Arbitrary and Necessary: Part 3 Educating Awareness

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David Wheeler and I shared a common influence: Caleb Gattegno. David knew Gattegno through the early years of the Association of Teachers of Mathematics, ATM, in the U.K. and then spent a brief period of time working with Gattegno in New York City before going to Concordia University in Montréal, Canada in 1975. I met Gattegno on many occasions during the 1980s in the U.K. at seminars held in Bristol and London. As I look back, I find it surprising that I did not meet David more often. Despite this, I felt a common language and kinship whenever we did meet, and I continued to feel David's influence on me through contact with mutual friends and colleagues in the mathematics education world.

However, it was through the power and style of his writing that I was to be most affected. The articles in the ATM journal, Mathematics Teaching, *entitled 'The role of the teacher' (MT number 50, 1970, 23-29), and the write-up of his lecture at an ATM conference (MT number 71, 1975, 4-9) entitled 'Humanising mathematical education' brought a clear focus on seeing mathematics as a human activity rather than a collection of topics on a syllabus and focusing on the role a teacher has in developing that mathematical activity. These articles had a considerable impact on me when I came to read them for the first time at a relatively young teacher in the 1980s, and the power in those articles remains with me as a re-read them today.*

The last time I saw David was at the Canadian Mathematics Education Study Group in 1998 in Vancouver. He gave a talk entitled 'The commonsense of teaching' (CMESG Proceedings, pp. 93-99), where he highlighted the relatively little attention given to the 'technical aspects' of teaching. Whatever meaning David may have had for this phrase, I hope that my third and final article in this sequence goes some way toward giving attention to possible techniques a teacher might employ, and that it also recognises students as people who already operate mathematically as a consequence of their everyday human activity.

1. Curriculum divide and working on what is necessary

In part 1 (Hewitt, 1999), I introduced a divide in the mathematics curriculum between those things which a learner can never know for sure without being informed by some external means, such as through a teacher, textbook or the internet, and those things which some students can know for sure without being informed from some external source. The former are names and conventions which have been socially agreed some time in the past and adopted within the mathematics community. I describe these as *arbitrary* since they can feel arbitrary for a learner ('Why is it this when it could just as well be that?'). The latter are properties and relationships which can be known to be *necessary* consequences of certain mathematical scenarios. The arbitrary, by its very nature, lies in the realm of memory. I restrict my use of the term 'memorisation' to describe a conscious process where someone, *at the time of receiving information and thereafter*, activates techniques to hold that information so that it will be available at a future time.

In part 2 (Hewitt, 2001), I addressed the pedagogic challenge of assisting students in their task of memorising the arbitrary and argued that although this is not the area where mathematics lies, there is an important and significant task to be done so that students can see and use the power of naming and adopting conventions whilst engaged in their work on where mathematics does lie - with what is necessary. In particular, I explored the fact that although a teacher's role is to inform a student of the arbitrary, there are different times when, and ways in which, this informing can take place and these have consequences for students' learning.

This article, the third and final part of this trilogy, considers the necessary, which lies in the realm of awareness. Even though some students can come to know something which is necessary without being informed by external means, there are some teachers who regularly inform students of the necessary. For example, Pythagoras' Theorem may be stated by a teacher, in which case a student is left with either having to accept its truth on the basis of trusting the teacher, or has to do some work themselves to work out why this theorem must be true.

In the first case, the theorem is treated as if it were arbitrary - 'This is something else I am telling you and that you will have to remember'. I describe such a situation as students getting 'received wisdom' from the teacher. This remains the case even if the teacher offers a proof - still the students are receiving the teacher's wisdom. The theorem and the proof remain in the realm of memory unless a student can use their own awareness to come to know why this theorem is true (if they are just presented with the theorem) or why this 'proof' proves that the theorem is true (if also offered a proof).

Pythagoras' Theorem only returns to its rightful place of being in the realm of awareness if a particular student uses their own awareness to be in a position where they know it must be true. What is significant within working on the necessary is that an individual student's awareness is educated rather than being purely a matter of whether that student can state and use a particular theorem. The former helps the latter, but the latter does not always help with the former. This article is concerned with considering practical ways in which a teacher can use and develop their sensitivity to the issue of educating their students' awarenesses

2. Implications for teaching: the realm of awareness

What is necessary can be known through awareness, a student does *not* have to be informed. If an aim is to reduce demands on memory, then all that is necessary needs to stay in the realm of awareness. So one reason to educate awareness is to reduce demands on memory However, there is also the positive reason which is that by educating awareness the mathematician inside a student is being educated, which would not be the case if everything were treated as if it were to be memorised Awareness informs decisions and how to act using information which is known. Memory only holds some information, not how to use that information in a novel situation (a situation which has not been memorised).

As Bruner (1960) remarked:

> Learning should not only take us somewhere; it should allow us later to go further more easily. (p. 17)

Memorising only takes us somewhere, it is by educating awareness that we can have the means to takes things further on our own accord and not be limited to reproducing only those things which we have been told

Claxton (1984) said that:

> The learning mechanism is fuelled not by energy but by awareness (p 53)

and that:

> I cannot learn if I am not attentive to, or aware of the success or failure of my actions at some, not necessarily conscious, level. (p. 45)

Becoming aware of the success or failure of actions provides an opportunity to educate the awareness which led to taking those actions in the first place.

The necessary *can* be known through awareness This does not mean that everybody will be able to know everything which is necessary. For example, my one-year-old daughter cannot know Pythagoras' Theorem at the moment through awareness However, there are many older students who can This is the reason why it is important for a teacher to be aware of that student's awareness, and what powers can be called upon (such as the power to abstract), so that a judgment can be made as to which properties are accessible for which students.

As Bruner (1966) commented:

> the sequence in which a learner encounters materials within a domain of knowledge affects the difficulty he will have in achieving mastery (p 49)

In other words, the accessibility or otherwise of a property for students depends upon the pedagogic approach taken by the teacher to that property. For example, the formula $\frac{n!}{r_1!r_2!...r_k!}$ for the number of permutations of n objects where there are k distinct objects, r_1 of one kind, r_2 of another kind, etc., is not considered to be accessible for most students until they study 'A'-level mathematics. Yet it is perfectly possible for a mixed-ability class of eleven-year-olds to derive this formula, where only the arbitrary notation is provided by the teacher

An approach I have taken in the past (see Hewitt *et al* (1992) for a description of the first of a sequence of lessons which lead to the development of this formula) was to ask the class to explore different ways in which the letters in someone's name can be re-arranged, initially working with a four-letter name where each letter is different, and working with awarenesses which arose about ways in which this task could be structured. Later, longer and shorter names were explored along with names which had repeated letters The formula came from students expressing the awarenesses they had come to about how to structure the counting of how many different permutations they got in the different scenarios.

