complete with physical interpretations and with a computer program to compute the derivative, before discussing the notion of limit. The treatment includes statements such as 'The smaller the value of Δx (of course $\Delta x = 0$ is not allowed), the better you would expect m_{sec} in Eq. (1) to approximate m_{tan}.' (p. 30). This claim is visually supported by a graph with two secants for which the statement is true; it is true in the example at hand but false in general! More precisely, it is false, in general, that the smaller the value of Δx, the better m_{sec} approximates m_{tan}. The qualification 'you would expect' is astutely placed. How would we as teachers react to a student explanation, similar to the ones in Section 2, in which a wrong statement is accompanied by an astutely placed ‘you would expect’?

In the next section of Fraleigh’s book, the ϵ-δ-definition of limit is introduced intuitively. Later in the book, students are expected to produce proofs, for example to ‘Show that a sequence can’t converge to two different limits’ (Exercise 19, in Section 10.1) and to give ϵ-N-proofs for the convergence of sequences. The transition between the intuitive and the formal stages is not clearly marked. Can and should students be expected to establish the distinction on their own?

And obviously, there is the ubiquitous ‘It is easy to see ...’ which is missing from few sources – textbooks and research papers alike; how could it fail to lead to explanations such as the following where AY ‘sees’ that two vectors span R^3.

Example.
Find the matrix A of the linear transformation T, and determine whether T is onto and/or 1-1-valued:
T: $R^2 \to R^3$, with

$$\mathrm{T}(e_1) = \begin{bmatrix} 3 \\ -6 \\ 0 \end{bmatrix} \qquad \mathrm{T}(e_2) = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

AY: [correctly constructs T and then continues:] One can see that T spans R^3 and therefore the transformation is ‘onto’.

In the above cases, choices have presumably been made intentionally and on the basis of didactic considerations by the authors. In other cases, choices may be less obvious and the reasons for them less conscious, even to authors and teachers. For example, many theorems usually taught in calculus courses including the mean value theorem, are based on the completeness of the real numbers. This is usually neither explicitly assumed in calculus textbooks, nor even discussed but taken as an intuitive fait acquis. Many calculus students hardly distinguish between rational and real numbers. Intuitively, the rationals are as complete to them as the reals (Bronner, 1997).

The previous paragraph concerns axiomatics, and one may make the point that mathematicians are often not explicit about their use of axioms, even in research papers. A similar point cannot be made, however, about the next issue, circularity of argumentation. Learning, even in mathematics, often proceeds in an order quite different from the logical one. We learn by establishing connections and relationships, by building a web of ideas rather than a linear and logical sequence of implications; ideas grow synergetically rather than strictly on top of each other. Thus many dilemmas about precedence arise for teachers and textbook authors, for example the dilemma whether to introduce limits before or after derivatives in calculus; the tensions between experimental and rigorous reasoning pointed out above for one calculus text, are common precisely because they are, at least partly, an effect of this dilemma.

This and similar dilemmas lead to circular reasoning on a global level which is usually not easy to identify. Circular reasoning in teaching occurs, however, also at the level of detailed and seemingly rigorous argument, such as the derivation of the important result that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. As pointed out by Richman (1993), this result is equivalent to the inequality sin x $<$ x $<$ tan x (in an appropriate interval) and this inequality is usually established on the basis of an argument which not only uses visual information in a crucial way but also uses the fact that the area of the unit circle is π; and the area of the unit circle, in turn, is usually established using the exact same inequality that sin x $<$ x $<$ tan x (unless it is taken as given on the authority of the elementary school teacher who happened to teach the students.) Do we, as teachers, have a case in criticizing students like AM (Example 5 in Section 2) for failing to well order his reasons?

Visual, intuitive, generic, experimental, and even circular justifications appear to be common in textbooks. It seems safe to assume that reasoning presented in class is usually less formal than that presented in textbooks. This is certainly so for most classrooms I have visited, including my own. I will illustrate this only by one sequence of events which repeatedly occurred in my own classes before I realized that it is historically documented and may well have occurred similarly in the majority of calculus classrooms taught all over the world during the past 100 years.

