Elementary Differential Equations and Boundary Value Problems

Eleventh Edition

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CHAPTER 5

Series Solutions of Second-Order Linear Equations

Finding the general solution of a linear differential equation depends on determining a fundamental set of solutions of the homogeneous equation. So far, we have given a systematic procedure for constructing fundamental solutions only when the equation has constant coefficients. To deal with the much larger class of equations that have variable coefficients, it is necessary to extend our search for solutions beyond the familiar elementary functions of calculus. The principal tool that we need is the representation of a given function by a power series. The basic idea is similar to that in the method of undetermined coefficients: we assume that the solutions of a given differential equation have power series expansions, and then we attempt to determine the coefficients so as to satisfy the differential equation.

5.1 Review of Power Series

In this chapter we discuss the use of power series to construct fundamental sets of solutions of second-order linear differential equations whose coefficients are functions of the independent variable. We begin by summarizing very briefly the pertinent results about power series that we need. Readers who are familiar with power series may go on to Section 5.2. Those who need more details than are presented here should consult a book on calculus.

1. A power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is said to *converge at a point* x if

$$\lim_{m\to\infty} \sum_{n=0}^{m} a_n(x-x_0)^n$$

exists for that x. The series certainly converges for $x = x_0$; it may converge for all x, or it may converge for some values of x and not for others.

2. The power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is said to *converge absolutely at a point* x if the associated power series

$$\sum_{n=0}^{\infty} |a_n(x-x_0)^n| = \sum_{n=0}^{\infty} |a_n||x-x_0|^n$$

converges. It can be shown that if the power series converges absolutely, then the power series also converges; however, the converse is not necessarily true.

3. One of the most useful tests for the absolute convergence of a power series is the ratio test: If $a_n \neq 0$, and if, for a fixed value of x,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n} \right| = |x-x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x-x_0|L,$$

then the power series converges absolutely at that value of x if $|x - x_0|L < 1$ and diverges if $|x - x_0|L > 1$. If $|x - x_0|L = 1$, the ratio test is inconclusive.

EXAMPLE 1

For which values of x does the power series

$$\sum_{n=1}^{\infty} (-1)^{n+1} n(x-2)^n = (x-2) - 2(x-2)^2 + 3(x-2)^3 - \cdots$$

converge?

Solution:

We first test for absolute convergence using the ratio test. We have

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2}(n+1)(x-2)^{n+1}}{(-1)^{n+1} n(x-2)^n} \right| = |x-2| \lim_{n\to\infty} \frac{n+1}{n} = |x-2|.$$

According to statement 3, the series converges absolutely for $|x - 2| < 1$, that is, for $1 < x < 3$, and diverges for $|x - 2| > 1$. The values of x corresponding to $|x - 2| = 1$ are $x = 1$ and $x = 3$. The series diverges for each of these values of x since the nth term of the series does not approach zero as $n \to \infty$. This power series converges (absolutely) for $1 < x < 3$ and diverges for $x \leq 1$ and for $x \geq 3$.

4. If the power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges at $x = x_1$, it converges absolutely for $|x - x_0| < |x_1 - x_0|$; and if it diverges at $x = x_1$, it diverges for $|x - x_0| > |x_1 - x_0|$.
5. For a typical power series, such as the one in Example 1, there is a positive number ρ, called the **radius of convergence**, such that $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges absolutely for $|x - x_0| < \rho$ and diverges for $|x - x_0| > \rho$. The interval $|x - x_0| < \rho$ is called the **interval of convergence**; it is indicated by the hatched lines in Figure 5.1.1. The series may either converge or diverge when $|x - x_0| = \rho$. Many important power series converge for all values of x. In this case it is customary to say that ρ is infinite and the interval of convergence is the entire real line. It is also possible for a power series to converge only at x_0. For such a series we say that $\rho = 0$ and the series has no interval of convergence. When these exceptional cases are taken into account, every power series has a nonnegative radius of convergence ρ, and if $\rho > 0$, then there is a (finite or infinite) interval of convergence centered at x_0.

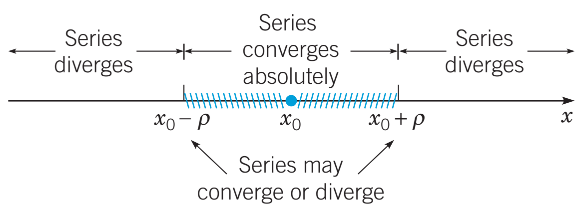

FIGURE 5.1.1 The interval of convergence of a power series.

EXAMPLE 2

Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n2^n}.$$

Solution:

We apply the ratio test:

$$\lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{(n+1)2^{n+1}}\frac{n2^n}{(x+1)^n}\right| = \frac{|x+1|}{2}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|x+1|}{2}.$$

Thus the series converges absolutely for $|x+1| < 2$, that is, for $-3 < x < 1$, and diverges for $|x+1| > 2$. The radius of convergence of the power series is $\rho = 2$. Finally, we check the endpoints of the interval of convergence. At $x = 1$ the series becomes the harmonic series

$$\sum_{n=1}^{\infty}\frac{1}{n},$$

which diverges. At $x = -3$ we have

$$\sum_{n=1}^{\infty}\frac{(-3+1)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}.$$

Recognizing this as the alternating harmonic series, we recall that it converges but does not converge absolutely. The power series is said to converge conditionally at $x = -3$. To summarize, the given power series converges for $-3 \le x < 1$ and diverges otherwise. It converges absolutely for $-3 < x < 1$ and has a radius of convergence of 2.

Suppose that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ and $\sum_{n=0}^{\infty} b_n(x-x_0)^n$ converge to $f(x)$ and $g(x)$, respectively, for $|x-x_0| < \rho$, $\rho > 0$.

6. The two series can be added or subtracted termwise, and

$$f(x) \pm g(x) = \sum_{n=0}^{\infty}(a_n \pm b_n)(x-x_0)^n;$$

the resulting series converges at least for $|x-x_0| < \rho$.

7. The two series can be formally multiplied, and

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n(x-x_0)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-x_0)^n\right) = \sum_{n=0}^{\infty} c_n(x-x_0)^n,$$

where $c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0$. The resulting series converges at least for $|x-x_0| < \rho$.

Further, if $b_0 \neq 0$, then $g(x_0) \neq 0$, and the series for $f(x)$ can be formally divided by the series for $g(x)$, and

$$\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} d_n(x-x_0)^n.$$

In most cases the coefficients d_n can be most easily obtained by equating coefficients in the equivalent relation

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = \left[\sum_{n=0}^{\infty} d_n(x-x_0)^n\right]\left[\sum_{n=0}^{\infty} b_n(x-x_0)^n\right]$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} d_kb_{n-k}\right)(x-x_0)^n.$$

In the case of division, the radius of convergence of the resulting power series may be less than ρ.

8. The function f is continuous and has derivatives of all orders for $|x-x_0| < \rho$. Moreover, $f', f'', \ldots$ can be computed by differentiating the series termwise; that is,

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x - x_0) + \cdots + na_n(x - x_0)^{n-1} + \cdots \\ &= \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}, \\ f''(x) &= 2a_2 + 6a_3(x - x_0) + \cdots + n(n-1)a_n(x - x_0)^{n-2} + \cdots \\ &= \sum_{n=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2}, \end{aligned}$$

and so forth, and each of the series converges absolutely for $|x - x_0| < \rho$.

9. The value of a_n is given by

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

The series is called the Taylor[1] series for the function f about $x = x_0$.

10. If $\sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} b_n(x - x_0)^n$ for each x in some open interval with center x_0, then $a_n = b_n$ for $n = 0, 1, 2, 3, \ldots$. In particular, if $\sum_{n=0}^{\infty} a_n(x - x_0)^n = 0$ for each such x, then $a_0 = a_1 = \cdots = a_n = \cdots = 0$.

A function f that has a Taylor series expansion about $x = x_0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

with a radius of convergence $\rho > 0$, is said to be **analytic** at $x = x_0$. All of the familiar functions of calculus are analytic except perhaps at certain easily recognized points. For example, $\sin x$ and e^x are analytic everywhere, $1/x$ is analytic except at $x = 0$, and $\tan x$ is analytic except at odd multiples of $\pi/2$. According to statements 6 and 7, if f and g are analytic at x_0, then $f \pm g$, $f \cdot g$, and f/g (provided that $g(x_0) \neq 0$) are also analytic at $x = x_0$. In many respects the natural context for the use of power series is the complex plane. The methods and results of this chapter nearly always can be directly extended to differential equations in which the independent and dependent variables are complex-valued.

Shift of Index of Summation. The index of summation in an infinite series is a dummy parameter just as the integration variable in a definite integral is a dummy variable. Thus it is immaterial which letter is used for the index of summation. For example,

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \sum_{j=0}^{\infty} \frac{2^j x^j}{j!}.$$

Just as we make changes of the variable of integration in a definite integral, we find it convenient to make changes of summation indices in calculating series solutions of differential equations. We illustrate by several examples how to shift the summation index.

[1]Brook Taylor (1685–1731), English mathematician, received his education at Cambridge University. His book *Methodus incrementorum directa et inversa*, published in 1715, includes a general statement of the expansion theorem that is named for him. This is a basic result in all branches of analysis, but its fundamental importance was not recognized until 1772 (by Lagrange). Taylor was also the first to use integration by parts, was one of the founders of the calculus of finite differences, and was the first to recognize the existence of singular solutions of differential equations.

EXAMPLE 3

Write $\sum_{n=2}^{\infty} a_n x^n$ as a series whose first term corresponds to $n = 0$ rather than $n = 2$.

Solution:

Let $m = n - 2$; then $n = m + 2$, and $n = 2$ corresponds to $m = 0$. Hence

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{m=0}^{\infty} a_{m+2} x^{m+2}. \tag{1}$$

By writing out the first few terms of each of these series, you can verify that they contain precisely the same terms. Finally, in the series on the right-hand side of equation (1), we can replace the dummy index m by n, obtaining

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_{n+2} x^{n+2}. \tag{2}$$

In effect, we have shifted the index upward by 2 and have compensated by starting to count at a level 2 lower than originally.

EXAMPLE 4

Write the series

$$\sum_{n=2}^{\infty} (n + 2)(n + 1) a_n (x - x_0)^{n-2} \tag{3}$$

as a series whose generic term involves $(x - x_0)^n$ rather than $(x - x_0)^{n-2}$.

Solution:

Again, we shift the index by 2 so that n is replaced by $n + 2$ and start counting 2 lower. We obtain

$$\sum_{n=0}^{\infty} (n + 4)(n + 3) a_{n+2} (x - x_0)^n. \tag{4}$$

You can readily verify that the terms in the series (3) and (4) are exactly the same.

EXAMPLE 5

Write the expression

$$x^2 \sum_{n=0}^{\infty} (r + n) a_n x^{r+n-1} \tag{5}$$

as a series whose generic term involves x^{r+n}.

Solution:

First, take the x^2 inside the summation, obtaining

$$\sum_{n=0}^{\infty} (r + n) a_n x^{r+n+1}. \tag{6}$$

Next, shift the index down by 1 and start counting 1 higher. Thus

$$\sum_{n=0}^{\infty}(r+n)a_n x^{r+n+1} = \sum_{n=1}^{\infty}(r+n-1)a_{n-1}x^{r+n}. \tag{7}$$

Again, you can easily verify that the two series in equation (7) are identical and that both are exactly the same as the expression (5).

EXAMPLE 6

Assume that

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n \tag{8}$$

for all x, and determine what this implies about the coefficients a_n.

Solution:

We want to use statement 10 to equate corresponding coefficients in the two series. In order to do this, we must first rewrite equation (8) so that the series display the same power of x in their generic terms. For instance, in the series on the left-hand side of equation (8), we can replace n by $n+1$ and start counting 1 lower. Thus equation (8) becomes

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=0}^{\infty} a_n x^n. \tag{9}$$

According to statement 10, we conclude that

$$(n+1)a_{n+1} = a_n, \quad n = 0, 1, 2, 3, \dots$$

or

$$a_{n+1} = \frac{a_n}{n+1}, \quad n = 0, 1, 2, 3, \dots . \tag{10}$$

Hence, choosing successive values of n in equation (10), we have

$$a_1 = a_0, \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2}, \quad a_3 = \frac{a_2}{3} = \frac{a_0}{3!},$$

and so forth. In general,

$$a_n = \frac{a_0}{n!}, \quad n = 1, 2, 3, \dots . \tag{11}$$

Thus the relation (8) determines all the following coefficients in terms of a_0. Finally, using the coefficients given by equation (11), we obtain

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\frac{a_0}{n!}x^n = a_0\sum_{n=0}^{\infty}\frac{x^n}{n!} = a_0 e^x,$$

where we have followed the usual convention that $0! = 1$, and recalled that $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ for all values of x. (See Problem 8.)

Problems

In each of Problems 1 through 6, determine the radius of convergence of the given power series.

1. $\sum_{n=0}^{\infty}(x-3)^n$

2. $\sum_{n=0}^{\infty}\frac{n}{2^n}x^n$

3. $\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$

4. $\sum_{n=0}^{\infty}2^n x^n$

5. $\sum_{n=1}^{\infty}\frac{(x-x_0)^n}{n}$

6. $\sum_{n=1}^{\infty}\frac{(-1)^n n^2(x+2)^n}{3^n}$

In each of Problems 7 through 13, determine the Taylor series about the point x_0 for the given function. Also determine the radius of convergence of the series.

7. $\sin x, \quad x_0 = 0$

8. $e^x, \quad x_0 = 0$

9. $x, \quad x_0 = 1$

10. $x^2, \quad x_0 = -1$

11. $\ln x, \quad x_0 = 1$

12. $\frac{1}{1-x}, \quad x_0 = 0$

13. $\frac{1}{1-x}, \quad x_0 = 2$

14. Let $y = \sum_{n=0}^{\infty} nx^n$.

a. Compute y' and write out the first four terms of the series.
b. Compute y'' and write out the first four terms of the series.

15. Let $y = \sum_{n=0}^{\infty} a_n x^n$.

a. Compute y' and y'' and write out the first four terms of each series, as well as the coefficient of x^n in the general term.
b. Show that if $y'' = y$, then the coefficients a_0 and a_1 are arbitrary, and determine a_2 and a_3 in terms of a_0 and a_1.
c. Show that $a_{n+2} = \frac{a_n}{(n+2)(n+1)}$, $n = 0, 1, 2, 3, \ldots$.

In each of Problems 16 and 17, verify the given equation.