3. What is awareness?

Gattegno (1987, p. 220) stated 'Only awareness is educable'. This provocative statement has created much discussion (for example, see Mason, 1987), and I have found myself examining the meaning of each of the words in this statement Throughout his extensive writing, and his equally extensive list of seminars which often lasted several days, to my knowledge Gattegno never defined his use of the word 'awareness' and it may well be the case that you will decide by the end of this article that I have done the same

I personally feel that Gattegno's and my use of the word are not that different from its everyday usage. As I sit by my computer writing this article, there are a number of things of which I can say that I am aware For example, I am aware, at the moment of typing this sentence, of some of the letters on my keyboard as I am not a touch typist and so do need to look at certain letters as I type I am also aware that I do not look for every letter that I type, the positions of some letters I know in relation to others and so do not need to look for them.

These two examples are of a different nature The first is an awareness which comes from the attention I place in my senses - in this case, my sight. Of course, there are many photons which hit my retina and so many things which I am seeing at some level; however, I was (at the time of typing that particular sentence) aware of only seeing certain letters For example, I was aware of frequently looking at the letter 'T'.

Awareness through my senses is guided by where I choose to place my attention. I could, for example, have become aware of the strain in my back after doing some heavy lifting the day before. My partner is frequently surprised how, for example, I could not comment on what someone was wearing earlier in the day. It is a matter of where I placed my attention whilst in their company At the same time, attention is guided by awareness, since following a conversation about a certain style of trousers, I found myself noticing people wearing that type of trousers. As Berger (1973) remarked:

> We only see what we look at To look is an act of choice. (p 8)

So there is a chicken-and-egg situation here; where I place my attention affects what I become aware of, and what I am already aware of affects where I place my attention

My second observation, that I was aware of not looking at every letter that I type and that the position of some letters

I know in relation to others, is a comment which leads to an awareness that awareness works at many levels. Firstly, my statement is a meta-level comment in that it is not something which is directly observable through my senses, such as becoming aware of seeing the letter 'T'. Thus, this is an awareness coming from *analysis* rather than a *description* of things which I was aware of through my senses.

Additionally, by entering into the kinesthetic movements of my fingers I was aware that my fingers appear to hold an awareness of where to move next when I am thinking at the level of sentences or ideas and so have my attention mainly somewhere different from the keyboard. Thus, I have awareness at an unconscious level as well - with the movement of my fingers. I wish to make a distinction here between automatisms which involve awareness at an unconscious level, and habits which are behavioural and which need to have awareness brought to them in order that they can be educated and so become changed. My finger movements when typing keys are informed, for example; I am not carrying out a habit of always going from one key to another.

So, awareness is present at the level of my senses, at a meta-level through conscious cognitive analysis, and also at the level of carrying out automated processes (which may at one time in the past have required both awareness at the level of my senses and awareness at the meta-level of analysis in order for them to become automated).

At a conference of Mathematics Advisors, Mike Askew related a story which caught for me an example of awareness being educated. A child was presented with two piles of four counters and was asked how many counters in all. He replied 'Eight' quite quickly. He appeared to know this without the need for counting. One counter was then moved from one pile to the other. When asked how many counters were there now in each pile he counted them and said 'Eight'. Another counter was moved to make piles of two and six. This time he said 'Eight' without counting. Another counter was moved and he said 'Eight, still eight'. At this point, he started rocking back and forward in his chair and biting his shirt. Mike reported that the boy appeared very excited.

For me, this story holds within it an example of awareness at the different levels I have discussed: the unconscious awareness of two fours making eight (provided he could, if called upon, account for why this is so); the conscious awareness involved with counting accurately; and the meta-level awareness of something staying the same whilst other things change. The story also reminds me that excitement can come from becoming aware that my awareness is being educated. The sense that I am a different person, that I have a different way of looking at things, that I have a different amount of control over things, is very exciting and is a result of educating my awareness. I believe this sense of personal growth to be the greatest and most effective form of motivation in the classroom, and yet I fear that many students experience it too rarely within their mathematics lessons.

4. Educating awareness

Educating awareness is not something I can do for someone else. The best I can do as a teacher is to use my awareness of pedagogy, the subject matter and the student, to make pedagogic decisions about what I do or do not offer. It is the student who must educate their own awareness. Sometimes this is helped by what a teacher offers, while at others well-intentioned offerings may not relate to a student's existing awareness and so little education takes place as a consequence.

For example, one teacher was offering an image of shading rectangles as an attempt to help students see that certain fractions were equivalent. He gave an example for $\frac{1}{4}=\frac{2}{8}$ as in Figure 1.

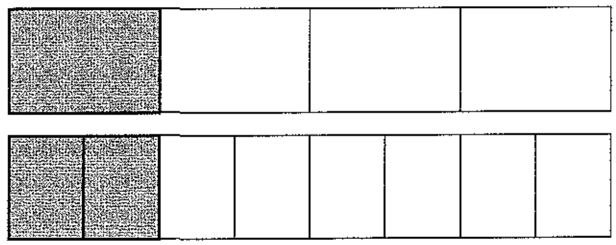

Figure 1 Teacher's shading for $\frac{1}{4}=\frac{2}{8}$

The class was given pairs of equivalent fractions and asked to draw a diagram to show why they are equivalent. Figures 2 and 3 show examples from two different students of their attempts at the first task set of $\frac{9}{12}=\frac{3}{4}$.

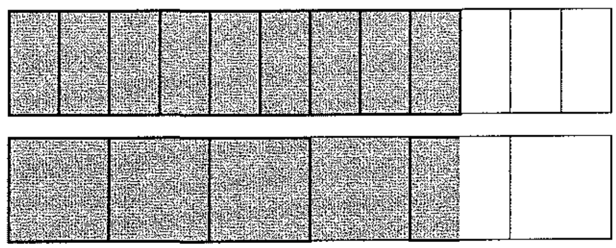

Figure 2 A girl's answer to $\frac{9}{12}=\frac{3}{4}$

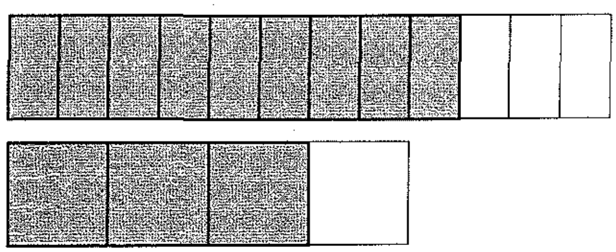

Figure 3 A boy's answer to $\frac{9}{12}=\frac{3}{4}$

These answers appear to show students trying to make sense of this task of shading rectangles. The story I have created for their answers is that both of the students had noticed the halving of the smaller rectangles in the teacher's example and considered that this was what was meant to be done in this task. Then they had different ways of making sense of the task The girl knew that the shaded areas were meant to be the same so made sure that they were. The boy knew that the second rectangle was meant to represent $\frac{3}{4}$ and so having shaded three parts knew that there was only one more to be left unshaded. Both of them brought their own particular awareness of fractions to the task, but the task itself seems to do little to help the students' awarenesses of fractions. Instead, *shading rectangles* seemed like a new and added topic in the mathematics curriculum rather than one which was meant to make the equivalence of fractions more accessible So my first consideration is to look at the link between teacher offerings and student awarenesses.