Almost every time I taught applications of integration to compute volumes and surfaces, one or two students came up with a variant of the following question. When computing the volume of a rotationally symmetric solid, one builds the corresponding integral after slicing the solid by planes perpendicular to the axis of rotation and approximating the volume of each slice by means of the volume of a 'straight' cylindrical slice of appropriate radius; why is it that the same method gives a wrong result when computing the surface area of such a solid or the length of an arc. (Slanted, conical slices have to be used to obtain the correct surface area, and slanted line segments to obtain the correct arc length.)

The answer I gave was a somewhat vague argument about the remainder (the volume being cut off and neglected) converging to zero in the three-dimensional case but not in the two-dimensional case. Whether the students were satisfied with this answer or not, I don't know – they did not return. Whether I was satisfied, however, I do know. I was not – but I was never able to find a better answer which would still be accessible to the students. And only much later did I realize that I was in very illustrious company. A remark by Young (1969, p. 152) drew my attention to Lebesgue's (1963) booklet 'En marge du calcul des variations' ['At the margin of the Calculus of Variations'] which was probably written in the 1920s. In it, Lebesgue recounts how strongly he has been influenced by an argument which purports to show that the length of one side of the triangle ABC is equal to the sum of the lengths of the two other sides; the argument proceeds by constructing a sequence of broken lines, each of length AB+AC, which approaches BC. Lebesgue writes: 'Tous mes travaux se rattachent à une plaisanterie de collégiens. Au Collège de Beauvais, nous démontrions que, dans un triangle, un côté est égal à la somme des deux autres. ... Mes camarades ne voyaient là qu'une bonne plaisanterie. Pour moi, ce raisonnement m'a paru extrêmement troublant, car je ne voyais aucune différence entre lui et les démonstrations relatives aux aires et surfaces des cylindres, cônes, sphères ou à la longueur de la circonférence' (p. 308). ['All my work is connected to a school boys' joke. At the Collège de Beauvais we proved that in a triangle, one side is equal to the sum of the two others. ... My friends considered this simply a joke. But to me, the argument was extremely troublesome because I did not see any difference between it and the proofs about areas and surfaces of cylinders, cones, spheres or the length of the circumference.'] The high school question under what conditions the length of a curve is the limit of the lengths of infinitely close curves has thus played a central role in Lebesgue's mathematical career.

Most of our students are not precisely like Lebesgue but rather like his classmates. They lack his ability to ask why a mathematical argument is or is not valid. And explanations like the one I gave to those of my students who came up with an excellent question, do little to help them acquire this ability, quite the contrary. I need to ask myself what the status of my explanation for the students is. Why did they accept it? And what does this imply for their willingness to continue to be critical with respect to the mathematical arguments I present? And finally, what right do I have not to accept an argument which is similar in style but given by one of them as answer to my assignment or examination question?

5. Theoretical approaches

Up to this point, we have avoided the theoretical question what constitutes an explanation, and what is a satisfactory, or an acceptable explanation.

We have also avoided a clear distinction between explanation, argument and proof.

There are reasons for avoiding such distinctions, and for stressing commonalties rather than differences. Hanna (1995) has made the point that 'While in mathematical practice the main function of proof is justification and verification, its main function in mathematics education is surely that of explanation' (p. 47). Others require proof to go beyond explanation. For example, when Ellen (Moore, 1994) was asked in a test to 'Prove that if A and B are sets satisfying $A \cap B = A$, then $A \cup B = B$', she wrote: '$A \cap B = A$ says – by definition of intersection – that the members of A and the members of B that are the same are all the members of A. Therefore by definition of subset $A \subseteq B$. If A is a subset of B, all of its members are contained in B. When there is a union of a set and it's subset the union then includes the whole set. Therefore $A \cup B = B$' (p. 258). Moore's interpretation is that 'In contrast to the professor's expectation, Ellen's proof was based on her intuitive understanding . . . she did not use the language and rules of inference that had been agreed upon in class. . . . Ellen needed to go beyond merely giving an explanation . . .' (p. 259).