16. $\sum_{n=0}^{\infty} a_n(x-1)^{n+1} = \sum_{n=1}^{\infty} a_{n-1}(x-1)^n$

17. $\sum_{k=0}^{\infty} a_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^{k+1} = a_1 + \sum_{k=1}^{\infty}(a_{k+1} + a_{k-1})x^k$

In each of Problems 18 through 22, rewrite the given expression as a single power series whose generic term involves x^n.

18. $\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$

19. $x\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{k=0}^{\infty} a_k x^k$

20. $\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{k=1}^{\infty} ka_k x^{k-1}$

21. $\sum_{n=1}^{\infty} na_n x^{n-1} + x\sum_{n=0}^{\infty} a_n x^n$

22. $x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$

23. Determine the a_n so that the equation

$$\sum_{n=1}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

is satisfied. Try to identify the function represented by the series $\sum_{n=0}^{\infty} a_n x^n$.

5.2 Series Solutions Near an Ordinary Point, Part I

In Chapter 3 we described methods of solving second-order linear differential equations with constant coefficients. We now consider methods of solving second-order linear equations when the coefficients are functions of the independent variable. In this chapter we will denote the independent variable by x. It is sufficient to consider the homogeneous equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0, \tag{1}$$

since the procedure for the corresponding nonhomogeneous equation is similar.

Many problems in mathematical physics lead to equations of the form (1) having polynomial coefficients; examples include the Bessel equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

where ν is a constant, and the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

where α is a constant. For this reason, as well as to simplify the algebraic computations, we primarily consider the case in which the functions P, Q, and R are polynomials. However, as we will see, the method of solution is also applicable when P, Q, and R are general analytic functions.

For the present, then, suppose that P, Q, and R are polynomials and that there is no factor $(x-c)$ that is common to all three of them. If there is such a common factor $(x-c)$, then divide it out before proceeding. Suppose also that we wish to solve equation (1) in the neighborhood of a point x_0. The solution of equation (1) in an interval containing x_0 is closely associated with the behavior of P in that interval.

A point x_0 such that $P(x_0) \neq 0$ is called an **ordinary point**. Since P is continuous, it follows that there is an open interval containing x_0 in which $P(x)$ is never zero. In that interval, which we will call I, we can divide equation (1) by $P(x)$ to obtain

$$y'' + p(x)y' + q(x)y = 0, \tag{2}$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$ are continuous functions on I. Hence, according to the existence and uniqueness theorem, Theorem 3.2.1, there exists a unique solution of equation (1) in the interval I that also satisfies the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$ for arbitrary values of y_0 and y_0'. In this and the following section, we discuss the solution of equation (1) in the neighborhood of an ordinary point.

On the other hand, if $P(x_0) = 0$, then x_0 is called a **singular point** of equation (1). In this case, because $(x - x_0)$ is not a factor of P, Q, and R, at least one of $Q(x_0)$ and $R(x_0)$ is not zero. Consequently, at least one of the coefficients p and q in equation (2) becomes unbounded as $x \to x_0$, and therefore Theorem 3.2.1 does not apply in this case. Sections 5.4 through 5.7 deal with finding solutions of equation (1) in the neighborhood of a singular point.

We now take up the problem of solving equation (1) in the neighborhood of an ordinary point x_0. We look for solutions of the form

$$y = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{3}$$

and assume that the series converges in the interval $|x - x_0| < \rho$ for some $\rho > 0$.

While at first sight it may appear unattractive to seek a solution in the form of a power series, this is actually a convenient and useful form for a solution. Within their intervals of convergence, power series behave very much like polynomials and are easy to manipulate both analytically and numerically. Indeed, even if we can obtain a solution in terms of elementary functions, such as exponential or trigonometric functions, we are likely to need a power series or some equivalent expression if we want to evaluate the solution numerically or to plot its graph.

The most practical way to determine the coefficients a_n is to substitute the series (3) and its derivatives for y, y', and y'' in equation (1). The following examples illustrate this process. The operations, such as differentiation, that are involved in the procedure are justified so long as we stay within the interval of convergence. The differential equations in these examples are also of considerable importance in their own right.

EXAMPLE 1

Find a series solution of the equation

$$y'' + y = 0, \quad -\infty < x < \infty. \tag{4}$$

Solution:

As we know, $\sin x$ and $\cos x$ form a fundamental set of solutions of this equation, so series methods are not needed to solve it. However, this example illustrates the use of power series in a relatively simple case. For equation (4), $P(x) = 1$, $Q(x) = 0$, and $R(x) = 1$; hence every point is an ordinary point.

We look for a solution in the form of a power series about $x_0 = 0$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n \tag{5}$$

and assume that the series converges in some interval $|x| < \rho$. Differentiating equation (5) term by term, we obtain

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1} + \cdots = \sum_{n=1}^{\infty} na_n x^{n-1} \tag{6}$$

and

$$y'' = 2a_2 + 3\cdot 2a_3 x + \cdots + n(n-1)a_n x^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}. \tag{7}$$

Substituting the series (5) and (7) for y and y'' in equation (4) gives

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

To combine the two series, we need to rewrite at least one of them so that both series display the same generic term. (See Problem 22 in Section 5.1.) Thus, in the first sum, we shift the index of summation by replacing n by $n+2$ and starting the sum at 0 rather than 2. We obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} \big((n+2)(n+1)a_{n+2} + a_n\big) x^n = 0.$$

For this equation to be satisfied for all x, the coefficient of each power of x must be zero; hence we conclude that

$$(n+2)(n+1)a_{n+2} + a_n = 0, \quad n = 0, 1, 2, 3, \ldots. \tag{8}$$

Equation (8) is referred to as a **recurrence relation**. The successive coefficients can be evaluated one by one by writing the recurrence relation first for $n = 0$, then for $n = 1$, and so forth. In this example equation (8) relates each coefficient to the second one before it. Thus the even-numbered coefficients ($a_0, a_2, a_4, \ldots$) and the odd-numbered ones ($a_1, a_3, a_5, \ldots$) are determined separately. For the even-numbered coefficients we have

$$a_2 = -\frac{a_0}{2\cdot 1} = -\frac{a_0}{2!}, \quad a_4 = -\frac{a_2}{4\cdot 3} = +\frac{a_0}{4!}, \quad a_6 = -\frac{a_4}{6\cdot 5} = -\frac{a_0}{6!}, \ \ldots.$$

These results suggest that in general, if $n = 2k$, then

$$a_n = a_{2k} = \frac{(-1)^k}{(2k)!} a_0, \quad k = 1, 2, 3, \ldots. \tag{9}$$

We can prove equation (9) by mathematical induction. First, observe that it is true for $k = 1$. Next, assume that it is true for an arbitrary value of k and consider the case $k+1$. We have

$$a_{2k+2} = -\frac{a_{2k}}{(2k+2)(2k+1)} = -\frac{(-1)^k}{(2k+2)(2k+1)(2k)!} a_0 = \frac{(-1)^{k+1}}{(2k+2)!} a_0.$$

Hence equation (9) is also true for $k+1$, and consequently it is true for all positive integers k.

Similarly, for the odd-numbered coefficients

$$a_3 = -\frac{a_1}{2\cdot 3} = -\frac{a_1}{3!}, \quad a_5 = -\frac{a_3}{5\cdot 4} = +\frac{a_1}{5!}, \quad a_7 = -\frac{a_5}{7\cdot 6} = -\frac{a_1}{7!}, \ \dots,$$

and in general, if $n = 2k+1$, then[2]

$$a_n = a_{2k+1} = \frac{(-1)^k}{(2k+1)!}a_1, \quad k = 1, 2, 3, \dots. \tag{10}$$

Substituting these coefficients into equation (5), we have

$$\begin{aligned} y &= a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 \\ &\quad + \cdots + \frac{(-1)^n a_0}{(2n)!}x^{2n} + \frac{(-1)^n a_1}{(2n+1)!}x^{2n+1} + \cdots \\ &= a_0\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots\right] \\ &\quad + a_1\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots\right] \\ &= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}. \end{aligned} \tag{11}$$

We identify two series solutions of equation (4):

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

Using the ratio test, we can show that the series for $y_1(x)$ and $y_2(x)$ converge for all x, and this justifies retroactively all of the steps used in obtaining these solutions. Indeed, the series for $y_1(x)$ is exactly the Taylor series for $\cos x$ about $x = 0$ and the series for $y_2(x)$ is the corresponding Taylor series for $\sin x$. Thus, as we anticipated in equation (11) we have obtained the general solution of equation (4) in the form $y = a_0 \cos x + a_1 \sin x$.

Notice that no conditions are imposed on a_0 and a_1; hence they are arbitrary. From equations (5) and (6) we see that y and y' evaluated at $x = 0$ are a_0 and a_1, respectively. Since the initial conditions $y(0)$ and $y'(0)$ can be chosen arbitrarily, it follows that a_0 and a_1 should be arbitrary until specific initial conditions are stated.

Figures 5.2.1 and 5.2.2 show how the partial sums of the series solutions $y_1(x)$ and $y_2(x)$ approximate $\cos x$ and $\sin x$, respectively. As the number of terms increases, the interval over which

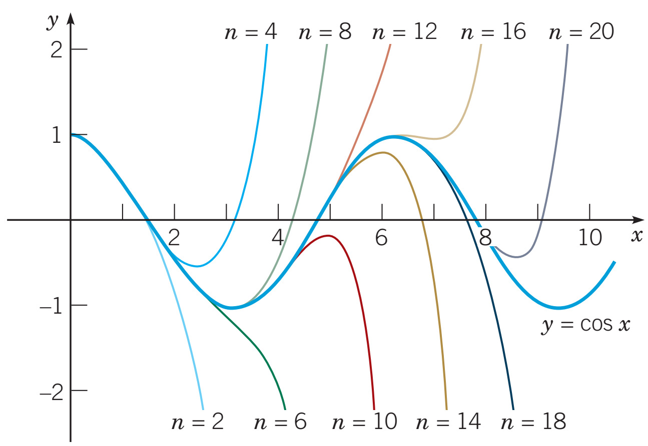

FIGURE 5.2.1 Polynomial approximations to $y = \cos x$. The value of n is the degree of the approximating polynomial.

[2]The result given in equation (10) and other similar formulas in this chapter can be proved by an induction argument resembling the one just given for equation (9). We assume that the results are plausible and omit the inductive argument hereafter. (See Problem 16.)

the approximation is satisfactory becomes longer, and for each x in this interval the accuracy of the approximation improves. However, you should always remember that a truncated power series provides only a local approximation of the solution in a neighborhood of the initial point $x = 0$; it cannot adequately represent the solution for large $|x|$.

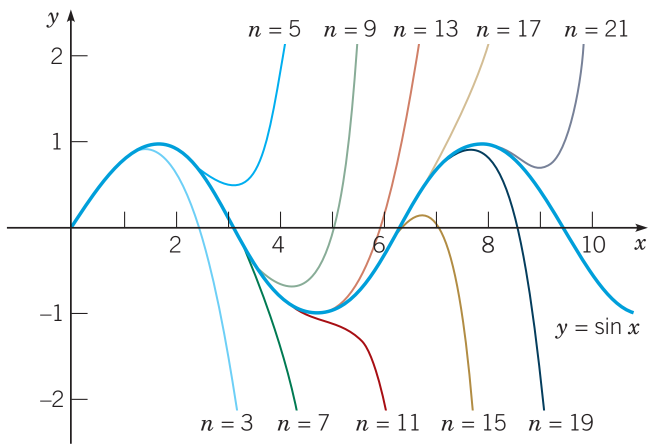

FIGURE 5.2.2 Polynomial approximations to $y = \sin x$. The value of n is the degree of the approximating polynomial.

In Example 1 we knew from the start that $\sin x$ and $\cos x$ form a fundamental set of solutions of equation (4). However, if we had not known this and had simply solved equation (4) using series methods, we would still have obtained the solution (11). In recognition of the fact that the differential equation (4) often occurs in applications, we might decide to give the two solutions of equation (11) special names, perhaps

$$C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}. \tag{12}$$

Then we might ask what properties these functions have. For instance, can we be sure that $C(x)$ and $S(x)$ form a fundamental set of solutions? It follows at once from the series expansions that $C(0) = 1$ and $S(0) = 0$. By differentiating the series for $C(x)$ and $S(x)$ term by term, we find that

$$S'(x) = C(x), \quad C'(x) = -S(x). \tag{13}$$

Thus at $x = 0$, we have $S'(0) = 1$ and $C'(0) = 0$. Consequently, the Wronskian of C and S at $x = 0$ is

$$W[C, S](0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \tag{14}$$

so these functions do indeed form a fundamental set of solutions. By substituting $-x$ for x in each of equations (12), we obtain $C(-x) = C(x)$ and $S(-x) = -S(x)$. Moreover, by calculating with the infinite series,[3] we can show that the functions $C(x)$ and $S(x)$ have all the usual analytical and algebraic properties of the cosine and sine functions, respectively.

Although you probably first saw the sine and cosine functions defined in a more elementary manner in terms of right triangles, it is interesting that these functions can be defined as solutions of a certain simple second-order linear differential equation. To be precise, the function $\sin x$ can be defined as the unique solution of the initial-value problem $y'' + y = 0$, $y(0) = 0$, $y'(0) = 1$; similarly, $\cos x$ can be defined as the unique solution of the initial-value

[3] Such an analysis is given in Section 24 of Knopp (see the References at the end of this chapter).

problem $y'' + y = 0$, $y(0) = 1$, $y'(0) = 0$. Many other functions that are important in mathematical physics are also defined as solutions of certain initial-value problems. For most of these functions there is no simpler or more elementary way to approach them.

EXAMPLE 2

Find a series solution in powers of x of Airy's[4] equation

$$y'' - xy = 0, \qquad -\infty < x < \infty. \tag{15}$$

Solution:

For this equation $P(x) = 1$, $Q(x) = 0$, and $R(x) = -x$; hence every point is an ordinary point. We assume that

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{16}$$

and that the series converges in some interval $|x| < \rho$. The series for y'' is given by equation (7); as explained in the preceding example, we can rewrite it as

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n. \tag{17}$$

Substituting the series (16) and (17) for y and y'' into the left-hand side of equation (15), we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1)a_{x+2}x^n - \sum_{n=0}^{\infty} a_n x^{n+1}. \tag{18}$$

Next, we shift the index of summation in the second series on the right-hand side of equation (18) by replacing n by $n-1$ and starting the summation at 1 rather than zero. Thus we write equation (15) as

$$2 \cdot 1 a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0.$$

Again, for this equation to be satisfied for all x in some interval, the coefficients of like powers of x must be zero; hence $a_2 = 0$, and we obtain the recurrence relation

$$(n+2)(n+1)a_{n+2} - a_{n-1} = 0 \quad \text{for } n = 1, 2, 3, \ldots. \tag{19}$$

Since a_{n+2} is given in terms of a_{n-1}, the a's are determined in steps of three. Thus a_0 determines a_3, which in turn determines a_6, ... ; a_1 determines a_4, which in turn determines a_7, ... ; and a_2 determines a_5, which in turn determines a_8, ... Since $a_2 = 0$, we immediately conclude that $a_5 = a_8 = a_{11} = \cdots = 0$.