I use the phrase '*working with awareness*' to describe a process where a teacher is aware of possible student awarenesses and makes pedagogic decisions which relate to them So although teachers cannot directly work with someone else's awareness, they can make decisions which are sensitive and relate to that awareness To explore this further, consider an example where the following was written on the board: $\frac{1}{4} = \frac{}{12}$ and a bottom set of thirteen-to-forteen-year-olds were asked what should go above the 12 One student suggested *two* and the teacher responded by saying no, because that would be $\frac{1}{6}$ wouldn't it? Here the offering from the teacher seems to be about his awareness of the mathematics rather than this sensitivity to the awareness of the student. If a student believes that $\frac{2}{12}$ might equal $\frac{1}{4}$ then I would consider it unlikely that they would necessarily see that $\frac{2}{12}$ is the same as $\frac{1}{6}$ and that this is different from $\frac{1}{4}$

How might I work with the student's awareness? Well, I do not know what their awareness is from this reply, so I would have to do some work in order to have an idea. Such work might involve considering whether the student's response might be a habit carried over from previous questions (such as those previous questions involving a multiplication factor of two), or asking the student to say more about why they feel the answer is two.

Asking questions does not immediately imply that a teacher is working with awareness: for example, 'What is a parallelogram?' or 'Where is the point (1, 2)?' are questions which concern arbitrary matters and so primarily concern memory rather than awareness. Of course, everything calls upon awareness at some level: however, I am considering here situations where the student's awareness is stressed by a teacher in their considerations of how to act and what to offer in the classroom.

So, questions such as 'What is a quarter of two hundred and forty?' do concern awareness; however, awareness is rarely revealed in a student's response and so it is not appropriate to describe a teacher offering such a question as working with awareness A shift in the question from focusing on an answer to focusing on a process – 'How can I work out a quarter of two hundred and forty? – does give the possibility of awareness being revealed.

Follow up questions such as 'Why?' can help to reveal even more of a student's awareness After all, someone might know a recipe for doing such a calculation without an awareness of why that process works. Mason (1994) asked the question:

> How can you distinguish between the student who understands, and the student who has mastered sufficient rules to get mostly correct answers? (p 52)

I recall talking with some girls who were about to take their GCSE examinations and, in fact, went on to get a high grade in mathematics They were drawing straight-line graphs at the time and were drawing graphs very quickly by plotting three points and joining them up with a line. I asked them why they plotted three points and they said that two would do but the third one was a check as they should all lie on a line

I asked how they knew it was going to be a line. They were not sure about this. So, looking at a particular graph (Figure 4), I asked them whether the line stopped at the top of their page They thought about this and replied that it would carry on. So I asked whether it would carry on in a straight line or whether it might begin bending at some point. They took some time considering this and came to the opinion that the line was going to begin to bend when it got further up

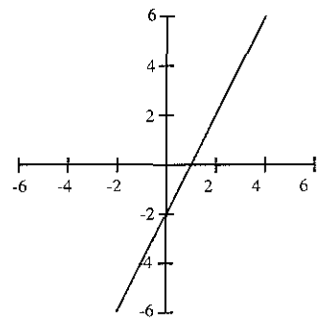

Figure 4 How does this graph of y = 2x-2 continue beyond the top of the drawing?

I subsequently asked students on other occasions this question and often found that students felt the line would bend as it carried on up beyond the page Invariably the direction of the bend would be consistent with the idea of the line being, for example, a fishing line bending under gravity and with gravity acting in the direction of down the page.

The first part of working with awareness is to have awareness revealed, in order that it can be worked with.

5. Opportunities for working with awareness

Opportunities for working on awareness arise through awareness being revealed by students. This can happen at any time during a lesson through an unexpected contribution by a student, or can happen through a planned task which is designed to create possibilities for students to reveal their awareness Since I am concerned with teaching and learning mathematics, as opposed to other subject areas, it is the students' awareness of *mathematics* that is of relevance

This may seem an obvious statement to make: however, I have been in many classrooms where the conversation in the room is mainly concerned with control issues, administration or non-mathematical descriptive statements. As an example of the latter, I recall arriving mid-year in a new school as a Head of Mathematics and finding my examination class students talking about 'Blue Book three' or 'Question 5' or 'I'm doing . . [and stating the title of the chapter]'. I found it extremely hard to get a comment from them that concerned their own awareness of the mathematics they were working on (Hewitt, 1987).

Awareness of mathematics is rarely revealed if a student is carrying out a routine task, such as re-producing a process again and again through an exercise, especially if their answers are correct. This is because awareness is what guides actions, and is not the actions themselves. So, if someone can add two fractions together successfully, I know little about whether they are following a mechanistic process described by a teacher (received wisdom) or whether they can articulate reasons why their answer is correct and why the process will inevitably lead to a correct answer.

Awareness is revealed not through actions themselves but through making explicit what is guiding those actions. It is this which needs to form the basis of conversation in the classroom. Unless this is done, there will be little awareness of mathematics revealed, and so few opportunities for a teacher to work with the students' awareness of mathematics.

A student's error offers an opportunity for a teacher to learn something about that student's awareness: for example, a shape was drawn on squared paper and a girl was counting the perimeter on the shape and wrote numbers around the outside of the shape (see Figure 5).

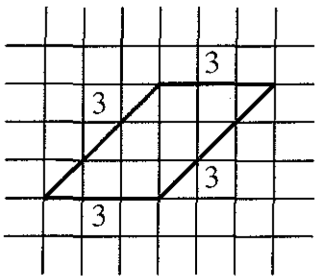

Figure 5 Working out the perimeter of a shape

The four threes are the result of her awareness, so a teacher cannot know just from this observation the awareness that student has concerning length and perimeter. However, a teacher also has awareness, and awareness not only about mathematics but also about students learning mathematics. A teacher can use their own awareness to make a judgement about that student's awareness of this mathematical situation.

For example, I have known many students treat the diagonal of a square as the same length as the side of a square, and so I develop a 'story' of what might have happened with this student around this notion. I have other stories about why this might happen, such as past experiences of trying to draw three-dimensional shapes on square grid paper (after all, how is someone going to draw a 3 x 3 x 3 cube on square grid paper?). At present, these are just stories: however, my experience as a teacher and educator might lead me to feel that certain stories are likely to be correct in such a situation (whilst remaining open to the possibility that they are not!). These will inform my decision about how to act as a teacher with this student, since I will be wanting to work with their awareness, not just the results of their awareness.

Opportunities to work on a student's awareness require a teacher to use their own awareness. Firstly, a teacher uses their awareness of mathematics (e.g. to notice an error) and secondly, that teacher has to use their awareness of teaching and learning issues, and of course the student, in order to work with the student's awareness. Without the second of these, a teacher may end up with a response such as 'No, that's wrong, those lengths are not three. You need to use Pythagoras'.

A response like this involves only the teacher's awareness of *mathematics*, and so such a teacher will fail to work with the student's awareness. The student will then be left on their own either to try to use the awareness they do have to work out why they were wrong and why Pythagoras is required (which some students may succeed in doing), or to give up trying to do this task through awareness and simply follow the teacher's instructions.