Moore thus raises the question of the relationship between explanation, proof and understanding. This same question led Sierpinska (1994) to analyze the epistemological differences between explanation and proof in the light of their role in the process of understanding mathematics. Her analysis, based on work of the philosopher Ajdukiewicz, acknowledges '. . . a close relationship between proving and explaining. Both when proving a theorem and when explaining a state of things we answer to one and the same 'why?' question' (p. 74). She identifies, however, a few important differences. The first is that 'proof aims at increasing the degree of firmness with which we accept a fact as basis for our understanding' (p. 75) whereas explanation 'does not serve as basis for our more positive acceptance of the derived statement'. The second difference is that explanations use examples, models, visualizations and similar means in order to express something about mathematics; explanatory discourse is more metamathematical than mathematical; it may, for example, include reasons why a certain fact is significant in mathematics, something which is clearly beyond the realm of a proof. In this sense, explanation goes beyond proof. Similarly, a proof may call for an explanation which highlights the central idea of the proof. Proof and explanation are thus interwoven in processes of understanding.

Duval (1992–93) takes a similar approach, in that he also uses epistemological and cognitive analysis. He distinguishes three forms of justification: explanation, argument and proof. Two criteria determine the acceptability of the reasons given to justify a claim: their pertinence (semantic coverage between reasons and claim) and their strength (resistance to objections). According to Duval, the main function of an explanation is descriptive; its aim is to produce reasons. Arguments and proofs, on the other hand, examine the strength of these reasons, in particular whether they are free of contradictions; their function is to determine and possibly change the truth value of the claim.

Whereas Sierpinska groups argument and proof together, Duval distinguishes them by their pertinence: In arguments, the semantic content of the reasons is important and determines the epistemic value of the claim; proofs, on the other hand, are detached from content; they must be valid rather than pertinent; the status of a proof, rather than its content, determines its epistemic value. As a consequence, the language of arguments is more natural than that of proofs; the discourse of arguments is globally and thematically coherent. Nevertheless, Duval states that there are no criteria that definitely distinguish arguments from either explanations or proofs. At best, one can use characteristics such as coherence of discourse.

In summary, for mathematics educators there appears to be a continuum reaching from explanation via argument and justification to proof, and the distinctions between the categories are not sharp. Maybe surprisingly, questions as to what constitutes a proof can be and occasionally are asked in mathematics itself. In a review of the history of proof, Kleiner (1991) identified two major themes, namely that the validity of a proof is a reflection of the overall mathematical climate of the time, and that transitions in both directions, from less to more or from more to less rigor usually had good mathematical reasons. Even contemporaries did not always agree on what does and what does not constitute a proof; in particular, formalists and intuitionists were irreconcilably divided over the legitimacy of non-constructive existence proofs at the beginning of the 20th century.

Even more fundamentally, Ernest (1999) notes that 'there is growing recognition that proofs do not follow the explicit rules of mathematical logic, and that acceptance is instead a fundamentally social act'. This recognition is based on recent work by philosophers of mathematics including Lakatos (1978) and Kitcher (1984).

Lakatos (1978) has analyzed the development of analysis at the beginning of the eighteenth century, and in particular Cauchy's proof that the limit of a converging sequence of continuous functions is continuous. How could Cauchy prove this, and publish and maintain his proof, in spite of the fact that he was very well aware of the fact that the limit of a Fourier series can be a step function? It is interesting to learn that according to Cauchy, the sequence does not converge at the jumps; according to Fourier, the limit (step) function is continuous; and according to Abel, Cauchy's 'theorem' had exceptions. Lakatos showed that it took almost 30 years until Cauchy's contemporaries sorted things out sufficiently to see that the confusion arose because their conceptions of the underlying notions, limit and continuity, were not yet sufficiently developed.