For the sequence $a_0, a_3, a_6, a_9, \ldots$ we set $n = 1, 4, 7, 10, \ldots$ in the recurrence relation:

$$a_3 = \frac{a_0}{2 \cdot 3}, \quad a_6 = \frac{a_3}{5 \cdot 6} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6}, \quad a_9 = \frac{a_6}{8 \cdot 9} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}, \ldots.$$

These results suggest the general formula

$$a_{3n} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1)(3n)}, \qquad n \geq 4.$$

[4] Sir George Biddell Airy (1801–1892), an English astronomer and mathematician, was director of the Greenwich Observatory from 1835 to 1881. He studied the equation named for him in an 1838 paper on optics. One reason why Airy's equation is of interest is that for x negative the solutions are similar to trigonometric functions, and for x positive they are similar to hyperbolic functions. Can you explain why it is reasonable to expect such behavior?

For the sequence $a_1, a_4, a_7, a_{10}, \ldots$, we set $n = 2, 5, 8, 11, \ldots$ in the recurrence relation:

$$a_4 = \frac{a_1}{3\cdot 4}, \quad a_7 = \frac{a_4}{6\cdot 7} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7}, \quad a_{10} = \frac{a_7}{9\cdot 10} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}, \ldots .$$

In general, we have

$$a_{3n+1} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdots(3n)(3n+1)}, \qquad n \geq 4.$$

Thus the general solution of Airy's equation is

$$\begin{aligned} y(x) &= a_0\left[1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \cdots + \frac{x^{3n}}{2\cdot 3\cdots(3n-1)(3n)} + \cdots\right] \\ &\quad + a_1\left[x + \frac{x^4}{3\cdot 4} + \frac{x^7}{3\cdot 4\cdot 6\cdot 7} + \cdots + \frac{x^{3n+1}}{3\cdot 4\cdots(3n)(3n+1)} + \cdots\right] \\ &= a_0 y_1(x) + a_1 y_2(x) \end{aligned} \tag{20}$$

where $y_1(x)$ and $y_2(x)$ are the first and second bracketed expressions in equation (20).

Having obtained these two series solutions, we can now investigate their convergence. Because of the rapid growth of the denominators of the terms in the series for $y_1(x)$ and for $y_2(x)$, we might expect these series to have a large radius of convergence. Indeed, it is easy to use the ratio test to show that both of these series converge for all x; see Problem 17.

Assume for the moment that the series for y_1 and y_2 do converge for all x. Then, by choosing first $a_0 = 1, a_1 = 0$ and then $a_0 = 0, a_1 = 1$, it follows that y_1 and y_2 are individually solutions of equation (15). Notice that y_1 satisfies the initial conditions $y_1(0) = 1$, $y_1'(0) = 0$ and that y_2 satisfies the initial conditions $y_2(0) = 0$, $y_2'(0) = 1$. Thus $W[y_1, y_2](0) = 1 \neq 0$, and consequently y_1 and y_2 are a fundamental set of solutions. Hence the general solution of Airy's equation is

$$y = a_0 y_1(x) + a_1 y_2(x) \qquad -\infty < x < \infty.$$

In Figures 5.2.3 and 5.2.4, respectively, we show the graphs of the solutions y_1 and y_2 of Airy's equation as well as graphs of several partial sums of the two series in equation (20). Again, the partial sums provide local approximations to the solutions in a neighborhood of the origin. Although the quality of the approximation improves as the number of terms increases, no polynomial can adequately represent y_1 and y_2 for large $|x|$. A practical way to estimate the interval in which a given partial sum is reasonably accurate is to compare the graphs of that partial sum and the next one, obtained by including one more term. As soon as the graphs begin to separate noticeably, we can be confident that the original partial sum is no longer accurate. For example, in Figure 5.2.3 the graphs for $n = 24$ and $n = 27$ begin to separate at about $x = -9/2$. Thus, beyond this point, the partial sum of degree 24 is worthless as an approximation to the solution.

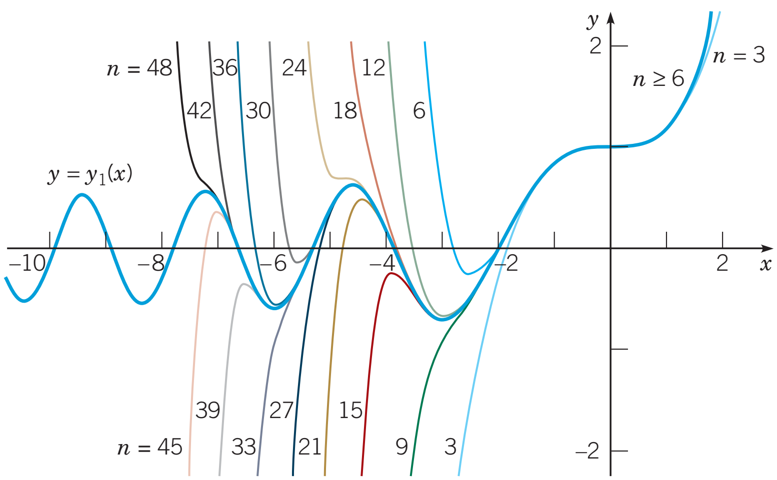

FIGURE 5.2.3 Polynomial approximations to the solution $y = y_1(x)$ of Airy's equation. The value of n is the degree of the approximating polynomial.

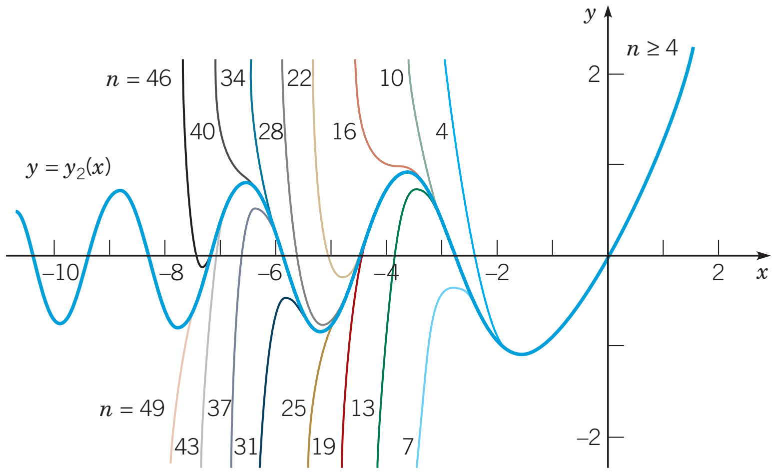

FIGURE 5.2.4 Polynomial approximations to the solution $y = y_2(x)$ of Airy's equation. The value of n is the degree of the approximating polynomial.

Observe that both y_1 and y_2 are monotone for $x > 0$ and oscillatory for $x < 0$. You can also see from the figures that the oscillations are not uniform but, rather, decay in amplitude and increase in frequency as the distance from the origin increases. In contrast to Example 1, the solutions y_1 and y_2 of Airy's equation are not elementary functions that you have already encountered in calculus. However, because of their importance in some physical applications, these functions have been extensively studied, and their properties are well known to applied mathematicians and scientists.

EXAMPLE 3

Find a solution of Airy's equation in powers of $x - 1$.

Solution:

The point $x = 1$ is an ordinary point of equation (15), and thus we look for a solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n,$$

where we assume that the series converges in some interval $|x - 1| < \rho$. Then

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n,$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n.$$

Substituting for y and y'' in equation (15), we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n = x \sum_{n=0}^{\infty} a_n (x-1)^n. \tag{21}$$

Now to equate the coefficients of like powers of $(x - 1)$, we must express x, the coefficient of y in equation (15), in powers of $x - 1$; that is, we write $x = 1 + (x - 1)$. Note that this is precisely the

Taylor series for x about $x = 1$. (See Problem 9 in Section 5.1.) Then equation (21) takes the form

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n = (1+(x-1))\sum_{n=0}^{\infty}a_n(x-1)^n$$

$$= \sum_{n=0}^{\infty}a_n(x-1)^n + \sum_{n=0}^{\infty}a_n(x-1)^{n+1}.$$

Shifting the index of summation in the second series on the right gives

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n = \sum_{n=0}^{\infty}a_n(x-1)^n + \sum_{n=1}^{\infty}a_{n-1}(x-1)^n.$$

Equating coefficients of like powers of $x-1$, we obtain

$$\begin{aligned}
2a_2 &= a_0,\\
(3\cdot 2)a_3 &= a_1 + a_0,\\
(4\cdot 3)a_4 &= a_2 + a_1,\\
(5\cdot 4)a_5 &= a_3 + a_2,\\
&\vdots
\end{aligned}$$

The general recurrence relation is

$$(n+2)(n+1)a_{n+2} = a_n + a_{n-1} \quad \text{for} \quad n \geq 1. \tag{22}$$

Solving for the first few coefficients a_n in terms of a_0 and a_1, we find that

$$a_2 = \frac{a_0}{2}, \quad a_3 = \frac{a_1}{6} + \frac{a_0}{6}, \quad a_4 = \frac{a_2}{12} + \frac{a_1}{12} = \frac{a_0}{24} + \frac{a_1}{12}, \quad a_5 = \frac{a_3}{20} + \frac{a_2}{20} = \frac{a_0}{30} + \frac{a_1}{120}.$$

Hence

$$y = a_0\left[1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \frac{(x-1)^5}{30} + \cdots\right]$$

$$+ a_1\left[(x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \frac{(x-1)^5}{120} + \cdots\right]. \tag{23}$$

In general, when the recurrence relation has more than two terms, as in equation (22), the determination of a formula for a_n in terms a_0 and a_1 will be fairly complicated, if not impossible. In this example such a formula is not readily apparent. Lacking such a formula, we cannot test the two series in equation (23) for convergence by direct methods such as the ratio test. However, we shall see in Section 5.3 that even without knowing the formula for a_n, it is possible to establish that the two series in equation (23) converge for all x. Further, they define functions y_3 and y_4 that are a fundamental set of solutions of the Airy equation (15). Thus

$$y = a_0 y_3(x) + a_1 y_4(x)$$

is the general solution of Airy's equation for $-\infty < x < \infty$.

While Airy's equation is not particularly complicated, Example 3 shows some of the complications encountered when looking for a power series solution expressed in powers of $x - x_0$ with $x_0 \neq 0$. There is an alternative. We can make the change of variable $x - x_0 = t$, obtaining a new differential equation for y as a function of t, and then look for solutions of this new equation of the form $\sum_{n=0}^{\infty} a_n t^n$. When we have finished the calculations, we replace t by $x - x_0$ (see Problem 15).

In Examples 2 and 3 we have found two sets of solutions of Airy's equation. The functions y_1 and y_2 defined by the series in equation (20) are a fundamental set of solutions of equation (15) for all x, and this is also true for the functions y_3 and y_4 defined by the series in equation (23). According to the general theory of second-order linear equations, each of the first two functions can be expressed as a linear combination of the latter two functions, and vice versa—a result that is certainly not obvious from an examination of the series alone.

Finally, we emphasize that it is not particularly important if, as in Example 3, we are unable to determine the general coefficient a_n in terms of a_0 and a_1. What is essential is that we can determine *as many coefficients as we want.* Thus we can find as many terms in the two series solutions as we want, even if we cannot determine the general term. While the task of calculating several coefficients in a power series solution is not difficult, it can be tedious. A symbolic manipulation package can be very helpful here; some are able to find a specified number of terms in a power series solution in response to a single command. With a suitable graphics package we can also produce plots such as those shown in the figures in this section.

Problems

In each of Problems 1 through 11:

a. Seek power series solutions of the given differential equation about the given point x_0; find the recurrence relation that the coefficients must satisfy.

b. Find the first four nonzero terms in each of two solutions y_1 and y_2 (unless the series terminates sooner).

c. By evaluating the Wronskian $W[y_1, y_2](x_0)$, show that y_1 and y_2 form a fundamental set of solutions.

d. If possible, find the general term in each solution.

1. $y'' - y = 0, \quad x_0 = 0$

2. $y'' + 3y' = 0, \quad x_0 = 0$

3. $y'' - xy' - y = 0, \quad x_0 = 0$

4. $y'' - xy' - y = 0, \quad x_0 = 1$

5. $y'' + k^2x^2y = 0, \quad x_0 = 0, \quad k$ a constant

6. $(1 - x)y'' + y = 0, \quad x_0 = 0$

7. $y'' + xy' + 2y = 0, \quad x_0 = 0$

8. $xy'' + y' + xy = 0, \quad x_0 = 1$

9. $(3 - x^2)y'' - 3xy' - y = 0, \quad x_0 = 0$

10. $2y'' + xy' + 3y = 0, \quad x_0 = 0$

11. $2y'' + (x + 1)y' + 3y = 0, \quad x_0 = 2$

In each of Problems 12 through 14:

a. Find the first five nonzero terms in the solution of the given initial-value problem.

G **b.** Plot the four-term and the five-term approximations to the solution on the same axes.

c. From the plot in part b, estimate the interval in which the four-term approximation is reasonably accurate.

12. $y'' - xy' - y = 0, \quad y(0) = 2, \quad y'(0) = 1$; see Problem 3

13. $y'' + xy' + 2y = 0, \quad y(0) = 4, \quad y'(0) = -1$; see Problem 7

14. $(1 - x)y'' + xy' - y = 0, \quad y(0) = -3, \quad y'(0) = 2$

15. **a.** By making the change of variable $x - 1 = t$ and assuming that y has a Taylor series in powers of t, find two series solutions of

$$y'' + (x - 1)^2y' + (x^2 - 1)y = 0$$

in powers of $x - 1$.

b. Show that you obtain the same result by assuming that y has a Taylor series in powers of $x - 1$ and also expressing the coefficient $x^2 - 1$ in powers of $x - 1$.

16. Prove equation (10).

17. Show directly, using the ratio test, that the two series solutions of Airy's equation about $x = 0$ converge for all x; see equation (20) of the text.