There are many unexpected events which happen during a lesson which are opportunities for a teacher to work on a student's awareness. Some events are completely unexpected, such as when a student teacher of mine asked her class what fraction was the same as 1 8, and got back 'Eight point one' and 'One and an eighth' from two of her students. She did not know what to do with these answers at the time and continued asking others until she got one of her desired answers of 'One and eight tenths', 'Eighteen tenths', 'One and four fifths', etc.

It takes a shift in thinking for a teacher to see the original responses as opportunities to work with student awareness rather than disappointing responses which interrupt the flow of a planned lesson. And opportunities they are indeed, since such errors are signals to a teacher indicating that there is something which needs work on. A few minutes given at this time to work on something which the student has revealed needs attention can be far more effective in terms of a student's learning than more time given at the convenience of a teacher's (or department's, or indeed state's or country's) planned curriculum.

Working with awareness may mean taking advantage of these opportunities when they arise and being prepared to put on hold the planned continuation of the lesson. In fact, planning lessons can take into account the possibility of such opportunities. Indeed, some errors might be expected and a response can be planned in advance. For example, a task [1] I devised early on in my teaching career involved students working in groups comparing areas of shapes on different grids. I found that many students made the assumption that two equilateral triangles forming a rhombus on the triangular grid had the same area as a square on the square grid (see Figure 6).

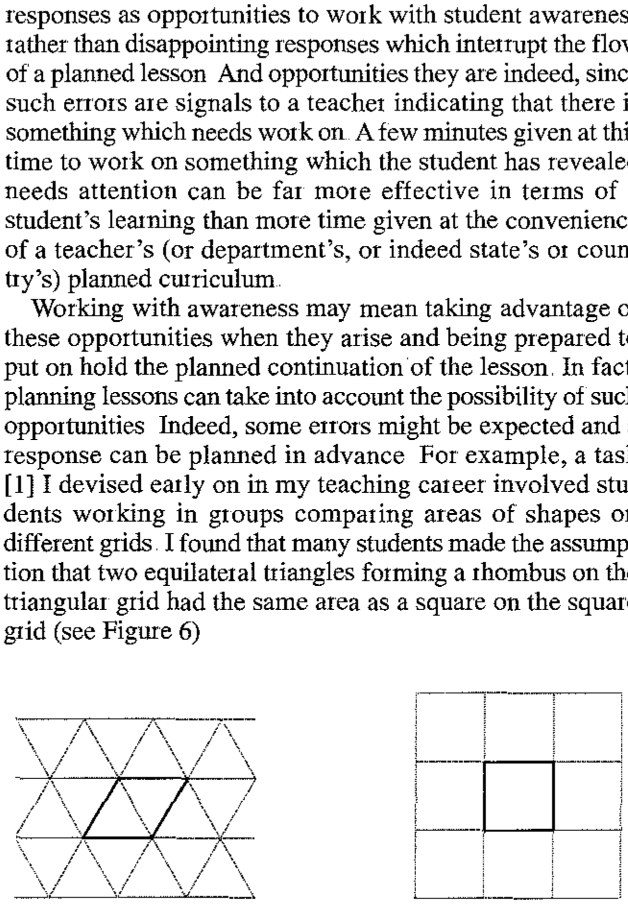

Figure 6 Are these areas the same?

On finding that this happened, I considered carefully what I might offer in future if and when this happened again I put four geo-strips together to form a rhombus I would ask them, after establishing that the lines were the same length, to look at the area inside my geo-strip rhombus when I had it arranged to look like a square. Then I would ask them to watch the area as I gradually changed the rhombus (see Figure 7). When I did this, I found that I did not need to say anything more, the students saw, through their own awareness, that the area changed and got less. This was sufficient for them to consider again the way in which they approached the task of comparing areas on different grids

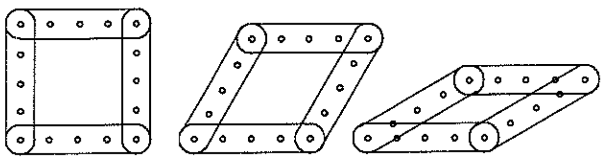

Figure 7 Changing area

This is an example of preparing in advance for a particular misconception which is likely to occur in a lesson. At other times, a lesson may be planned not so much for a particular misconception, but to reveal the awarenesses students have related to a certain topic. The planning of such a lesson involves how to get students to reveal their awarenesses around a certain topic, just as much as then considering how to work with those awarenesses.

One example concerns fractions. I draw on the board a line with 0 at one end and 1 at the other. I write up on the board a fraction, say $\frac{1}{2}$, and say that I am going to point my metre ruler on 0 and will begin moving it towards 1. I ask for a volunteer to say 'Stop' when I am pointing to where that fraction is on the line. As I move my ruler, I deliberately do not look at the board and stop the moment I hear the word 'stop' said by my volunteer. I then carefully keep the ruler where I stopped it and mark $\frac{1}{2}$ at that position. I write another fraction on the board and repeat the task

I may vary the speed at which my ruler travels along the line, since I wish there to be mistakes made I never allow someone to have a second go, and always ignore any complaints made by students, saying that I am not doing what I said I would do. At this stage I am not concerned with what is right or wrong After a while, I may have several fractions marked on the number line with many in incorrect positions (see Figure 8)

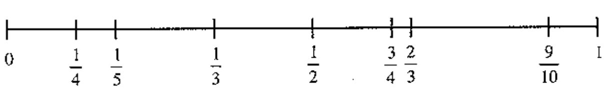

Figure 8 A possible resultant fraction number line

By this time there are usually many students who are wanting to say that some of these are wrong, and I ask someone to let me know whether there is a fraction they would like to move. I invite that person to come up and state the fraction and point to where it should be moved. I ask them to explain why they feel it should be moved there and effectively chair a class discussion on whether the reasons for the new position are agreed or not within the class. The planning of this lesson is centred around students revealing the awarenesses they have of fractions and beginning to work on those awarenesses. The unexpected is thus planned for, and is indeed central to the planning of the lesson. In fact, the lesson would not be so successful if the students got everything 'right'.

Opportunities to work with awareness also arise when a teacher notices what students *can* do, just as much as noticing any errors they make: for example, finding out that a young (or indeed older) student can double numbers can lead to a teacher considering how they might ask a student to use this ability in order to work on a new challenge. In this case, the ability to double a number can lead to working on the four- and eight-times tables through successive doubling. The awareness employed here is that something done once can be done again and again. When this awareness is applied to the skill of doubling, a new awareness can be gained that such iteration can lead to a method for finding the four- and eight- (and indeed sixteen-, etc) times tables.