Cauchy's argument is convincing; it used the following convergence criterion for series: 'It is sufficient for convergence that, for infinitely large values of the number n, the partial sums S(n), S(n+1), ... differ from the limit S and consequently among themselves by an infinitely small quantity'. This formulation not only uses infinitesimals and infinities in a manner that today makes us raise eyebrows but it is also unclear about the logical status of the crucial phrase 'among themselves'. Example 6 in Section 2 uses a very similar criterion and is equally convincing; it has the additional advantage that the result is true whereas Cauchy's is not, at least not from today's point of view.

In view of these and other recent challenges to the role and status of proof in mathematics (Hanna, 1995; Fallis, 1996; Velleman, 1997) one may legitimately raise the question what, then, do mathematicians consider to be a proof? In an imaginary dialogue between a mathematics professor – the Ideal Mathematician – and a philosophy student who came to ask him what a proof is, Davis and Hersh (1981) convey that the mathematician might recognize a proof when (s)he sees one but is unable to define it or even to improve its description beyond 'Well, it's an argument that convinces someone who knows the subject' (p. 40). The problems inherent in this description are well underlined by the fact the (inadequate) arguments in Examples 7 and 8 (Section 2) did, at least for some time, convince the experts!

Two tasks arise from the theoretical analysis: As didacticians, we must sharpen our awareness of the distinctions between explanation, argument and proof, and we must reflect on what we can and what we should expect from students in different age groups, levels and courses. And as teachers, we must attempt the difficult task of helping students to understand what we expect from them. The examples in Section 2 provide ample room for questioning what is expected by the different formulations used, including 'explain' (Examples 1, 2), 'justify' (Example 3), 'prove' (Example 5), and 'show that' (Example 6). Does 'show that' mean 'formally prove' or 'use an example to demonstrate that' (or something intermediate between these two)? Does 'explain' mean explain to a fellow student or explain in such a way as to convince the teacher that you understand the reasoning behind the claim?

6. STUDENTS' EXPLANATIONS: WHAT (NOT) TO EXPECT

The two preceding papers in this special issue on forms of knowledge implicitly contribute their share to our understanding why a large part of students' knowledge is not of the kind which supports mathematical justifications: According to Ernest, much of our students' mathematical knowledge is tacit; and while tacit knowledge is likely to be used correctly in applications, it cannot be used explicitly in reasoning. Mason and Spence, on the other hand, show that even students' explicit mathematical knowledge is, to a large extent, not deductive but inductive, abductive or generalized from experience.

As shown in Section 4, teachers and textbooks make extensive use of a great variety of forms of knowledge, and for good reasons. The opportunity to acquire knowledge in a variety of forms, and to establish connections between different forms of knowledge are apt to contribute to the flexibility of students' thinking (Dreyfus and Eisenberg, 1996). The same variety, however, also tends to blur students' appreciation of the difference in status which different means of establishing mathematical knowledge bestow upon that knowledge.

It thus appears that, at least in some measure, the task of learning and teaching mathematical justification conflicts with the pursuit of learning and teaching mathematical relationships, concepts and procedures in a flexible manner. Kline, from whose book (Kline, 1973) I have adapted the title of this paper, has argued convincingly that making logic the guiding principle of curriculum design, as tried by the New Math movement, does not solve this (nor any other) problem. And while recognizing proof as the hallmark of mathematics, he has strongly argued that its place is at the end rather than at the beginning, and that even in proof, rigor should play a lesser role than motivation: 'In no case should one start with the deductive approach, even after students have come to know what this means. The deductive proof is the final step. ... [The student] should be allowed to accept and use any facts that are so obvious to him that he does not realize he is using them. ... Proofs of whatever nature should be invoked only where the students think they are required. The proof is meaningful when it answers the student's doubts, when it proves what is not obvious.' (p. 195).

So where does this leave the students? They have few if any means to distinguish between different forms of reasoning and to appreciate the consequences for the resulting knowledge; nor can they be expected to distinguish between explanation, argument and proof (Section 5). And what means do they have to judge the validity of mathematical arguments? Even for mathematicians it is not always clear cut what a proof is, both philosophically and practically; we should therefore not be surprised if students find it difficult to make such judgments even at the level of the simple, short proofs likely to appear in high school and college classrooms. Hence, there is little reason to be surprised at the findings presented in Section 3 which rather clearly show that most students have at best a very vague notion of what constitutes a mathematical proof.