18. **The Hermite Equation.** The equation

$$y'' - 2xy' + \lambda y = 0, \quad -\infty < x < \infty,$$

where λ is a constant, is known as the Hermite[5] equation. It is an important equation in mathematical physics.

a. Find the first four nonzero terms in each of two solutions about $x = 0$ and show that they form a fundamental set of solutions.

b. Observe that if λ is a nonnegative even integer, then one or the other of the series solutions terminates and becomes a polynomial. Find the polynomial solutions for $\lambda = 0, 2, 4, 6, 8$, and 10. Note that each polynomial is determined only up to a multiplicative constant.

c. The Hermite polynomial $H_n(x)$ is defined as the polynomial solution of the Hermite equation with $\lambda = 2n$ for which the coefficient of x^n is 2^n. Find $H_0(x), H_1(x), \ldots, H_5(x)$.

19. Consider the initial-value problem $y' = \sqrt{1 - y^2}$, $y(0) = 0$.

a. Show that $y = \sin x$ is the solution of this initial-value problem.

b. Look for a solution of the initial-value problem in the form of a power series about $x = 0$. Find the coefficients up to the term in x^3 in this series.

In each of Problems 20 through 23, plot several partial sums in a series solution of the given initial-value problem about $x = 0$, thereby obtaining graphs analogous to those in Figures 5.2.1 through 5.2.4 (except that we do not know an explicit formula for the actual solution).

G **20.** $y'' + xy' + 2y = 0, \; y(0) = 0, \; y'(0) = 1$; see Problem 7

G **21.** $(4 - x^2)y'' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1$

G **22.** $y'' + x^2y = 0, \quad y(0) = 1, \quad y'(0) = 0$; see Problem 5

G **23.** $(1 - x)y'' + xy' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 1$

[5]Charles Hermite (1822–1901) was an influential French analyst and algebraist. An inspiring teacher, he was professor at the École Polytechnique and the Sorbonne. He introduced the Hermite functions in 1864 and showed in 1873 that e is a transcendental number (that is, e is not a root of any polynomial equation with rational coefficients). His name is also associated with Hermitian matrices (see Section 7.3), some of whose properties he discovered.

5.3 Series Solutions Near an Ordinary Point, Part II

In the preceding section we considered the problem of finding solutions of

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{1}$$

where P, Q, and R are polynomials, in the neighborhood of an ordinary point x_0. Assuming that equation (1) does have a solution $y = \phi(x)$ and that ϕ has a Taylor series

$$\phi(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{2}$$

that converges for $|x - x_0| < \rho$, where $\rho > 0$, we found that the a_n can be determined by directly substituting the series (2) for y in equation (1).

Let us now consider how we might justify the statement that if x_0 is an ordinary point of equation (1), then there exist solutions of the form (2). We also consider the question of the radius of convergence of such a series. In doing this, we are led to a generalization of the definition of an ordinary point.

Suppose, then, that there is a solution of equation (1) of the form (2). By differentiating equation (2) m times and setting x equal to x_0, we obtain

$$m!a_m = \phi^{(m)}(x_0). \tag{3}$$

Hence, to compute a_n in the series (2), we must show that we can determine $\phi^{(n)}(x_0)$ for $n = 0, 1, 2, \ldots$ from the differential equation (1).

Suppose that $y = \phi(x)$ is a solution of equation (1) satisfying the initial conditions $y(x_0) = y_0$, $y'(x_0) = y'_0$. Then $a_0 = y_0$ and $a_1 = y'_0$. If we are solely interested in finding a solution of equation (1) without specifying any initial conditions, then a_0 and a_1 remain arbitrary. To determine $\phi^{(n)}(x_0)$ and the corresponding a_n for $n = 2, 3, \ldots$, we turn to equation (1) with the goal of finding a formula for $\phi''(x), \phi'''(x), \ldots$. Since ϕ is a solution of equation (1), we have

$$P(x)\phi''(x) + Q(x)\phi'(x) + R(x)\phi(x) = 0.$$

For the interval about x_0 for which P is nonzero, we can write this equation in the form

$$\phi''(x) = -p(x)\phi'(x) - q(x)\phi(x), \tag{4}$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$. Observe that, at $x = x_0$, the right-hand side of equation (4) is known, thus allowing us to compute $\phi''(x_0)$: Setting x equal to x_0 in equation (4) gives

$$\phi''(x_0) = -p(x_0)\phi'(x_0) - q(x_0)\phi(x_0) = -p(x_0)a_1 - q(x_0)a_0.$$

Hence, using equation (3) with $m = 2$, we find that a_2 is given by

$$2!a_2 = \phi''(x_0) = -p(x_0)a_1 - q(x_0)a_0. \tag{5}$$

To determine a_3, we differentiate equation (4) and then set x equal to x_0, obtaining

$$\begin{aligned} 3!a_3 = \phi'''(x_0) &= -\left(p(x)\phi'(x) + q(x)\phi(x)\right)'\Big|_{x=x_0} \\ &= -2!p(x_0)a_2 - \left(p'(x_0) + q(x_0)\right)a_1 - q'(x_0)a_0. \end{aligned} \tag{6}$$

Substituting for a_2 from equation (5) gives a_3 in terms of a_1 and a_0.

Since P, Q, and R are polynomials and $P(x_0) \neq 0$, all the derivatives of p and q exist at x_0. Hence we can continue to differentiate equation (4) indefinitely, determining after each differentiation the successive coefficients $a_4, a_5, \ldots$ by setting x equal to x_0.

EXAMPLE 1

Let $y = \phi(x)$ be a solution of the initial value problem $(1 + x^2)y'' + 2xy' + 4x^2y = 0$, $y(0) = 0$, $y'(0) = 1$. Determine $\phi''(0)$, $\phi'''(0)$, and $\phi^{(4)}(0)$.

Solution:

To find $\phi''(0)$, simply evaluate the differential equation when $x = 0$:

$$(1 + 0^2)\phi''(0) + 2 \cdot 0 \cdot \phi'(0) + 4 \cdot 0^2 \cdot \phi(0) = 0,$$

so $\phi''(0) = 0$.

To find $\phi'''(0)$, differentiate the differential equation with respect to x:

$$(1 + x^2)\phi'''(x) + 2x\phi''(x) + 2x\phi''(x) + 2\phi'(x) + 4x^2\phi'(x) + 8x\phi(x) = 0. \tag{7}$$

Then evaluate the resulting equation (7) at $x = 0$:

$$\phi'''(0) + 2\phi'(0) = 0.$$

Thus $\phi'''(0) = -2\phi'(0) = -2$ (because $\phi'(0) = 1$).

Finally, to find $\phi^{(4)}(0)$, first differentiate equation (7) with respect to x:

$$\left(1 + x^2\right)\phi^{(4)}(x) + 2x\phi'''(x) + 4x\phi'''(x) + 4\phi''(x) + \left(2 + 4x^2\right)\phi''(x) + 8x\phi'(x) + 8x\phi'(x) + 8\phi(x) = 0.$$

Evaluating this equation at $x = 0$ we find

$$\phi^{(4)}(0) + 6\phi''(0) + 8\phi(0) = 0.$$

Finally, using $\phi(0) = 0$ and $\phi''(0) = 0$, we conclude that $\phi^{(4)}(0) = 0$.

Notice that the important property that we used in determining the a_n was that we could compute infinitely many derivatives of the functions p and q. It might seem reasonable to relax our assumption that the functions p and q are ratios of polynomials and simply require that they be infinitely differentiable in the neighborhood of x_0. Unfortunately, this condition is too weak to ensure that we can prove the convergence of the resulting series expansion for $y = \phi(x)$. What is needed is to assume that the functions p and q are *analytic* at x_0; that is, they have Taylor series expansions that converge to them in some interval about the point x_0:

$$p(x) = p_0 + p_1(x - x_0) + \cdots + p_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} p_n(x - x_0)^n, \tag{8}$$

$$q(x) = q_0 + q_1(x - x_0) + \cdots + q_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} q_n(x - x_0)^n. \tag{9}$$

With this idea in mind, we can generalize the definitions of an ordinary point and a singular point of equation (1) as follows: if the functions $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$ are analytic at x_0, then the point x_0 is said to be an **ordinary point** of the differential equation (1); otherwise, it is a **singular point**.

Now let us turn to the question of the interval of convergence of the series solution. One possibility is actually to compute the series solution for each problem and then to apply one of the tests for convergence of an infinite series to determine its radius of convergence. Unfortunately, these tests require us to obtain an expression for the general coefficient a_n as a function of n, and this task is often quite difficult, if not impossible; recall Example 3 in Section 5.2. However, the question can be answered at once for a wide class of problems by the following theorem.

Theorem 5.3.1

If x_0 is an ordinary point of the differential equation (1)

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

that is, if $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$ are analytic at x_0, then the general solution of equation (1) is

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 y_1(x) + a_1 y_2(x),$$

where a_0 and a_1 are arbitrary, and y_1 and y_2 are two power series solutions that are analytic at x_0. The solutions y_1 and y_2 form a fundamental set of solutions. Further, the radius of convergence for each of the series solutions y_1 and y_2 is at least as large as the minimum of the radii of convergence of the series for p and q.

To see that y_1 and y_2 are a fundamental set of solutions, note that they have the form $y_1(x) = 1+b_2(x-x_0)^2+\cdots$ and $y_2(x) = (x-x_0)+c_2(x-x_0)^2+\cdots$, where $b_2+c_2 = a_2$. Hence y_1 satisfies the initial conditions $y_1(x_0) = 1$, $y_1'(x_0) = 0$, and y_2 satisfies the initial conditions $y_2(x_0) = 0$, $y_2'(x_0) = 1$. Thus $W[y_1, y_2](x_0) = 1$.

Also note that although calculating the coefficients by successively differentiating the differential equation is excellent in theory, it is usually not a practical computational procedure. Rather, you should substitute the series (2) for y in the differential equation (1) and determine the coefficients so that the differential equation is satisfied, as in the examples in the preceding section.

We will not prove this theorem, which in a slightly more general form was established by Fuchs.[6] What is important for our purposes is that there is a series solution of the form (2) and that the radius of convergence of the series solution cannot be less than the smaller of the radii of convergence of the series for p and q; hence we need only determine these.

This can be done in either of two ways. Again, one possibility is simply to compute the power series for p and q and then to determine the radii of convergence by using one of the convergence tests for infinite series. However, there is an easier way when $P(x)$, $Q(x)$, and $R(x)$ are polynomials. It is shown in the theory of functions of a complex variable that the ratio of two polynomials, say, $Q(x)/P(x)$, has a convergent power series expansion about a point $x = x_0$ if $P(x_0) \neq 0$. Further, if we assume that any factors common to $Q(x)$ and $P(x)$ have been canceled, then the radius of convergence of the power series for $Q(x)/P(x)$ about the point x_0 is precisely the distance from x_0 to the nearest zero of $P(x)$. In determining this distance, we must remember that $P(x) = 0$ may have complex roots, and these must also be considered.

EXAMPLE 2

What is the radius of convergence of the Taylor series for $(1 + x^2)^{-1}$ about $x = 0$?

Solution:

One way to proceed is to find the Taylor series in question, namely,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots.$$

Then it can be verified by the ratio test that $\rho = 1$. Another approach is to note that the zeros of $1 + x^2$ are $x = \pm i$. Since the distance in the complex plane from 0 to i or to $-i$ is 1, the radius of convergence of the power series about $x = 0$ is 1.

[6] Lazarus Immanuel Fuchs (1833–1902), a German mathematician, was a student and later a professor at the University of Berlin. He proved the result of Theorem 5.3.1 in 1866. His most important research was on singular points of linear differential equations. He recognized the significance of regular singular points (Section 5.4), and equations whose only singularities, including the point at infinity, are regular singular points are known as Fuchsian equations.

EXAMPLE 3

What is the radius of convergence of the Taylor series for $(x^2 - 2x + 2)^{-1}$ about $x = 0$? about $x = 1$?

Solution:

First notice that

$$x^2 - 2x + 2 = 0$$

has solutions $x = 1 \pm i$. The distance in the complex plane from $x = 0$ to either $x = 1 + i$ or $x = 1 - i$ is $\sqrt{2}$; hence the radius of convergence of the Taylor series expansion $\sum_{n=0}^{\infty} a_n x^n$ about $x = 0$ is $\sqrt{2}$.

The distance in the complex plane from $x = 1$ to either $x = 1 + i$ or $x = 1 - i$ is 1; hence the radius of convergence of the Taylor series expansion $\sum_{n=0}^{\infty} b_n (x-1)^n$ about $x = 1$ is 1.

According to Theorem 5.3.1, the series solutions of the Airy equation in Examples 2 and 3 of the preceding section converge for all values of x and $x - 1$, respectively, since in each problem $P(x) = 1$ and hence is never zero.

A series solution may converge for a wider range of x than indicated by Theorem 5.3.1, so the theorem actually gives only a lower bound on the radius of convergence of the series solution. This is illustrated by the Legendre polynomial solution of the Legendre equation given in the next example.

EXAMPLE 4

Determine a lower bound for the radius of convergence of series solutions about $x = 0$ for the Legendre equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

where α is a constant.

Solution:

Note that $P(x) = 1 - x^2$, $Q(x) = -2x$, and $R(x) = \alpha(\alpha + 1)$ are polynomials, and that the zeros of P, namely, $x = \pm 1$, are a distance 1 from $x = 0$. Hence a series solution of the form $\sum_{n=0}^{\infty} a_n x^n$ converges at least for $|x| < 1$, and possibly for larger values of x. Indeed, it can be shown that if α is a positive integer, one of the series solutions terminates after a finite number of terms, that is, one solution is a polynomial, and hence converges not just for $|x| < 1$ but for all x. For example, if $\alpha = 1$, the polynomial solution is $y = x$. See Problems 17 through 23 at the end of this section for a further discussion of the Legendre equation.

EXAMPLE 5

Determine a lower bound for the radius of convergence of series solutions of the differential equation

$$(1 + x^2)y'' + 2xy' + 4x^2 y = 0 \tag{10}$$

about the point $x = 0$; about the point $x = -\frac{1}{2}$.

Solution:

Again P, Q, and R are polynomials, and P has zeros at $x = \pm i$. The distance in the complex plane from 0 to $\pm i$ is 1, and from $-\dfrac{1}{2}$ to $\pm i$ is $\sqrt{1 + \dfrac{1}{4}} = \dfrac{\sqrt{5}}{2}$. Hence in the first case the series $\sum_{n=0}^{\infty} a_n x^n$ converges at least for $|x| < 1$, and in the second case the series $\sum_{n=0}^{\infty} b_n \left(x + \dfrac{1}{2}\right)^n$ converges at least for $\left|x + \dfrac{1}{2}\right| < \dfrac{\sqrt{5}}{2}$.