This awareness of repeating something is not new for students. Indeed, they have employed such a process since they were very young. Learning to walk involves repetition – not only can I make a step, but I can do this again and again and this then becomes walking Listening to young children learning to talk involves them often repeating syllables – *Dada, Mama*, etc As well as the awarenesses students have about mathematical content, they also have awareness as a consequence of the learning they have already done as a human being. Some of these, such as repetition, order and inverse, are available to be worked with by a teacher and are applicable to a student's learning of mathematics

To summarise this section, opportunities for working on awareness require students to reveal their own awareness of mathematics This can come unexpectedly as a consequence of noticing errors or seeing opportunities for extending something a student can do correctly Such observations can lead to a teacher preparing how to act in the future if and when such misconceptions are repeated or particular successes are revealed Lessons can also be planned with an appropriately designed task which is likely to result in students revealing the awareness they have of a particular area of mathematics.

6. Working with awareness

As previously mentioned, a teacher cannot directly manipulate a student's awareness nor give a student awareness. It is the students' work to educate their own awareness A teacher can, however, show sensitivity to the fact that a student's learning of mathematics is concerned with them working on their own awareness, and so a teacher can use techniques which help the student carry out this work I will discuss three such techniques here:

- helping students see the consequences of their actions/decisions;
- directing attention;
- forcing awareness

Seeing the consequences of actions/decisions

If a teacher notices that a student is carrying out a mathematical process incorrectly, one option is for the teacher to try to help the student become aware of this incorrectness, since it may well be the case that the student considers what they are doing to be correct For example, a common misconception is for students to add fractions in a similar way to the one by which fractions are multiplied:

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$$

If a teacher asks the student to add together $\frac{1}{2}+\frac{1}{2}$ using their algorithm, the following result will be obtained:

$$\frac{1}{2} + \frac{1}{2} = \frac{2}{4}$$

It is invariably the case that the student does know that two halves make one and so are confused at this result. This is often sufficient for the student to become aware that the algorithm they are using is incorrect. This example offered by the teacher effectively brings the student's awareness into play, whereas previously the student was using their memory by employing a mixture of two partially memorised processes (addition and multiplication of fractions). In this case, the teacher does not attempt to offer the student received wisdom, but offers a carefully chosen example instead.

The notion of seeing the consequences of our actions is a technique which we have all employed within our own learning in everyday life As a writer, I have become quite skilled at throwing screwed up old drafts of my writing into a bin a few feet away from me. My skill has been developed through throwing the paper with a certain speed and angle and then noticing where it lands It is only through seeing the consequences of my actions that I can inform myself how to change the speed or angle of my next attempt.

With respect to learning mathematics, Mandler (1989) commented that:

> we must keep in mind that people frequently engage in actions that they believe to be correct (i.e., they proceed as intended, but the actions are in fact false, incorrect, illogical, etc.) In the case of unintended errors, the discrepancy arises because of a mismatch between what is intended and what occurs; in other cases, the mismatch is between an expected outcome ("I thought I did what would solve the problem") and the real-world response ("It didn't work"). (p. 13)

If the feedback I got from throwing my piece of paper was only whether it did or did not land in the bin, then that would do little to inform *how* I might change my throw next time even though I may experience a mis-match between my expected outcome of it landing in the bin. Suppose a student worked on the following problem: 23 + ? = 71 and they wrote down: 23 + 58 = 71. Response along the lines of 'No, that's wrong' would provide little information for the student to work out their next attempt. Instead, if the feedback were to provide the mathematical consequences of adding 23 and 58 together by saying that 'I make 23 and 58 equal to 81', then the student could realise that their answer was wrong but further to this the feedback would help them to employ their awareness in considering what their next answer to this question would be

The nature of mathematics is such that a student does not always have the awareness necessary to see, on their own, the consequences of what they are doing,. Hence, this is a useful role a teacher can play by feeding back consequences This is also a role which can be designed into computer software, with the nature of feedback provided being the consequences of actions/decisions rather than actually a yes/no or right/wrong type of feedback As Ainley (1997) commented:

> the image on the screen can be used directly to see how successfully instructions have been given (p 93)

Logo is an example of a piece of software which does this. Students may try drawing an equilateral triangle by typing in the following:

Forward 100

Right 60

Forward 100

Right 60

Forward 100

Right 60

The feedback provided by Logo is the geometric consequences of these instructions (see Figure 9)

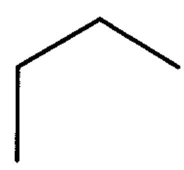

Figure 9 An attempt to draw an equilateral triangle

There are times when feeding back the consequences of actions/decisions is relatively straightforward and can indeed be written into software. On other occasions, it is not clear how such consequences can be revealed For example, I was talking with an adult about an image of a rotating point moving round a circle and how this can be used to introduce trigonometry I described the image of a dot moving anti-clockwise round the circumference of a circle and asked him to attend to the height of the dot I said that the dot will always start on the extreme right point on the circumference, where its height is 0. As the dot moves round, at its highest point the height is 1, and at its lowest, it is -1.

I asked through what angle the dot would turn from its starting position to reach a height of 0.5. He replied 45 degrees. The issue for me as a teacher was how I could show him the consequences of his reply so that he might know for himself that 45 degrees could not be the right answer. A possible response similar to the number example above would be to say 'The height at 45 degrees is about 0.707'. However, unlike the number example, I could not see how he could use his awareness to see why the height would be this and not 0.5.

So, if I wanted to stay with awareness, I needed to consider a different offering. In the end, I asked him to imagine the triangle made from the radius going to the dot, the horizontal diameter and the vertical from the dot to this horizontal. When asked about the angles and length of sides, he said they were 45, 90 and 45 degrees; and the length of the sides were 1 (the radius - this had come out earlier), the height 0.5 (as I had asked) and the horizontal was 0.5. Then, I gave him 4 matchsticks, saying each one was of length 0.5, and asked him to make that triangle (see Figure 10).

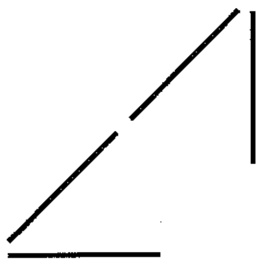

Figure 10 An attempt at making a matchstick right-angled triangle with sides 1, 0.5 and 0.5

After a while, he commented that such a triangle was not possible. I pushed him further and said that we know the radius is of length 1 and that the height must be 0.5, so could he ensure that this was so. He found himself pushing the radius back clockwise and as he did so stated that the angle must be less than 45 degrees (see Figure 11).

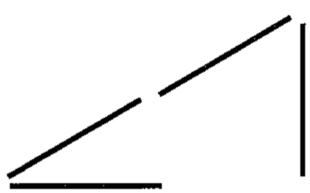

Figure 11 Modification of the triangle

The task I introduced was the result of pedagogic considerations and was not a mechanical feedback which would be easy to program computer software to do. This is another example of how the pedagogic awareness of a teacher, as well as mathematical awareness, plays a crucial role when working with a student's awareness.

Directing attention

I have recently been on holiday with my one-year-old daughter where both her mother and myself have pointed to birds in the sky, saying 'There's a bird'. By the end of the holiday I kept hearing a grunt from behind my head as my daughter pointed from her position in a backpack to yet another bird in the sky. Drawing attention to something can be a tool for a teacher and what attention is drawn to, and when, are pedagogic decisions.