In spite of this, many teachers, including this author, frequently request students to explain their reasoning, show why a statement is true, justify a claim or even prove a result. What can we realistically expect from students when we ask them to 'explain why', when we ask them to construct an argument? What criteria do we have to judge their productions, and what criteria can we use with good conscience? Is it even realistic to expect high school and college teachers to make judgments and decisions as to whether or not their students' mathematical arguments are acceptable, and to make such judgments in real time in a classroom?

Teachers have to decide how to relate to experimentally based reasoning and to visually based reasoning, and have to adapt their reaction to whether such reasonings are presented as justifications, as explanations or as basis for conjectures. And under what circumstances should one accept a student's justification based on: 'Because my teacher said so'; or 'I can just see it'? My personal experience reported in Section 4 may well lead to a 'because the teacher said so'; and we have all experienced students 'seeing' blatantly wrong things. But what if the conclusion which the student 'can see' is correct? What about a student who 'can see' that 26/65 equals 2/5? Maybe the student 'can see' the digit 6 disappear from numerator and denominator? So correctness of the answer is not the issue, certainly not the main issue.

On the other hand teachers are often willing to accept the slightest sign of a student's understanding as satisfactory explanation, even if the student's words and actions leave much to be desired. The teacher may recognize that the student has more or less consciously established some connections between what is given and what is to be justified. In other cases, visual reasoning can be deep and go far beyond vaguely seeing some connections. The term 'visual reasoning' is used here to refer to arguments based on analysis of a diagrammatic situation (Dreyfus, 1994). Visual reasoning is often analytic in the sense that the thinking subject consciously analyzes the visual images, and reflects on them. Such reasoning may include analyzing, acting on and transforming images, mental or external ones, and drawing conclusions about mathematical relationships from these actions. It is apt to underlie detailed justifications of mathematical statements, even rigorous proofs (Barwise and Etchemendy, 1995).

The question under what conditions, and according to which criteria, visually based explanations can and should be accepted has received little attention and it is thus left up to the individual teacher to intuitively decide on a case to case basis.

The situation is similar with respect to experimentally based reasoning – and the reference is not to the use of experiment in exploration, and in the generation of conjectures but in students' justificative explanations. Should a teacher accept the argument that a sequence converges because numerical experiment shows so? Or rather, in which situations should the teacher accept such an argument? And if not, what arguments are acceptable? Is a Cauchy type argument such as in Example 4 of Section 2 preferable? Why? And when? Should students be allowed to use infinitesimals in a similar manner as Cauchy did? Why or why not? How about a visual argument showing how successive elements of the sequence come closer to each other? How close does such a visual argument need to mimic a proof in the Weierstrass (epsilon-delta) sense? And to what extent, in what respect, does an epsilon-delta argument reveal more (or less) about a student's understanding than an experimental one?

Just as for visual arguments, the question arises under what conditions experimentally based explanations can and should be accepted. What criteria can and should a teacher apply and what general considerations can help mathematics educators and teachers to establish such criteria?

In conclusion, the requirement to explain and justify their reasoning requires students to make the difficult transition from a computational view of mathematics to a view that conceives of mathematics as a field of intricately related structures. This implies acquiring new attitudes and conceiving of new tasks: The central question changes from 'What is the result?' to 'Is it true that . . .?'. Students thus need to develop new and more sophisticated forms of knowledge.

Although it has been known for some time how complex and difficult this transition is, only a few attempts to directly deal with it have been reported in the literature (e.g., Movshowitz-Hadar, 1988; Dreyfus and Hadas, 1996), and even these have made little or no attempt to assess changes in students' views of mathematics and their ability to explain and justify. The question how to sensitize students to this change and help them achieve it, remains open.

Of equal importance, and equally open is the development of criteria which can be used by teachers to judge the acceptability of their students' mathematical arguments, and of principles on which the development and examination of such criteria can be based.