An interesting observation that we can make about equation (10) follows from Theorems 3.2.1 and 5.3.1. Suppose that initial conditions $y(0) = y_0$ and $y'(0) = y'_0$ are given. Since $1 + x^2 \neq 0$ for all x, we know from Theorem 3.2.1 that there exists a unique solution of the initial-value problem on $-\infty < x < \infty$. On the other hand, Theorem 5.3.1 only guarantees a series solution of the form $\sum_{n=0}^{\infty} a_n x^n$ (with $a_0 = y_0$, $a_1 = y'_0$) for $-1 < x < 1$. The unique solution on the interval $-\infty < x < \infty$ may not have a power series about $x = 0$ that converges for all x.

EXAMPLE 6

Can we determine a series solution about $x = 0$ for the differential equation

$$y'' + (\sin x)y' + (1 + x^2)y = 0,$$

and if so, what is the radius of convergence?

Solution:

For this differential equation, $p(x) = \sin x$ and $q(x) = 1 + x^2$. Recall from calculus that $\sin x$ has a Taylor series expansion about $x = 0$ that converges for all x. Further, q also has a Taylor series expansion about $x = 0$, namely, $q(x) = 1 + x^2$, that converges for all x. Thus there is a series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ with a_0 and a_1 arbitrary, and the series converges for all x.

Problems

In each of Problems 1 through 3, determine $\phi''(x_0)$, $\phi'''(x_0)$, and $\phi^{(4)}(x_0)$ for the given point x_0 if $y = \phi(x)$ is a solution of the given initial-value problem.

1. $y'' + xy' + y = 0; \quad y(0) = 1, \quad y'(0) = 0$

2. $x^2y'' + (1 + x)y' + 3(\ln x)y = 0; \quad y(1) = 2, \quad y'(1) = 0$

3. $y'' + x^2y' + (\sin x)y = 0; \quad y(0) = a_0, \quad y'(0) = a_1$

In each of Problems 4 through 6, determine a lower bound for the radius of convergence of series solutions about each given point x_0 for the given differential equation.

4. $y'' + 4y' + 6xy = 0; \quad x_0 = 0, \quad x_0 = 4$

5. $(x^2 - 2x - 3)y'' + xy' + 4y = 0; \quad x_0 = 4, \quad x_0 = -4, \quad x_0 = 0$

6. $(1 + x^3)y'' + 4xy' + y = 0; \quad x_0 = 0, \quad x_0 = 2$

7. Determine a lower bound for the radius of convergence of series solutions about the given x_0 for each of the differential equations in Problems 1 through 11 of Section 5.2.

8. **The Chebyshev Equation.** The Chebyshev[7] differential equation is

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a constant.

a. Determine two solutions in powers of x for $|x| < 1$, and show that they form a fundamental set of solutions.

b. Show that if α is a nonnegative integer n, then there is a polynomial solution of degree n. These polynomials, when properly normalized, are called the **Chebyshev polynomials**. They are very useful in problems that require a polynomial approximation to a function defined on $-1 \leq x \leq 1$.

c. Find a polynomial solution for each of the cases $\alpha = n = 0$, 1, 2, 3.

For each of the differential equations in Problems 9 through 11, find the first four nonzero terms in each of two power series solutions about the origin. Show that they form a fundamental set of solutions. What do you expect the radius of convergence to be for each solution?

9. $y'' + (\sin x)y = 0$

10. $e^x y'' + xy = 0$

11. $(\cos x)y'' + xy' - 2y = 0$

12. Let $y = x$ and $y = x^2$ be solutions of a differential equation $P(x)y'' + Q(x)y' + R(x)y = 0$. Can you say whether the point $x = 0$ is an ordinary point or a singular point? Prove your answer.

First-Order Equations. The series methods discussed in this section are directly applicable to the first-order linear differential equation $P(x)y' + Q(x)y = 0$ at a point x_0, if the function $p = Q/P$ has a Taylor series expansion about that point. Such a point is called an ordinary point, and further, the radius of convergence of the series $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ is at least as large as the radius of convergence of the series for Q/P. In each of Problems 13 through 16, solve the given differential equation by a series in powers of x and verify that a_0 is arbitrary in each case. Problem 17 involves a nonhomogeneous differential equation to which series methods can be easily extended. Where possible, compare the series solution with the solution obtained by using the methods of Chapter 2.

[7] Pafnuty L. Chebyshev (1821–1894), the most influential nineteenth-century Russian mathematician, was for 35 years professor at the University of St. Petersburg, which produced a long line of distinguished mathematicians. His study of Chebyshev polynomials began in about 1854 as part of an investigation of the approximation of functions by polynomials. Chebyshev is also known for his work in number theory and probability.

13. $y' - y = 0$

14. $y' - xy = 0$

15. $(1 - x)y' = y$

16. $y' - y = x^2$

The Legendre Equation. Problems 17 through 23 deal with the Legendre[8] equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.$$

As indicated in Example 4, the point $x = 0$ is an ordinary point of this equation, and the distance from the origin to the nearest zero of $P(x) = 1 - x^2$ is 1. Hence the radius of convergence of series solutions about $x = 0$ is at least 1. Also notice that we need to consider only $\alpha > -1$ because if $\alpha \le -1$, then the substitution $\alpha = -(1 + \gamma)$, where $\gamma \ge 0$, leads to the Legendre equation $(1 - x^2)y'' - 2xy' + \gamma(\gamma + 1)y = 0$.

17. Show that two solutions of the Legendre equation for $|x| < 1$ are

$$y_1(x) = 1 - \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha-2)(\alpha+1)(\alpha+3)}{4!}x^4 + \sum_{m=3}^{\infty}(-1)^m \frac{\alpha\cdots(\alpha-2m+2)(\alpha+1)\cdots(\alpha+2m-1)}{(2m)!}x^{2m},$$

$$y_2(x) = x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)}{5!}x^5 + \sum_{m=3}^{\infty}(-1)^m \times \frac{(\alpha-1)\cdots(\alpha-2m+1)(\alpha+2)\cdots(\alpha+2m)}{(2m+1)!}x^{2m+1}.$$

18. Show that if α is zero or a positive even integer $2n$, the series solution y_1 reduces to a polynomial of degree $2n$ containing only even powers of x. Find the polynomials corresponding to $\alpha = 0$, 2, and 4. Show that if α is a positive odd integer $2n + 1$, the series solution y_2 reduces to a polynomial of degree $2n + 1$ containing only odd powers of x. Find the polynomials corresponding to $\alpha = 1$, 3, and 5.

19. The Legendre polynomial $P_n(x)$ is defined as the polynomial solution of the Legendre equation with $\alpha = n$ that also satisfies the condition $P_n(1) = 1$.

a. Using the results of Problem 18, find the Legendre polynomials $P_0(x), \ldots, P_5(x)$.

G **b.** Plot the graphs of $P_0(x), \ldots, P_5(x)$ for $-1 \le x \le 1$.

N **c.** Find the zeros of $P_0(x), \ldots, P_5(x)$.

20. The Legendre polynomials play an important role in mathematical physics. For example, in solving Laplace's equation (the potential equation) in spherical coordinates, we encounter the equation

$$\frac{d^2F(\varphi)}{d\varphi^2} + \cot\varphi \frac{dF(\varphi)}{d\varphi} + n(n+1)F(\varphi) = 0, \quad 0 < \varphi < \pi,$$

where n is a positive integer. Show that the change of variable $x = \cos\varphi$ leads to the Legendre equation with $\alpha = n$ for $y = f(x) = F(\arccos x)$.

21. Show that for $n = 0, 1, 2, 3$, the corresponding Legendre polynomial is given by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n.$$

This formula, known as Rodrigues's formula,[9] is true for all positive integers n.

22. Show that the Legendre equation can also be written as

$$\big((1 - x^2)y'\big)' = -\alpha(\alpha + 1)y.$$

Then it follows that

$$\big((1 - x^2)P_n'(x)\big)' = -n(n + 1)P_n(x)$$

and

$$\big((1 - x^2)P_m'(x)\big)' = -m(m + 1)P_m(x).$$

By multiplying the first equation by $P_m(x)$ and the second equation by $P_n(x)$, integrating by parts, and then subtracting one equation from the other, show that

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0 \quad \text{if } n \ne m.$$

This property of the Legendre polynomials is known as the orthogonality property. If $m = n$, it can be shown that the value of the preceding integral is $2/(2n + 1)$.

23. Given a polynomial f of degree n, it is possible to express f as a linear combination of $P_0, P_1, P_2, \ldots, P_n$:

$$f(x) = \sum_{k=0}^{n} a_k P_k(x).$$

Using the result of Problem 22, show that

$$a_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)P_k(x)\,dx.$$

[8] Adrien-Marie Legendre (1752–1833) held various positions in the French Académie des Sciences from 1783 onward. His primary work was in the fields of elliptic functions and number theory. The Legendre functions, solutions of Legendre's equation, first appeared in 1784 in his study of the attraction of spheroids.

[9] Benjamin Olinde Rodrigues (1795–1851) published this result as part of his doctoral thesis from the University of Paris in 1815. He then became a banker and social reformer but retained an interest in mathematics. Unfortunately, his later papers were not appreciated until the late twentieth century.

5.4 Euler Equations; Regular Singular Points

In this section we will begin to consider how to solve equations of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1}$$

in the neighborhood of a singular point x_0. Recall that if the functions P, Q, and R are polynomials having no factors common to all three of them, then the singular points of equation (1) are the points for which $P(x) = 0$.

Euler Equations. A relatively simple differential equation that has a singular point is the **Euler equation**[10]

$$L[y] = x^2y'' + \alpha xy' + \beta y = 0, \tag{2}$$

where α and β are real constants. Then $P(x) = x^2$, $Q(x) = \alpha x$, and $R(x) = \beta$. If $\beta \neq 0$, then $P(x)$, $Q(x)$, and $R(x)$ have no common factors, so the only singular point of equation (2) is $x = 0$; all other points are ordinary points. For convenience we first consider the interval $x > 0$; later we extend our results to the interval $x < 0$.

Observe that $(x^r)' = rx^{r-1}$ and $(x^r)'' = r(r-1)x^{r-2}$. Hence, if we assume that equation (2) has a solution of the form

$$y = x^r, \tag{3}$$

then we obtain

$$\begin{aligned} L[x^r] &= x^2(x^r)'' + \alpha x(x^r)' + \beta x^r \\ &= x^2 r(r-1)x^{r-2} + \alpha x(rx^{r-1}) + \beta x^r \\ &= x^r(r(r-1) + \alpha r + \beta). \end{aligned} \tag{4}$$

If r is a root of the quadratic equation

$$F(r) = r(r-1) + \alpha r + \beta = 0, \tag{5}$$

then $L[x^r]$ is zero, and $y = x^r$ is a solution of equation (2). The roots of equation (5) are

$$r_1, r_2 = \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}, \tag{6}$$

and the quadratic polynomial $F(r)$ defined in equation (5) can also be written as $F(r) = (r-r_1)(r-r_2)$. Mirroring the treatment of second-order linear differential equations with constant coefficients, we consider separately the cases in which the roots are real and different, real but equal, and complex conjugates. Indeed, the entire discussion of Euler equations is similar to the treatment of second-order linear equations with constant coefficients in Chapter 3, with e^{rx} replaced by x^r.

Real, Distinct Roots. If $F(r) = 0$ has real roots r_1 and r_2, with $r_1 \neq r_2$, then $y_1(x) = x^{r_1}$ and $y_2(x) = x^{r_2}$ are solutions of equation (2). Since

$$W[x^{r_1}, x^{r_2}] = (r_2 - r_1)x^{r_1+r_2-1}$$

is nonzero for $r_1 \neq r_2$ and $x > 0$, it follows that the general solution of equation (2) is

$$y = c_1x^{r_1} + c_2x^{r_2}, \quad x > 0. \tag{7}$$

Note that if r is not a rational number, then x^r is defined by $x^r = e^{r\ln x}$.

EXAMPLE 1

Solve

$$2x^2y'' + 3xy' - y = 0, \quad x > 0. \tag{8}$$

[10]This equation is sometimes called the Cauchy–Euler equation or the equidimensional equation. Euler studied it in about 1740, but its solution was known to Johann Bernoulli before 1700.

Solution:

Substituting $y = x^r$ in equation (8) gives

$$x^r(2r(r-1) + 3r - 1) = x^r(2r^2 + r - 1) = x^r(2r-1)(r+1) = 0.$$

Hence $r_1 = \frac{1}{2}$ and $r_2 = -1$, so the general solution of equation (8) is

$$y = c_1 x^{1/2} + c_2 x^{-1}, \quad x > 0. \tag{9}$$

Equal Roots. If the roots r_1 and r_2 are equal, then we obtain only one solution $y_1(x) = x^{r_1}$ of the assumed form. A second solution can be obtained by the method of reduction of order, but for the purpose of our future discussion we consider an alternative method. Since $r_1 = r_2$, it follows that $F(r) = (r - r_1)^2$. Thus in this case, not only does $F(r_1) = 0$ but also $F'(r_1) = 0$. This suggests differentiating equation (4) with respect to r and then setting r equal to r_1. By differentiating equation (4) with respect to r, we obtain

$$\begin{aligned}\frac{\partial}{\partial r}L[x^r] &= \frac{\partial}{\partial r}[x^r F(r)] = \frac{\partial}{\partial r}\left[x^r(r-r_1)^2\right] \\ &= (r-r_1)^2 x^r \ln x + 2(r-r_1)x^r.\end{aligned} \tag{10}$$

However, by interchanging differentiation with respect to x and with respect to r, we also obtain

$$\frac{\partial}{\partial r}L[x^r] = L\left[\frac{\partial}{\partial r}x^r\right] = L[x^r \ln x].$$

The right-hand side of equation (10) is zero for $r = r_1$; consequently, $L[x^{r_1} \ln x] = 0$ also. Therefore, a second solution of equation (2) is

$$y_2(x) = x^{r_1} \ln x, \quad x > 0. \tag{11}$$

By evaluating the Wronskian of y_1 and y_2, we find that

$$W[x^{r_1}, x^{r_1} \ln x] = x^{2r_1 - 1}.$$

Hence x^{r_1} and $x^{r_1} \ln x$ are a fundamental set of solutions for $x > 0$, and the general solution of equation (2) is

$$y = (c_1 + c_2 \ln x)x^{r_1}, \quad x > 0. \tag{12}$$

EXAMPLE 2

Solve

$$x^2 y'' + 5xy' + 4y = 0, \quad x > 0. \tag{13}$$

Solution:

Substituting $y = x^r$ in equation (13) gives

$$x^r(r(r-1) + 5r + 4) = x^r\left(r^2 + 4r + 4\right) = 0.$$

Hence $r_1 = r_2 = -2$, and

$$y = x^{-2}(c_1 + c_2 \ln x), \quad x > 0 \tag{14}$$

is the general solution of equation (13).