During one lesson, I observed a teacher skilfully use seemingly simple and yet highly effective use of directing attention. The students were working on finding the maximum volume of an open box which can be made from a sheet of paper with squares cut out from the corners. They were working in groups and were arranging cubes in their boxes to find the volumes. This was taking some time and the teacher came over and asked 'What do you notice when cubes go like that?'. One student responded 'They go in lines'. The teacher then continued to develop the idea of counting how many lines there were.

Here, directing attention to the lines - of which the students were aware - not only helped them find the volume more quickly, but also began to develop some mathematics of how volumes can be calculated. Another group asked the teacher about surface area. The teacher held up a piece of A4 paper asking 'How would you find the surface area of that piece of paper?' A student responded 'Length times width'. The teacher then held pieces of paper to form the sides of a box and asked what the surface area of the box was. This showed a pedagogic awareness of finding something which the students did know and relating that to the current situation which they were not so sure about.

Rousseau (1762/1986) in the mid-eighteenth century gave an example of this technique:

> If he does not know how the sun gets from the place where it sets to where it rises, he knows at least how it travels from sunrise to sunset, his eyes teach him that. Use the second question to throw light on the first. (p. 132)

Figure 12 represents this process from a mathematics teacher's perspective.

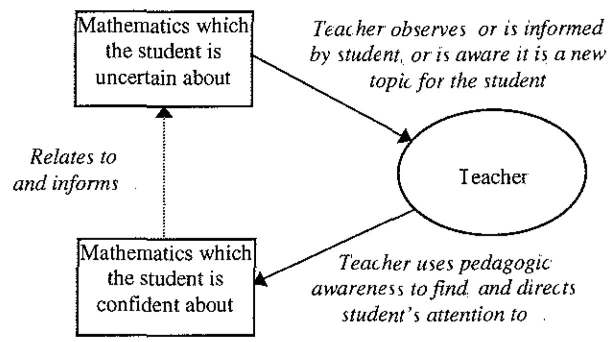

Figure 12 One use for directing attention

Directing attention is also a technique in order to stress some things and ignore others. It is a common task for students in the U.K., particularly in relation to the National Curriculum attainment target *Using and applying mathematics* and in preparation for GCSE coursework elements, to be asked to 'investigate' a situation and try to find a general rule relating to that situation Students are encouraged to go through a process of trying out different cases - from the simplest onwards - and putting those results in a table. If for the moment we consider the problem of how many handshakes take place when a certain number of people meet and all shake hands with each other, students might end up with a table like the one in Figure 13.

Number of people	Number of handshakes
1	0
2	1
3	3
4	6
5	10

Figure 13 A common way of tacking the handshake problem

A problem I have found with such an approach is that the way in which the numbers appear in the table encourages students to stress connections between one line and the next line. So, a student might end up noticing that to get from one row to the next you add on the number of people in that row This is a general rule, yet it is not the general rule many teachers and examiners are after! To combat this, some teacher's worksheets, textbooks and examination board tasks give students a pre-prepared table which looks something like Figure 14.

Number of people	Number of handshakes
1	
2	
3	
4	
10	
n	

Figure 14 An attempt to encourage students to find the rule the teacher wants them to find

However, the same pattern is often spotted within the first few cases and students can continue to find out the results for five, six, seven, eight and nine people and that way arrive at a figure for ten people. Some students are then stuck when it comes to considering n.

I argue that one of the consequences of results being placed in their numerical order in the table is that students' attention is drawn to a rule going from one case to the next, rather than seeking a generality *within* the way of counting handshakes. I have criticised such use of tables and the transference from consideration of a mathematical problem to one which becomes a task of number pattern spotting elsewhere (Hewitt, 1992), but here I wish to consider how students' attention can be directed as a consequence of the visual presentation of the results within a table.

If deliberately non-consecutive results are entered into a pre-prepared table (see Figure 15 - which has results entered for clarity of the point being made), then attention is immediately taken away from the previously obvious connection *down* the table, and placed onto consideration of results *across* the table.

Number of people	Number of handshakes
12	66
4	6
7	21
5	10
10	45

Figure 15 An alternative collection and arrangement of results

The arrangement of results in a table can direct attention to certain aspects of that table. If educational value is placed on students finding a rule going across the table, then I fail to understand why students are given a pre-prepared table which encourages their attention to be placed elsewhere.

Forcing awareness

The phrase 'forcing awareness' is a useful one for me. It comes from Gattegno (1987) who indicated that:

> Forcing awareness does not imply violence to the state of the person. On the contrary, once lightened in a certain way, awareness becomes a source of illumination for oneself. (p 219)

The act of forcing awareness is an inner individual activity carried out by a student but one which has been made more accessible through the efforts of a teacher Slightly earlier in that book, Gattegno also stated:

> there is no other way of forcing awareness than making one person get hold of that which is capable of forcing individual self-awareness. (p. 214)

The use of the word 'forcing' indicates for me that the learning which takes place is often much faster and more assured than is a normal occurrence in classrooms, or that what is learned is considered to be unusually advanced for the students compared with common expectations of students of their age and attainment. As Gattegno also observed:

> But when we enter into other people's lives, we can find something to do which will make the ordinary become what the extraordinary appears to be We can force awareness beyond what one's circumstances would have brought to oneself (p. 213)

I will offer one example of forcing awareness in detail and make reference to a second One significant factor in these examples is that students are worked with in such a way that the students are more likely to be able to educate their own awareness of the mathematics involved than if they worked on their own. It is not the teacher who does any explaining: it is the students who explain if any explaining takes place. The teacher works with the students' awareness of mathematics rather than expressing their own awareness of mathematics.

Another general feature is that students will not necessarily know where they are being taken in the line of questioning which occurs. This may go against what is often considered as 'good practice' by inspectors from Ofsted (the U.K. Office for Standards in Education), where a teacher is expected to outline the aims of a lesson at the start of a lesson. However, I consider it a significant factor that students stay with the questions being asked at any particular point in time, rather than concern themselves with how this questioning is leading to an expressed aim. In fact, some students might have a negative initial reaction if they are told that the aim of the lesson is to divide fractions or solve complex linear equations with strange Greek letters appearing in them.

Sadly, I have sat in too many lessons and observed the negative reaction many students have when a title of *Fractions* or *Algebra* gets put up on the board. Not uncommon responses are along the lines of 'But I can't do fractions/algebra'. The lesson then begins with students in a negative frame of mind and feeling as if they are not going to understand this lesson even before it starts. Instead of this, forcing awareness has a feature of starting with questions/tasks which most, if not all, students in the class find straightforward and gradually the questions develop into more challenging/advanced areas.

These steps are only taken by a teacher if the teacher feels that there is an accompanying development of awareness, since this developing awareness supports later work as the questioning continues. Thus, at any point in time, it is still the case that most students (I never claim *all* students, although sometimes it is true for all students as well) find the current question accessible with the awareness they have developed so far.