REFERENCES

Alibert, D. and Thomas, M.: 1991, 'Research on mathematical proof', in D. Tall (ed.), *Advanced Mathematical Thinking,* Kluwer, Dordrecht, The Netherlands, pp. 215–230.

Anton, H.: 1994, *Elementary Linear Algebra,* Wiley, New York, USA.

Arnold, V. I.: 1991, 'A Mathematical Trivium', *Russian Mathematical Surveys* 46(1), 271–278.

Balacheff, N.: 1987, 'Processus de preuve et situations de validation', *Educational Studies in Mathematics* 18(2), 147–176.

Barwise, J. and Etchemendy, J.: 1995, *Hyperproof,* CSLI Lecture Notes, No. 42, Stanford University, Stanford, CA, USA.

Bronner, A.: 1997, 'Les rapports d'enseignants de troisième et de seconde aux objets 'nombre réel' et 'racine carrée', *Recherches en didactique des mathématiques* 17(3), 55–80.

Cauchy, A.-L.: 1821, *Cours d'analyse de l'école royale polytechnique*, de Bure, Paris, France.

Coe, R. and Ruthven, K.: 1994, 'Proof practices and constructs of advanced mathematics students', *British Educational Research Journal* 20(1), 41–53.

Davis, P. J. and Hersh, R.: 1981, *The Mathematical Experience,* Birkhäuser, Boston.

Dreyfus, T.: 1994, 'Imagery and reasoning in mathematics and mathematics education', in D. Robitaille, D. Wheeler and C. Kieran (eds.), *Selected Lectures from the 7th International Congress on Mathematical Education,* Les presses de l'université Laval, Sainte-Foy, Québec, Canada, pp. 107–122.

Dreyfus, T.: in press, 'La demostracion a lo largo del curriculum' ('Experiencing proof throughout the curriculum'), in J. Deulofeu and N. Gorgorio (eds.), *Educación matemática: retos y cambios desde una perspectiva internacional*, ICE/GRAO pub., Barcelona, Spain.

Dreyfus, T. and Eisenberg, T.: 1996, 'On different facets of mathematical thinking', in R. J. Sternberg and T. Ben-Zeev (eds.), *The Nature of Mathematical Thinking*, Lawrence Erlbaum, Mahwah, NJ, USA.

Dreyfus, T. and Hadas, N.: 1996, 'Proof as answer to the question why', *Zentralblatt für Didaktik der Mathematik* 28(1), 1–5.

Duval, R.: 1992–1993, 'Argumenter, démontrer, expliquer: continuité ou rupture cognitive?', *petit x* 31, 37–61.

Ernest, P.: 1999, 'Forms of knowledge in mathematics and mathematics education: Philosophical and rhetorical perspectives', *Educational Studies in Mathematics* 38, 67–83.

Fallis, D.: 1996, 'Mathematical Proof and the Reliability of DNA Evidence', *American Mathematical Monthly* 103(6), 491–497.

Ferrari, P. L.: 1997, 'Action-based strategies in advanced algebraic problem solving', in E. Pehkonen (ed.), *Proceedings of the Twenty-First International Conference on the Psychology of Mathematics Education,* University of Helsinki, Finland, Vol. 2, pp. 257–264.

Finlow-Bates, K., Lerman, S. and Morgan, C.: 1993, 'A survey of current concepts of proof help by first year mathematics students', in I. Hirabayashi, N. Nohda, K. Shigematsu and F.-L. Lin (eds.), *Proceedings of the Seventeenth International Conference on the Psychology of Mathematics Education,* University of Tsukuba, Japan, Vol. I, pp. 252–259.

Fischbein, E.: 1982, 'Intuition and proof', *For the Learning of Mathematics* 3(2), 9–24.

Fraleigh, J. B.: 1990, *Calculus with Analytic Geometry,* Addison-Wesley, Menlo Park, CA, USA.

Hanna, G.: 1995, 'Challenges to the importance of proof', *For the Learning of Mathematics* 15(3), 42–49.