Complex Roots. Finally, suppose that the roots r_1 and r_2 of equation (5) are complex conjugates, say, $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, with $\mu \neq 0$. We must now explain what is meant by x^r when r is complex. Remembering that

$$x^r = e^{r \ln x} \tag{15}$$

when $x > 0$ and r is real, we can use this equation to *define* x^r when r is complex. Then, using Euler's formula for $e^{i\mu \ln x}$, we obtain

$$\begin{aligned} x^{\lambda+i\mu} &= e^{(\lambda+i\mu)\ln x} = e^{\lambda \ln x} e^{i\mu \ln x} = x^\lambda e^{i\mu \ln x} \\ &= x^\lambda (\cos(\mu \ln x) + i \sin(\mu \ln x)), \quad x > 0. \end{aligned} \tag{16}$$

With this definition of x^r for complex values of r, it can be verified that the usual laws of algebra and differential calculus hold, and hence x^{r_1} and x^{r_2} are indeed solutions of equation (2). The general solution of equation (2) is

$$y = c_1 x^{\lambda+i\mu} + c_2 x^{\lambda-i\mu}. \tag{17}$$

The disadvantage of this expression is that the functions $x^{\lambda+i\mu}$ and $x^{\lambda-i\mu}$ are complex-valued. Recall that we had a similar situation for the second-order differential equation with constant coefficients when the roots of the characteristic equation were complex. Just as we did then, we can use Theorem 3.2.6 to obtain real-valued solutions of equation (2) by taking the real and imaginary parts of $x^{\lambda+i\mu}$, namely,

$$x^\lambda \cos(\mu \ln x) \quad \text{and} \quad x^\lambda \sin(\mu \ln x). \tag{18}$$

A straightforward calculation shows (see Problem 29) that

$$W\left[x^\lambda \cos(\mu \ln x), x^\lambda \sin(\mu \ln x)\right] = \mu x^{2\lambda-1}.$$

Hence these solutions form a fundamental set of solutions for $x > 0$, and the general solution of the Euler equation (2) is

$$y = c_1 x^\lambda \cos(\mu \ln x) + c_2 x^\lambda \sin(\mu \ln x), \quad x > 0. \tag{19}$$

EXAMPLE 3

Solve

$$x^2 y'' + xy' + y = 0. \tag{20}$$

Solution:

Substituting $y = x^r$ in equation (20) gives

$$x^r(r(r-1) + r + 1) = x^r(r^2 + 1) = 0.$$

Hence $r = \pm i$, and the general solution is

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x), \quad x > 0. \tag{21}$$

The factor x^λ does not appear explicitly in equation (21) because in this example $\lambda = 0$ and $x^\lambda = 1$.

Now let us consider the qualitative behavior of the solutions of equation (2) near the singular point $x = 0$. This depends entirely on the values of the exponents r_1 and r_2. First, if r is real and positive, then $x^r \to 0$ as x tends to zero through positive values. On the other hand, if r is real and negative, then x^r becomes unbounded. Finally, if $r = 0$, then $x^r = 1$. Figure 5.4.1 shows these possibilities for various values of r. If r is complex, then a typical solution is $x^\lambda \cos(\mu \ln x)$. This function becomes unbounded or approaches zero if λ is negative or positive, respectively, and also oscillates more and more rapidly as $x \to 0$. This behavior is shown in Figures 5.4.2 and 5.4.3 for selected values of λ and μ. If $\lambda = 0$, the oscillation is of constant amplitude. Finally, if there are repeated roots, then one solution is of the form $x^r \ln x$, which tends to zero if $r > 0$ and becomes unbounded if $r \leq 0$. An example of each case is shown in Figure 5.4.4.

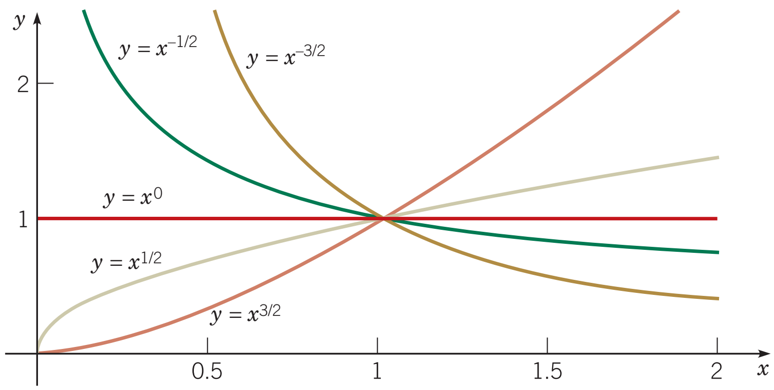

FIGURE 5.4.1 Solutions of an Euler equation; real roots ($\mu = 0$).

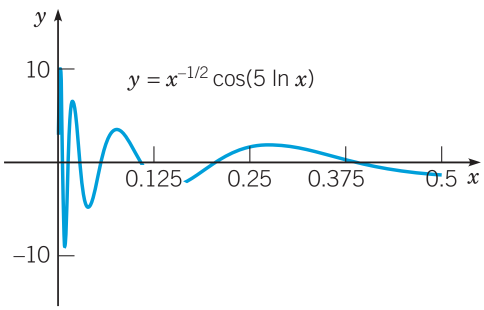

FIGURE 5.4.2 Solution of an Euler equation; complex roots with negative real part.

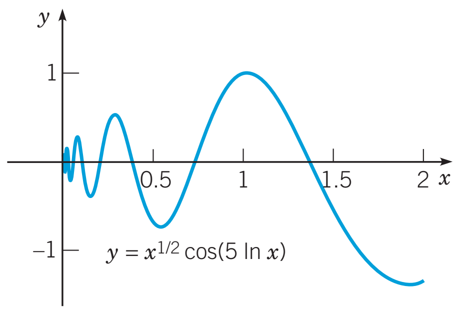

FIGURE 5.4.3 Solution of an Euler equation; complex roots with positive real part.

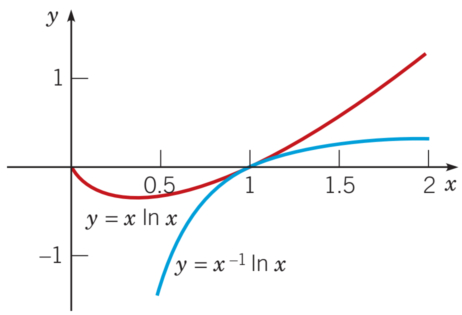

FIGURE 5.4.4 The two typical second solutions of an Euler equation with equal roots: $r > 0$ (red), $r < 0$ (blue).

The extension of the solutions of equation (2) into the interval $x < 0$ can be carried out in a relatively straightforward manner. The difficulty lies in understanding what is meant by x^r when x is negative and r is not an integer; similarly, $\ln x$ has not been defined for $x < 0$. The solutions of the Euler equation that we have given for $x > 0$ can be shown to be valid for $x < 0$, but in general they are complex-valued. Thus in Example 1 the solution $x^{1/2}$ is imaginary for $x < 0$.

It is always possible to obtain real-valued solutions of the Euler equation (2) in the interval $x < 0$ by making the following change of variable. Let $x = -\xi$, where $\xi > 0$, and let $y = u(\xi)$. Then we have

$$\frac{dy}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx} = -\frac{du}{d\xi}, \quad \frac{d^2y}{dx^2} = \frac{d}{d\xi}\left(-\frac{du}{d\xi}\right)\frac{d\xi}{dx} = \frac{d^2u}{d\xi^2}. \tag{22}$$

Thus, for $x < 0$, equation (2) takes the form

$$\xi^2 \frac{d^2u}{d\xi^2} + \alpha\xi\frac{du}{d\xi} + \beta u = 0, \quad \xi > 0. \tag{23}$$

But except for names of the variables, this is exactly the same as equation (2); from equations (7), (12), and (19), we have

$$u(\xi) = \begin{cases} c_1\xi^{r_1} + c_2\xi^{r_2} & \text{if } r_1 \text{ and } r_2 \text{ are real-valued and different} \\ (c_1 + c_2 \ln\xi)\xi^{r_1} & \text{if } r_1 \text{ and } r_2 \text{ are real-valued with } r_1 = r_2 \\ c_1\xi^{\lambda}\cos(\mu\ln\xi) + c_2\xi^{\lambda}\sin(\mu\ln\xi) & \text{if } r_{1,2} = \lambda \pm i\mu \text{ are complex-valued} \\ & \quad (\mu \neq 0), \end{cases} \tag{24}$$

depending on the nature of the zeros of $F(r) = r(r-1) + \alpha r + \beta = 0$. To obtain u in terms of x, we replace ξ by $-x$ in equations (24).

We can combine the results for $x > 0$ and $x < 0$ by recalling that $|x| = x$ when $x > 0$ and that $|x| = -x$ when $x < 0$. Thus we need only replace x by $|x|$ in equations (7), (12), and (19) to obtain real-valued solutions valid in any interval not containing the origin.

Hence the general solution of the Euler equation (2)

$$x^2y'' + \alpha xy' + \beta y = 0$$

in any interval not containing the origin is determined by the roots r_1 and r_2 of the equation

$$F(r) = r(r-1) + \alpha r + \beta = 0$$

as follows. If the roots r_1 and r_2 are real and different, $r_{1,2} = \lambda \pm i\mu$, then

$$y = c_1|x|^{r_1} + c_2|x|^{r_2}. \tag{25}$$

If the roots are real and equal, then

$$y = (c_1 + c_2 \ln|x|)|x|^{r_1}. \tag{26}$$

If the roots are complex conjugates, $r_{1,2} = \lambda \pm i\mu$, then

$$y = |x|^{\lambda}\big(c_1\cos(\mu\ln|x|) + c_2\sin(\mu\ln|x|)\big). \tag{27}$$

The solutions of an Euler equation of the form

$$(x-x_0)^2y'' + \alpha(x-x_0)y' + \beta y = 0 \tag{28}$$

are similar. If we look for solutions of the form $y = (x-x_0)^r$, then the general solution is given by equation (25), equation (26), or equation (27) with x replaced by $x - x_0$. Alternatively, we can reduce equation (28) to the form of equation (2) by making the change of independent variable $t = x - x_0$.

Regular Singular Points. We now return to a consideration of the general equation (1)

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

where x_0 is a singular point. This means that $P(x_0) = 0$ and that at least one of Q and R is not zero at x_0.

Unfortunately, if we attempt to use the methods of the preceding two sections to solve equation (1) in the neighborhood of a singular point x_0, we find that these methods fail. This is because the solution of equation (1) is often not analytic at x_0 and consequently cannot be represented by a Taylor series in powers of $x - x_0$. Examples 1, 2, and 3 illustrate this fact; in each of these examples, the solution fails to have a power series expansion about the singular point $x = 0$. Therefore, to have any chance of solving equation (1) in the neighborhood of a singular point we must use a more general type of series expansion.

Since the singular points of a differential equation are usually few in number, we might ask whether we can simply ignore them, especially since we already know how to construct solutions about ordinary points. However, this is not feasible. The singular points determine the principal features of the solution to a much larger extent than you might at first suspect. In the neighborhood of a singular point the solution often becomes large in magnitude or experiences rapid changes in magnitude. For example, the solutions found in Examples 1, 2, and 3 are illustrations of this fact. Thus the behavior of a physical system modeled by a differential equation frequently is most interesting in the neighborhood of a singular point. Often geometric singularities in a physical problem, such as corners or sharp edges, lead to singular points in the corresponding differential equation. Thus, although at first we might want to avoid the few points where a differential equation is singular, it is precisely at these points that it is necessary to study the solution most carefully.

As an alternative to analytical methods, we can consider the use of numerical methods, which are discussed in Chapter 8. However, these methods are ill suited for the study of solutions near a singular point. Thus, even if we adopt a numerical approach, it is advantageous to combine it with the analytical methods of this chapter in order to examine the behavior of solutions near singular points.

Without any additional information about the behavior of Q/P and R/P in the neighborhood of the singular point, it is impossible to describe the behavior of the solutions of equation (1) near $x = x_0$. It may be that there are two distinct solutions of equation (1) that remain bounded as $x \to x_0$ (as in Example 3); or there may be only one, with the other becoming unbounded as $x \to x_0$ (as in Example 1); or they may both become unbounded as $x \to x_0$ (as in Example 2). If equation (1) has solutions that become unbounded as $x \to x_0$, it is often important to determine how these solutions behave as $x \to x_0$. For example, does $y \to \infty$ in the same way as $(x - x_0)^{-1}$ or $|x - x_0|^{-1/2}$, or in some other manner?

Our goal is to extend the method already developed for solving equation (1) near an ordinary point so that it also applies to the neighborhood of a singular point x_0. To do this in a reasonably simple manner, it is necessary to restrict ourselves to cases in which the singularities in the functions Q/P and R/P at $x = x_0$ are not too severe—that is, to what we might call "weak singularities." At this stage it is not clear exactly what is an acceptable singularity. However, as we develop the method of solution, you will see that the appropriate conditions (see also Section 5.6, Problem 16) to distinguish "weak singularities" are

$$\lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)} \quad \text{is finite} \tag{29}$$

and

$$\lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} \quad \text{is finite.} \tag{30}$$

This means that the singularity in Q/P can be no worse than $(x - x_0)^{-1}$ and the singularity in R/P can be no worse than $(x - x_0)^{-2}$. Such a point is called a **regular singular point** of equation (1). For equations with more general coefficients than polynomials, x_0 is a regular singular point of equation (1) if it is a singular point and if both[11]

$$(x - x_0) \frac{Q(x)}{P(x)} \quad \text{and} \quad (x - x_0)^2 \frac{R(x)}{P(x)} \tag{31}$$

have convergent Taylor series about x_0—that is, if the functions in equation (31) are analytic at $x = x_0$. Equations (29) and (30) imply that this will be the case when P, Q, and R are polynomials. Any singular point of equation (1) that is not a regular singular point is called an **irregular singular point** of equation (1).

[11]The functions given in equation (31) may not be defined at x_0, in which case their values at x_0 are to be assigned as their limits as $x \to x_0$.

Observe that the conditions in equations (29) and (30) are satisfied by the Euler equation (28). Thus the singularity in an Euler equation is a regular singular point. Indeed, we will see that all equations of the form (1) behave very much like Euler equations near a regular singular point. That is, solutions near a regular singular point may include powers of x with negative or nonintegral exponents, logarithms, or sines or cosines of logarithmic arguments.