A possible metaphor occurs to me as I write since I currently have some people putting a new roof on my house and I have just been on the roof myself to talk with them despite having a poor head for heights. I was aware as I went up the ladder that I only concerned myself with stepping onto the next rung of the ladder. After a while, to my surprise I found myself on the roof of my house - something I would never have thought I would be able to do without a panic attack.

Likewise, students are invited just to concern themselves with the current question and, at the end of the lesson, they might be surprised where they have ended up. Sometimes I have known students get an immense surge of excitement as they stop for breath at the end of a lesson and reflect upon what they have been doing. Vygotsky (1978) said that:

> the only 'good learning' is that which is in advance of development (p. 89)

This is what forcing awareness tries to achieve. It is also a feature of the way in which young children appear to learn. I have heard many parents, as well as noticing this myself as a parent, that their child always tries to do things which are seemingly beyond the child's current skill level.

The skill of a teacher is to find a route whereby it is actually possible for students to achieve something which at first sight may not appear possible with their current level of attainment in mathematics. This route, if it is about forcing awareness, will be a journey which only calls upon the students' awareness and does not involve some kind of explanation from a teacher (received wisdom). I can, for example, always 'explain' the calculus to a five-year-old and not concern myself with what the five-year-old makes of my explanation. With forcing awareness, it is the learning which is in advance of development, not any explanations.

The first example, which I only refer to briefly here since I have discussed it more fully elsewhere (Hewitt, 1996), concerns a way of approaching the introduction of algebraic notation and the solving of linear equations. This approach, the beginning of which can be seen in an Open University videotape (OU, 1991), can result in a mixed-ability class of eleven-year-olds solving relatively complex linear equations (such as the one below) after only about three hours' work. Most, if not all, students will be able to solve algebraically for p, x, β, and μ, writing down solutions in one line of work in correct algebraic notation, and the students will numerically solve the equations if values are given for the other unknowns.

$$\frac{p\left[\frac{2(x-3)+\beta}{7}+\mu\right]-4.3}{31}+2=100$$

This will all have come from a line of questioning which helps students articulate the way in which inverses can be applied. The only thing which students are informed about by the teacher is the arbitrary issue of notation - how things are written. Even then, this is provided in written form without any verbal description or explanation.

The second example I give in detail here and concerns dividing fractions. I have sometimes asked graduate mathematicians coming for an interview for our PGCE teacher education course to tell me what $\frac{3}{8} \div \frac{2}{5}$ is. I gave them a pen and paper and with confidence they wrote:

$$\frac{3}{8} \div \frac{2}{5} = \frac{3}{8} \times \frac{5}{2} = \frac{3 \times 5}{8 \times 2} = \frac{15}{16}$$

I then asked them why they turned the second fraction upside down? Why did they multiply when I asked them to do a division? Why, for example, did they not turn the first fraction upside down and add? With very few exceptions, these graduates were unable to give me any explanation for what they did, but instead just said that this was what they were told to do at school. So the question is whether it is possible for students to educate their own awareness to an extent where they know how to divide fractions and this knowing is based upon their own awareness: also whether it is possible for a teacher to find a route through which this awareness can be forced.

The series of questions below gives a flavour for such a route. It does not, however, show the sensitivities which are required when working with students, such as deciding when to ask the next question, when to proceed with a new line of questioning, to whom to ask each question, whether each question is asked to just one student or more than one, whether students are asked to explain why their answer is correct, how many examples of a certain kind of question to ask, whether a 'deviation' from a particular line of question is required and what that diversion might be. These are all vital considerations and will affect whether or not students' awarenesses are educated or whether they end up feeling frustrated and confused.

What I invite here is for you - the reader - to consider the line of questioning given below and see whether you can see:

(i) what awareness is being called upon at a particular time;

(ii) what awareness might be developed/educated as the questioning proceeds and so be available to be called upon as the questioning continues.

I leave you to consider the way in which you would address the issues I have raised above when working with a particular class. I do this not to avoid such issues but to recognise that such decisions need to be made in the moment with sensitivity to the individuals within a class.

The following notes indicate some significant features in the questioning:

(a) Three dots (...) represent a continuation of questioning along similar lines. The number of questions and their exact content will be decided by the teacher based upon the teacher's assessment of the students' developing awareness

(b) Stars (***) indicate a potential generalisation which may have been established for students through the preceding series of questions. I also use a deliberately silly word, in this case *flinkerty-floo*, to represent a general number *before* I use such things as x or n, etc This is partly because by this stage the questioning and answering is likely to have become being a word game This, for me, is an indicator of students gaining awareness of a generality It is usual leading up to this point that the questions are arriving at quite a fast speed and answers may be said by the whole class in chant-like fashion.

(c) Answers to questions may involve students carrying out a calculation As a general rule, I suggest the teacher asks students not to carry out any arithmetic but to state what arithmetic they would do to get the answer This will bring out the generalisation more explicitly than if the results of calculations were stated. Thus, a teacher may need to work on this with students during their activity. For example, when asking 'How many halves are there in five?', a student who answers 'Ten' may be questioned further by the teacher so that it is re-expressed as 'Two times five'. A corollary to this is that there is a relatively small demand for arithmetic fluency in order to participate successfully in this task.

(d) Notation, for what students have expressed in words, can be provided by the teacher as and when it is felt appropriate This can be provided since it is arbitrary and provides only a conventional way of writing what students have already expressed Note that the expressions I write below are for you the reader and are not necessarily a suggestion of notation to be provided to students at these points

(e) I would expect the questions listed to be broken up with times of reflection and times of conscious consideration and checking. At other times, I would expect the flow of questions to be quite fast when leading to a generality, as mentioned above It might be the case that this task goes over more than one lesson, although it is possible to achieve the final generalisation within a lesson of about an hour.

How many halves are there in one?
How many quarters are there in one?
How many tenths are there in one?
How many thirds are there in one?
...
How many twenty-fourths are there in one?
How many two-hundred-and-forty-ninths are there in one?
How many five-million-eight-hundred-and-sixths are there in one?
..
How many flinkerty-flooths are there in one?
How many $n$ths are there in one?

$$***1 \div \frac{1}{n} = n***$$

How many halves are there in one?
How many halves are there in two?
How many halves are there in five?
...
How many halves are there in nine hundred and fifty-two?
How many halves are there in twenty trillion, five million and seven?
How many halves are there in flinkerty-floo?
How many halves are there in x?

$$*** x \div \frac{1}{2} = x \times 2***$$

How many quarters are there in one?
How many quarters are there in two?
How many quarters are there in eight?
...
How many quarters are there in twenty-seven?
...
How many tenths are there in one?
How many tenths are there in fourteen?
How many tenths are there in two thousand and fifty-one?
..
How many sixteenths are there in one?
How many sixteenths are there in six?
..
How many four-hundred-and-twentieths are there in one?
How many four-hundred-and-twentieths are there in nine?
..