Harel, G. and Sowder, L.: 1998, 'Students' proof schemes: Results from Exploratory Studies', in E. Dubinsky, A. H. Schoenfeld and J. J. Kaput (eds.), *Research on Collegiate Mathematics Education*, Vol. III, American Mathematical Society, Providence, RI, USA, pp. 234–283.

Hillel, J. and Alvarez, J. C.: 1996, 'University mathematics, report of topic group 3′, in C. Alsina, J. Alvarez, M. Niss, A. Perez, L. Rico and A. Sfard (eds.), *Proceedings of the 8th International Congress on Mathematical Education*, S.A.E.M. 'Thales', Sevilla, pp. 245–248.

Kitcher, P.: 1984, *The Nature of Mathematical Knowledge,* Oxford University Press, New York, USA.

Kleiner, I.: 1991, 'Rigor and proof in mathematics: a historical perspective', *Mathematics Magazine* 64(5), 291–314.

Kline, M.: 1973, *Why Johnny Can't Add*, Random House, New York, USA.

Lakatos, I.: 1978, 'Cauchy and the continuum', *The Mathematical Intelligencer* 1(3), 151–161.

Lampert, M.: 1990, 'When the problem is not the question and the solution is not the answer: mathematical knowing and teaching', *American Educational Research Journal* 27(1), 29–63.

Lay, D. C.: 1994, *Linear Algebra and Its Applications*, Addison-Wesley, Reading, MA, USA.

Lebesgue, H.: 1963, En marge du calcul des variations', *L'enseignement mathématique* 9(4), 212–326.

Maher, C. A. and Martino, A. M.: 1996, 'The development of the idea of mathematical proof: a 5-year case study', *Journal for Research in Mathematics Education* 27(2), 194–214.

Margolinas, C.: 1992, 'Eléments pour l'analyse du rôle du maître: les phases de conclusion', *Recherches en Didactique des Mathématiques* 12(1), 113–158.

Martin, G. and Harel, G.: 1989, 'Proof frames of preservice elementary teachers', *Journal for Research in Mathematics Education* 20(1), 41–51.

Mason, J. and Spence, M.: 1999, 'Beyond mere knowledge of mathematics: The importance of Knowing-to Act in the moment', *Education Studies in Mathematics* 38, 135–161.

Moore, R. C.: 1994, 'Making the transition to formal proof', *Educational Studies in Mathematics* 27(3), 249–266.

Movshowitz-Hadar, N.: 1988, 'Stimulating presentation of theorems followed by responsive proofs', *For the Learning of Mathematics* 8(2), 12–19.

Richman, F.: 1993, 'A Circular Argument', *The College Mathematics Journal* 24(2), 160–162.

Sierpinska, A.: 1994, *Understanding in Mathematics,* Falmer Press, London, UK.

Silver, E. A.: 1994, 'Mathematical thinking and reasoning for all students: moving from rhetoric to reality', in D. Robitaille, D. Wheeler and C. Kieran (eds.), *Selected Lectures from the 7th International Congress on Mathematical Education,* Les presses de l'université Laval, Sainte-Foy, Québec, Canada, pp. 311–326.

Velleman, D. J.: 1997, 'Fermat's last theorem and Hilbert's program', *The Mathematical Intelligencer* 19(1), 64–67.

Vinner, S.: 1997, 'The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning', *Educational Studies in Mathematics* 34(2), 97–129.

Yackel, E. and Cobb, P.: 1996, 'Sociomathematical norms, argumentation, and autonomy in mathematics', *Journal for Research in Mathematics Education* 27(4), 458–477.

Young, L. C.: 1969, *Calculus of Variations and Optimal Control Theory*, Saunders, Philadelphia, USA.

Zack, V.: 1997, ' "You have to prove us wrong": Proof at the elementary school level', in E. Pehkonen (ed.), *Proceedings of the Twenty-First International Conference on the Psychology of Mathematics Education,* University of Helsinki, Finland, Vol. 4, pp. 291–298.

Center for Technological Education, Holon
(affiliated with Tel Aviv University)
P.O. Box 305, Holon 58102, Israel
E-mail: nttommy@weizmann.weizmann.ac.il