In the following sections we discuss how to solve equation (1) in the neighborhood of a regular singular point. A discussion of the solutions of differential equations in the neighborhood of irregular singular points is more complicated and may be found in more advanced books.

EXAMPLE 4

Determine the singular points of the Legendre equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0 \tag{32}$$

and determine whether they are regular or irregular.

Solution:

In this case $P(x) = 1 - x^2$, so the singular points are $x = 1$ and $x = -1$. Observe that when we divide equation (32) by $1-x^2$, the coefficients of y' and y are $-2x/(1-x^2)$ and $\alpha(\alpha+1)/(1-x^2)$, respectively. We consider the point $x = 1$ first. Thus, from equations (29) and (30), we calculate

$$\lim_{x\to 1}(x-1)\frac{-2x}{1-x^2} = \lim_{x\to 1}\frac{(x-1)(-2x)}{(1-x)(1+x)} = \lim_{x\to 1}\frac{2x}{1+x} = 1$$

and

$$\begin{aligned}\lim_{x\to 1}(x-1)^2\frac{\alpha(\alpha+1)}{1-x^2} &= \lim_{x\to 1}\frac{(x-1)^2\alpha(\alpha+1)}{(1-x)(1+x)} \\ &= \lim_{x\to 1}\frac{(x-1)(-\alpha)(\alpha+1)}{1+x} = 0.\end{aligned}$$

Since these limits are finite, the point $x = 1$ is a regular singular point.

It can be shown in a similar manner that $x = -1$ is also a regular singular point.

EXAMPLE 5

Determine the singular points of the differential equation

$$2x(x-2)^2y'' + 3xy' + (x-2)y = 0$$

and classify them as regular or irregular.

Solution:

Dividing the differential equation by $2x(x-2)^2$, we have

$$y'' + \frac{3}{2(x-2)^2}y' + \frac{1}{2x(x-2)}y = 0,$$

so $p(x) = \dfrac{Q(x)}{P(x)} = \dfrac{3}{2(x-2)^2}$ and $q(x) = \dfrac{R(x)}{P(x)} = \dfrac{1}{2x(x-2)}$. The singular points are $x = 0$ and $x = 2$. Consider $x = 0$. We have

$$\lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{3}{2(x-2)^2} = 0,$$

and

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{1}{2x(x-2)} = 0.$$

Since these limits are finite, $x = 0$ is a regular singular point.

For $x = 2$ we have

$$\lim_{x\to 2}(x-2)\,p(x) = \lim_{x\to 2}(x-2)\frac{3}{2(x-2)^2} = \lim_{x\to 2}\frac{3}{2(x-2)},$$

so the limit does not exist; hence $x = 2$ is an irregular singular point.

EXAMPLE 6

Determine the singular points of

$$\left(x - \frac{\pi}{2}\right)^2 y'' + (\cos x)\,y' + (\sin x)\,y = 0$$

and classify them as regular or irregular.

Solution:

The only singular point is $x = \frac{\pi}{2}$. To study it, we consider the functions

$$\left(x - \frac{\pi}{2}\right)p(x) = \left(x - \frac{\pi}{2}\right)\frac{Q(x)}{P(x)} = \frac{\cos x}{x - \pi/2}$$

and

$$\left(x - \frac{\pi}{2}\right)^2 q(x) = \left(x - \frac{\pi}{2}\right)^2 \frac{R(x)}{P(x)} = \sin x.$$

Starting from the Taylor series for $\cos x$ about $x = \frac{\pi}{2}$, we find that

$$\frac{\cos x}{x - \pi/2} = -1 + \frac{(x-\pi/2)^2}{3!} - \frac{(x-\pi/2)^4}{5!} + \cdots,$$

which converges for all x. Similarly, $\sin x$ is analytic at $x = \frac{\pi}{2}$. Therefore, we conclude that $\frac{\pi}{2}$ is a regular singular point for this equation.

Problems

In each of Problems 1 through 8, determine the general solution of the given differential equation that is valid in any interval not including the singular point.

1. $x^2y'' + 4xy' + 2y = 0$
2. $(x+1)^2y'' + 3(x+1)y' + 0.75y = 0$
3. $x^2y'' - 3xy' + 4y = 0$
4. $x^2y'' - xy' + y = 0$
5. $x^2y'' + 6xy' - y = 0$
6. $2x^2y'' - 4xy' + 6y = 0$
7. $x^2y'' - 5xy' + 9y = 0$
8. $(x-2)^2y'' + 5(x-2)y' + 8y = 0$

In each of Problems 9 through 11, find the solution of the given initial-value problem. Plot the graph of the solution and describe how the solution behaves as $x \to 0$.

G 9. $2x^2y'' + xy' - 3y = 0, \quad y(1) = 1, \quad y'(1) = 4$

G 10. $4x^2y'' + 8xy' + 17y = 0, \quad y(1) = 2, \quad y'(1) = -3$

G 11. $x^2y'' - 3xy' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 3$

In each of Problems 12 through 23, find all singular points of the given equation and determine whether each one is regular or irregular.

12. $xy'' + (1-x)y' + xy = 0$
13. $x^2(1-x)^2y'' + 2xy' + 4y = 0$

14. $x^2(1-x)y'' + (x-2)y' - 3xy = 0$

15. $x^2(1-x^2)y'' + \left(\frac{2}{x}\right)y' + 4y = 0$

16. $(1-x^2)^2y'' + x(1-x)y' + (1+x)y = 0$

17. $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ (Bessel equation)

18. $(x+2)^2(x-1)y'' + 3(x-1)y' - 2(x+2)y = 0$

19. $x(3-x)y'' + (x+1)y' - 2y = 0$

20. $xy'' + e^xy' + (3\cos x)y = 0$

21. $y'' + (\ln|x|)y' + 3xy = 0$

22. $(\sin x)y'' + xy' + 4y = 0$

23. $(x\sin x)y'' + 3y' + xy = 0$

24. Find all values of α for which all solutions of $x^2y'' + \alpha xy' + \frac{5}{2}y = 0$ approach zero as $x \to 0$.

25. Find all values of β for which all solutions of $x^2y'' + \beta y = 0$ approach zero as $x \to 0$.

26. Find γ so that the solution of the initial-value problem $x^2y'' - 2y = 0$, $y(1) = 1$, $y'(1) = \gamma$ is bounded as $x \to 0$.

27. Consider the Euler equation $x^2y'' + \alpha xy' + \beta y = 0$. Find conditions on α and β so that:

a. All solutions approach zero as $x \to 0$.
b. All solutions are bounded as $x \to 0$.
c. All solutions approach zero as $x \to \infty$.
d. All solutions are bounded as $x \to \infty$.
e. All solutions are bounded both as $x \to 0$ and as $x \to \infty$.

28. Using the method of reduction of order, show that if r_1 is a repeated root of

$$r(r-1) + \alpha r + \beta = 0,$$

then x^{r_1} and $x^{r_1}\ln x$ are solutions of $x^2y'' + \alpha xy' + \beta y = 0$ for $x > 0$.

29. Verify that $W[x^\lambda\cos(\mu\ln x), x^\lambda\sin(\mu\ln x)] = \mu x^{2\lambda-1}$.

In each of Problems 30 and 31, show that the point $x = 0$ is a regular singular point. In each problem try to find solutions of the form $\sum_{n=0}^{\infty} a_nx^n$. Show that (except for constant multiples) there is only one nonzero solution of this form in Problem 30 and that there are no nonzero solutions of this form in Problem 31. Thus in neither case can the general solution be found in this manner. This is typical of equations with singular points.

30. $2xy'' + 3y' + xy = 0$

31. $2x^2y'' + 3xy' - (1+x)y = 0$

32. **Singularities at Infinity.** The definitions of an ordinary point and a regular singular point given in the preceding sections apply only if the point x_0 is finite. In more advanced work in differential equations, it is often necessary to consider the point at infinity. This is done by making the change of variable $\xi = 1/x$ and studying the resulting equation at $\xi = 0$. Show that, for the differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

the point at infinity is an ordinary point if

$$\frac{1}{P(1/\xi)}\left(\frac{2P(1/\xi)}{\xi} - \frac{Q(1/\xi)}{\xi^2}\right) \quad \text{and} \quad \frac{R(1/\xi)}{\xi^4P(1/\xi)}$$

have Taylor series expansions about $\xi = 0$. Show also that the point at infinity is a regular singular point if at least one of the above functions does not have a Taylor series expansion, but both

$$\frac{\xi}{P(1/\xi)}\left(\frac{2P(1/\xi)}{\xi} - \frac{Q(1/\xi)}{\xi^2}\right) \quad \text{and} \quad \frac{R(1/\xi)}{\xi^2P(1/\xi)}$$

do have such expansions.

In each of Problems 33 through 37, use the results of Problem 32 to determine whether the point at infinity is an ordinary point, a regular singular point, or an irregular singular point of the given differential equation.

33. $y'' + y = 0$

34. $x^2y'' + xy' - 4y = 0$

35. $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$ (Legendre equation)

36. $y'' - 2xy' + \lambda y = 0$ (Hermite equation)

37. $y'' - xy = 0$ (Airy equation)

5.5 Series Solutions Near a Regular Singular Point, Part I

We now consider the question of solving the general second-order linear differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1}$$

in the neighborhood of a regular singular point $x = x_0$. For convenience we assume that $x_0 = 0$. If $x_0 \neq 0$, the equation can be transformed into one for which the regular singular point is at the origin by letting $x - x_0$ equal t.

The assumption that $x = 0$ is a regular singular point of equation (1) means that $xQ(x)/P(x) = xp(x)$ and $x^2R(x)/P(x) = x^2q(x)$ have finite limits as $x \to 0$ and are analytic at $x = 0$. Thus they have convergent power series expansions of the form

$$xp(x) = \sum_{n=0}^{\infty} p_nx^n, \quad x^2q(x) = \sum_{n=0}^{\infty} q_nx^n, \tag{2}$$

on some interval $|x| < \rho$ about the origin, where $\rho > 0$. To make the quantities $xp(x)$ and $x^2q(x)$ appear in equation (1), it is convenient to divide equation (1) by $P(x)$ and then to multiply by x^2, obtaining

$$x^2y'' + x(xp(x))y' + \left(x^2q(x)\right)y = 0, \tag{3}$$

or

$$\begin{aligned} x^2y'' + x(p_0 + p_1x + \cdots + p_nx^n + \cdots)y' \\ + (q_0 + q_1x + \cdots + q_nx^n + \cdots)y = 0. \end{aligned} \tag{4}$$

Notice that the first terms of $xp(x)$ and of $x^2q(x)$ are

$$p_0 = \lim_{x\to 0} \frac{xQ(x)}{P(x)} \quad \text{and} \quad q_0 = \lim_{x\to 0} \frac{x^2R(x)}{P(x)}. \tag{5}$$

If all other coefficients p_n and q_n for $n \geq 1$ in equation (2) are zero, then equation (4) reduces to the Euler equation

$$x^2y'' + p_0xy' + q_0y = 0, \tag{6}$$

which was discussed in the preceding section.

In general, of course, some of the coefficients p_n and q_n, $n \geq 1$, are not zero. However, the essential character of solutions of equation (4) in the neighborhood of the singular point is identical to that of solutions of the Euler equation (6). The presence of the terms $p_1x + \cdots + p_nx^n + \cdots$ and $q_1x + \cdots + q_nx^n + \cdots$ merely complicates the calculations.

We restrict our discussion primarily to the interval $x > 0$. The interval $x < 0$ can be treated, just as for the Euler equation, by making the change of variable $x = -\xi$ and then solving the resulting equation for $\xi > 0$.

The coefficients in equation (4) can be viewed as "Euler coefficients" times power series. To see this, you can write the coefficient of y' in equation (4) as

$$p_0x\left(1 + \frac{p_1}{p_0}x + \frac{p_2}{p_0}x^2 + \cdots + \frac{p_n}{p_0}x^n + \cdots\right),$$

and similarly for the coefficient of y. Thus it may seem natural to seek solutions of equation (4) in the form of "Euler solutions" times power series. Hence we assume that

$$y = x^r(a_0 + a_1x + \cdots + a_nx^n + \cdots) = x^r\sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_nx^{r+n}, \tag{7}$$

where $a_0 \neq 0$. In other words, r is the exponent of the first nonzero term in the series, and a_0 is its coefficient. As part of the solution, we have to determine:

1. The values of r for which equation (1) has a solution of the form (7)
2. The recurrence relation for the coefficients a_n
3. The radius of convergence of the series $\sum_{n=0}^{\infty} a_nx^n$

The general theory was constructed by Frobenius[12] and is fairly complicated. Rather than trying to present this theory, we simply assume, in this and the next two sections, that there does exist a solution of the stated form. In particular, we assume that any power series in an expression for a solution has a nonzero radius of convergence and concentrate on showing how to determine the coefficients in such a series. To illustrate the method of Frobenius, we first consider an example.

[12]Ferdinand Georg Frobenius (1849–1917) grew up in the suburbs of Berlin, received his doctorate in 1870 from the University of Berlin, and returned as professor in 1892. For most of the intervening years he was professor at the Eidgenössische Polytechnikum at Zürich. He showed how to construct series solutions about regular singular points in 1874. His most distinguished work, however, was in algebra, where he was one of the foremost early developers of group theory.