How many flinkerty-flooths are there in one?
How many flinkerty-flooths are there in ten?
How many flinkerty-flooths are there in seven thousand?
How many flinkerty-flooths are there in nine million, two hundred and eight?
How many flinkerty-flooths are there in zipperly-bond?
How many flinkerty-flooths are there in x?
How many nths are there in x?

$$*** \; x \div \frac{1}{n} = x \times n \; ***$$

How many times does one-quarter go into one?
How many times does one-quarter go into seventeen?
How many times does two-quarters go into one?
How many times does two-quarters go into three?
How many times does two-quarters go into twenty-five?

...

How many times does two-quarters go into three?
How many times does four-quarters go into three?
How many times does six-quarters go into three?
How many times does five-quarters go into three?
How many times does seventeen-quarters go into three?

...

How many times does six-quarters go into three?
How many times does six-quarters go into nine?
How many times does six-quarters go into forty-one?

...

How many times does six-quarters go into three?
How many times does six-quarters go into thirty-seven?
How many times does nine-quarters go into thirty-seven?
How many times does two-hundred-and-seven-quarters go into thirty-seven?
How many times does two-hundred-and-seven-quarters go into nine million and six?

...

How many times does one-tenth go into one?
How many times does one-tenth go into twenty?
How many times does two-tenths go into twenty?
How many times does five-tenths go into twenty?
How many times does sixty-three-tenths go into twenty?
How many times does sixty-three-tenths go into ninety-four?

..

How many times does one-flinkerty-flooth go into one?
How many times does one-*n*th go into one?
How many times does one-*n*th go into seventy-five?
How many times does one-*n*th go into x?
How many times does two-*n*ths go into x?
How many times does sixty-one-*n*ths go into x?
How many times does p-*n*ths go into x?

$$*** \; x \div \frac{p}{n} = x \times \frac{n}{p} \; ***$$

How many times does p-*n*ths go into x?
How many times does p-*n*ths go into four?
How many times does p-*n*ths go into eighty-seven?
How many times does p-*n*ths go into sixty-nine?
How many times does p-*n*ths go into zipperly-bond?
How many times does p-*n*ths go into two-quarters?
How many times does p-*n*ths go into seven-twelfths?

...

How many times does p-*n*ths go into x-*y*ths?

$$*** \; \frac{x}{y} \div \frac{p}{n} = \frac{x}{y} \times \frac{n}{p} \; ***$$

7. Summary

The division of the mathematics curriculum into arbitrary and necessary brings implications for the respective jobs that teachers and students have in the classroom. The necessary is in the realm of awareness To know that something is necessary is in itself a statement of awareness All students have awareness, the role of a teacher is to educate that awareness so that students can come to know what is necessary through their own awareness

The only other alternative is for students to be informed of what is necessary just as they must be informed of what is arbitrary. This would ignore the arbitrary/necessary divide and result in all 'learning' becoming an exercise in memorisation. This would not only place a large burden on memorisation but also would fail to educate the mathematician which resides with all students. I repeat Gattegno's assertion that 'Only awareness is educable' and the education of awareness cannot be done by teachers on behalf of students.

Educating a student's awareness is a job for that student and that student alone. A teacher can, however, work with the awareness a student reveals and so one role a teacher has is to create a culture in the classroom where awareness of mathematics is the currency of communication Once awareness has been revealed, teachers can help students educate their awareness through helping them see *the consequences of their actions/decisions*, *directing their attention* to certain aspects of what they have revealed, or *forcing awareness* through a carefully constructed sequence of questioning which builds on this awareness Most, if not all, students already possess this awareness in order for the students to find themselves doing something which they have never done before, or doing something faster than they have ever done before

Mathematics lies in what is necessary, so learning mathematics is concerned primarily with the education of awareness I finish with a challenge which still lies ahead for us today and which was articulated by Gattegno in 1971:

> Awareness of relationships *per se* is what distinguishes mathematical from other thinking. Is it possible to offer a complete mathematics curriculum in terms of awareness? Is it possible to replace the linear presentation of mathematical ideas by a variety of entries into the field, all starting from scratch and each calling for special awarenesses, and have our students reach at least as good a grasp of mathematics as is currently attained by the best learners? (p 91)

Note

[1] *Greatest and Smallest*, part of the *Area and Perimeter* pack which was developed through the *Resources for Learning Development Unit* (RLDU) in Bristol is now available from the Association of Teachers of Mathematics, 7 Shaftesbury Street, Derby DE23 8YB, U K

References

Ainley, J (1997) 'Roles for teachers, and computers', in Pehkonen, E. (ed.), *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education*, Helsinki, Lahti Research and Training Centre, University of Helsinki, vol 1, pp 90-98

Berger, J (1973) *Ways of Seeing*, Harmondsworth, Middlesex BBC/ Penguin Books Ltd.

Bruner, J S (1960) *The Process of Education*, Cambridge, MA, Harvard University Press.

Bruner, J. (1966) *Toward a Theory of Instruction*, Cambridge, MA, The Belknap Press of Harvard University Press

Claxton, G (1984) *Live and Learn: an Introduction to the Psychology of Growth and Change in Everyday Life*, London, Harper and Row

Gattegno, C. (1971) *What We Owe Children the Subordination of Teaching to Learning*. London, Routledge and Kegan Paul Ltd

Gattegno, C (1987) *The Science of Education Part 1 – Theoretical Considerations*, New York, NY, Educational Solutions.

Hewitt, D (1987) 'Involvement', *Mathematics Teaching* **120**, 60-61.

Hewitt, D. (1992) 'Train spotters' paradise', *Mathematics Teaching* **140**, 6-8

Hewitt, D (1996) 'Mathematical fluency: the nature of practice and the role of subordination', *For the Learning of Mathematics* **16**(2), 28-35

Hewitt, D (1999) Arbitrary and necessary: part 1 a way of viewing the mathematics curriculum', *For the Learning of Mathematics* **19**(3), 2-9

Hewitt, D (2001) 'Arbitrary and necessary: part 2 assisting memory', *For the Learning of Mathematics* **21**(1), 44-51

Hewitt, D. *et al.* (1992) 'First lessons', *Mathematics Teaching* **139**, 3-9.

Mandler, G (1989) 'Affect and learning: causes and consequences of emotional interactions', in McLeod, D B and Adams, V M (eds), *Affect and Mathematical Problem Solving. a New Perspective*, New York, NY, Springer-Verlag, pp 3-19

Mason, J (1987) 'Only awareness is educable', *Mathematics Teaching* **120**, 30-31

Mason, J. (1994) 'Researching from the inside in mathematics education: locating an I-you relationship', expanded version of a plenary presentation to the Psychology of Mathematics Education Conference XVII, Lisbon. (Available through the Centre for Mathematics Education, The Open University, Milton Keynes, Bucks, MK7 6AA, U.K.)

OU (1991) *Working Mathematically on Symbols in Key Stage 3* (Video) Milton Keynes, Bucks, The Open University.

Rousseau, J. (1762/1986) *Emile*, London, Dent Everyman's Library

Vygotsky, L. S. (1978) *Mind in Society: the Development of Higher Psychological Processes*, Cambridge, MA, Harvard University Press