EXAMPLE 1

Solve the differential equation

$$2x^2y'' - xy' + (1+x)y = 0. \tag{8}$$

Solution:

It is easy to show that $x = 0$ is a regular singular point of equation (8). Further, $xp(x) = -1/2$ and $x^2q(x) = (1+x)/2$. Thus $p_0 = -1/2$, $q_0 = 1/2$, $q_1 = 1/2$, and all other p_n's and q_n's are zero. Then, from equation (6), the Euler equation corresponding to equation (8) is

$$2x^2y'' - xy' + y = 0. \tag{9}$$

To solve equation (8), we assume that there is a solution of the form (7). Then y' and y'' are given by

$$y' = \sum_{n=0}^{\infty} a_n(r+n)x^{r+n-1} \tag{10}$$

and

$$y'' = \sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{r+n-2}. \tag{11}$$

By substituting the expressions for y, y', and y'' in equation (8), we obtain

$$\begin{aligned} 2x^2y'' - xy' + (1+x)y &= \sum_{n=0}^{\infty} 2a_n(r+n)(r+n-1)x^{r+n} \\ &- \sum_{n=0}^{\infty} a_n(r+n)x^{r+n} + \sum_{n=0}^{\infty} a_nx^{r+n} + \sum_{n=0}^{\infty} a_nx^{r+n+1}. \end{aligned} \tag{12}$$

The last term in equation (12) can be written as $\sum_{n=1}^{\infty} a_{n-1}x^{r+n}$, so by combining the terms in equation (12), we obtain

$$\begin{aligned} 2x^2y'' - xy' + (1+x)y &= a_0[2r(r-1) - r + 1]x^r \\ &+ \sum_{n=1}^{\infty}\big((2(r+n)(r+n-1) - (r+n) + 1)a_n + a_{n-1}\big)x^{r+n} = 0. \end{aligned} \tag{13}$$

If equation (13) is to be satisfied for all x, the coefficient of each power of x in equation (13) must be zero. From the coefficient of x^r we obtain, since $a_0 \neq 0$,

$$2r(r-1) - r + 1 = 2r^2 - 3r + 1 = (r-1)(2r-1) = 0. \tag{14}$$

Equation (14) is called the **indicial equation** for equation (8). Note that it is exactly the polynomial equation we would obtain for the Euler equation (9) associated with equation (8). The roots of the indicial equation are

$$r_1 = 1, \quad r_2 = \frac{1}{2}. \tag{15}$$

These values of r are called the **exponents at the singularity** for the regular singular point $x = 0$. They determine the qualitative behavior of the solution (7) in the neighborhood of the singular point.

Now we return to equation (13) and set the coefficient of x^{r+n} equal to zero. This gives the relation

$$(2(r+n)(r+n-1) - (r+n) + 1)a_n + a_{n-1} = 0, \quad n \geq 1, \tag{16}$$

or

$$\begin{aligned} a_n &= -\frac{a_{n-1}}{2(r+n)^2 - 3(r+n) + 1} \\ &= -\frac{a_{n-1}}{((r+n)-1)(2(r+n)-1)}, \quad n \geq 1. \end{aligned} \tag{17}$$

For each root r_1 and r_2 of the indicial equation, we use the recurrence relation (17) to determine a set of coefficients $a_1, a_2, \ldots$. For $r = r_1 = 1$, equation (17) becomes

$$a_n = -\frac{a_{n-1}}{(2n+1)n}, \qquad n \geq 1.$$

Thus

$$a_1 = -\frac{a_0}{3 \cdot 1},$$

$$a_2 = -\frac{a_1}{5 \cdot 2} = \frac{a_0}{(3 \cdot 5)(1 \cdot 2)},$$

and

$$a_3 = -\frac{a_2}{7 \cdot 3} = -\frac{a_0}{(3 \cdot 5 \cdot 7)(1 \cdot 2 \cdot 3)}.$$

In general, we have

$$a_n = \frac{(-1)^n}{(3 \cdot 5 \cdot 7 \cdots (2n+1))n!} a_0, \qquad n \geq 4. \tag{18}$$

If we multiply both the numerator and denominator of the right-hand side of equation (18) by $2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n = 2^n n!$, we can rewrite a_n as

$$a_n = \frac{(-1)^n 2^n}{(2n+1)!} a_0, \qquad n \geq 1.$$

Hence, if we omit the constant multiplier a_0, one solution of equation (8) is

$$y_1(x) = x\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} x^n\right), \qquad x > 0. \tag{19}$$

To determine the radius of convergence of the series in equation (19), we use the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \frac{2|x|}{(2n+2)(2n+3)} = 0$$

for all x. Thus the series converges for all x.

Corresponding to the second root $r = r_2 = \frac{1}{2}$, we proceed similarly. From equation (17) we have

$$a_n = -\frac{a_{n-1}}{2n\left(n - \frac{1}{2}\right)} = -\frac{a_{n-1}}{n(2n-1)}, \qquad n \geq 1.$$

Hence

$$a_1 = -\frac{a_0}{1 \cdot 1},$$

$$a_2 = -\frac{a_1}{2 \cdot 3} = \frac{a_0}{(1 \cdot 2)(1 \cdot 3)},$$

$$a_3 = -\frac{a_2}{3 \cdot 5} = -\frac{a_0}{(1 \cdot 2 \cdot 3)(1 \cdot 3 \cdot 5)},$$

and, in general,

$$a_n = \frac{(-1)^n}{n!(1 \cdot 3 \cdot 5 \cdots (2n-1))} a_0, \qquad n \geq 4. \tag{20}$$

Just as in the case of the first root r_1, we multiply the numerator and denominator by $2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n = 2^n n!$. Then we have

$$a_n = \frac{(-1)^n 2^n}{(2n)!} a_0, \qquad n \geq 1.$$

Again omitting the constant multiplier a_0, we obtain the second solution

$$y_2(x) = x^{1/2}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(2n)!} x^n\right), \qquad x > 0. \tag{21}$$

As before, we can show that the series in equation (21) converges for all x. Since y_1 and y_2 behave like x and $x^{1/2}$, respectively, near $x = 0$, they are linearly independent and so they form a fundamental set of solutions. Hence the general solution of equation (8) is

$$y = c_1 y_1(x) + c_2 y_2(x), \quad x > 0.$$

The preceding example illustrates that if $x = 0$ is a regular singular point, then sometimes there are two solutions of the form (7) in the neighborhood of this point. Similarly, if there is a regular singular point at $x = x_0$, then there may be two solutions of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{22}$$

that are valid near $x = x_0$. However, just as an Euler equation may not have two solutions of the form $y = x^r$, so a more general equation with a regular singular point may not have two solutions of the form (7) or (22). In particular, we show in the next section that if the roots r_1 and r_2 of the indicial equation are equal or differ by an integer, then the second solution normally has a more complicated structure. In all cases, though, it is possible to find at least one solution of the form (7) or (22); if r_1 and r_2 differ by an integer, this solution corresponds to the larger value of r. If there is only one such solution, then the second solution involves a logarithmic term, just as for the Euler equation when the roots of the characteristic equation are equal. The method of reduction of order or some other procedure can be invoked to determine the second solution in such cases. This is discussed in Sections 5.6 and 5.7.

If the roots of the indicial equation are complex, then they cannot be equal or differ by an integer, so there are always two solutions of the form (7) or (22). Of course, these solutions are complex-valued functions of x. However, as for the Euler equation, it is possible to obtain real-valued solutions by taking the real and imaginary parts of the complex solutions.

Finally, we mention a practical point. If P, Q, and R are polynomials, it is often much better to work directly with equation (1) than with equation (3). This avoids the necessity of expressing $xQ(x)/P(x)$ and $x^2R(x)/P(x)$ as power series. For example, it is more convenient to consider the equation

$$x(1 + x)y'' + 2y' + xy = 0$$

than to write it in the form

$$x^2 y'' + \frac{2x}{1 + x} y' + \frac{x^2}{1 + x} y = 0,$$

which would entail expanding $\dfrac{2x}{1 + x}$ and $\dfrac{x^2}{1 + x}$ in power series.

Problems

In each of Problems 1 through 6:

a. Show that the given differential equation has a regular singular point at $x = 0$.
b. Determine the indicial equation, the recurrence relation, and the roots of the indicial equation.
c. Find the series solution ($x > 0$) corresponding to the larger root.
d. If the roots are unequal and do not differ by an integer, find the series solution corresponding to the smaller root also.

1. $2xy'' + y' + xy = 0$

2. $x^2 y'' + xy' + \left(x^2 - \frac{1}{9}\right) y = 0$

3. $xy'' + y = 0$

4. $xy'' + y' - y = 0$

5. $x^2 y'' + xy' + (x - 2)y = 0$

6. $xy'' + (1 - x)y' - y = 0$

7. The Legendre equation of order α is

$$(1-x^2)y''-2xy'+\alpha(\alpha+1)y=0.$$

The solution of this equation near the ordinary point $x=0$ was discussed in Problems 17 and 18 of Section 5.3. In Example 4 of Section 5.4, it was shown that $x=\pm 1$ are regular singular points.

a. Determine the indicial equation and its roots for the point $x=1$.

b. Find a series solution in powers of $x-1$ for $x-1>0$.
Hint: Write $1+x=2+(x-1)$ and $x=1+(x-1)$. Alternatively, make the change of variable $x-1=t$ and determine a series solution in powers of t.

8. The Chebyshev equation is

$$(1-x^2)y''-xy'+\alpha^2y=0,$$

where α is a constant; see Problem 8 of Section 5.3.

a. Show that $x=1$ and $x=-1$ are regular singular points, and find the exponents at each of these singularities.

b. Find two solutions about $x=1$.

9. The Laguerre[13] differential equation is

$$xy''+(1-x)y'+\lambda y=0.$$

a. Show that $x=0$ is a regular singular point.

b. Determine the indicial equation, its roots, and the recurrence relation.

c. Find one solution (for $x>0$). Show that if $\lambda=m$, a positive integer, this solution reduces to a polynomial. When properly normalized, this polynomial is known as the **Laguerre polynomial**, $L_m(x)$.

10. The Bessel equation of order zero is

$$x^2y''+xy'+x^2y=0.$$

a. Show that $x=0$ is a regular singular point.

b. Show that the roots of the indicial equation are $r_1=r_2=0$.

c. Show that one solution for $x>0$ is

$$J_0(x)=1+\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

The function J_0 is known as the **Bessel function of the first kind of order zero**.

d. Show that the series for $J_0(x)$ converges for all x.

11. Referring to Problem 10, use the method of reduction of order to show that the second solution of the Bessel equation of order zero contains a logarithmic term.
Hint: If $y_2(x)=J_0(x)v(x)$, then

$$y_2(x)=J_0(x)\int\frac{dx}{x\big(J_0(x)\big)^2}.$$

Find the first term in the series expansion of $\dfrac{1}{x\big(J_0(x)\big)^2}$.

12. The Bessel equation of order one is

$$x^2y''+xy'+(x^2-1)y=0.$$

a. Show that $x=0$ is a regular singular point.

b. Show that the roots of the indicial equation are $r_1=1$ and $r_2=-1$.

c. Show that one solution for $x>0$ is

$$J_1(x)=\frac{x}{2}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(n+1)!\,n!\,2^{2n}}.$$

The function J_1 is known as the **Bessel function of the first kind of order one**.

d. Show that the series for $J_1(x)$ converges for all x.

e. Show that it is impossible to determine a second solution of the form

$$x^{-1}\sum_{n=0}^{\infty}b_nx^n,\quad x>0.$$

[13] Edmond Nicolas Laguerre (1834–1886), a French geometer and analyst, studied the polynomials named for him about 1879. He is also known for an algorithm for calculating roots of polynomial equations.

5.6 Series Solutions Near a Regular Singular Point, Part II

Now let us consider the general problem of determining a solution of the equation

$$L[y]=x^2y''+x(xp(x))y'+\big(x^2q(x)\big)y=0, \tag{1}$$

where

$$xp(x)=\sum_{n=0}^{\infty}p_nx^n,\quad x^2q(x)=\sum_{n=0}^{\infty}q_nx^n, \tag{2}$$

and both series converge in an interval $|x|<\rho$ for some $\rho>0$. The point $x=0$ is a regular singular point, and the corresponding Euler equation is

$$x^2y''+p_0xy'+q_0y=0. \tag{3}$$

We seek a solution of equation (1) for $x > 0$ and assume that it has the form

$$y = \phi(r, x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n}, \tag{4}$$

where $a_0 \neq 0$, and we have written $y = \phi(r, x)$ to emphasize that ϕ depends on r as well as x. It follows that

$$y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}, \quad y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}. \tag{5}$$

Then, substituting from equations (2), (4), and (5) in equation (1) gives

$$\begin{aligned} L[\phi](r, x) &= a_0 r(r-1)x^r + a_1(r+1)rx^{r+1} + \cdots + a_n(r+n)(r+n-1)x^{r+n} + \cdots \\ &\quad + \left(p_0 + p_1 x + \cdots + p_n x^n + \cdots\right)\left(a_0 r x^r + a_1(r+1)x^{r+1} + \cdots + a_n(r+n)x^{r+n} + \cdots\right) \\ &\quad + \left(q_0 + q_1 x + \cdots + q_n x^n + \cdots\right)\left(a_0 x^r + a_1 x^{r+1} + \cdots + a_n x^{r+n} + \cdots\right) \\ &= 0. \end{aligned}$$

Multiplying the infinite series together and then collecting terms, we obtain

$$\begin{aligned} L[\phi](r, x) = a_0 F(r) x^r &+ \left[a_1 F(r+1) + a_0(p_1 r + q_1)\right] x^{r+1} \\ &+ \left[a_2 F(r+2) + a_0(p_2 r + q_2) + a_1\left(p_1(r+1) + q_1\right)\right] x^{r+2} \\ &+ \cdots + \left[a_n F(r+n) + a_0(p_n r + q_n) + a_1\left(p_{n-1}(r+1) + q_{n-1}\right)\right. \\ &\qquad \left. + \cdots + a_{n-1}\left(p_1(r+n-1) + q_1\right)\right] x^{r+n} + \cdots = 0, \end{aligned}$$

or, in a more compact form,

$$\begin{aligned} L[\phi] &= a_0 F(r) x^r \\ &\quad + \sum_{n=1}^{\infty} \left(F(r+n) a_n + \sum_{k=0}^{n-1} a_k \left((r+k) p_{n-k} + q_{n-k}\right) \right) x^{r+n} = 0, \end{aligned} \tag{6}$$

where

$$F(r) = r(r-1) + p_0 r + q_0. \tag{7}$$

For equation (6) to be satisfied for all $x > 0$, the coefficient of each power of x must be zero.

Since $a_0 \neq 0$, the term involving x^r yields the equation $F(r) = 0$. This equation is called the *indicial equation*; note that it is exactly the equation we would obtain in looking for solutions $y = x^r$ of the Euler equation (3). Let us denote the roots of the indicial equation by r_1 and r_2 with $r_1 \geq r_2$ if the roots are real. If the roots are complex, the designation of the roots is immaterial. Only for these values of r can we expect to find solutions of equation (1) of the form (4). The roots r_1 and r_2 are called the *exponents at the singularity*; they determine the qualitative nature of the solution in the neighborhood of the singular point.

Setting the coefficient of x^{r+n} in equation (6) equal to zero gives the **recurrence relation**

$$F(r+n) a_n + \sum_{k=0}^{n-1} a_k \left((r+k) p_{n-k} + q_{n-k}\right) = 0, \quad n \geq 1. \tag{8}$$

Equation (8) shows that, in general, a_n depends on the value of r and all the preceding coefficients $a_0, a_1, \ldots, a_{n-1}$. It also shows that we can successively compute $a_1, a_2, \ldots, a_n, \ldots$ in terms of a_0 and the coefficients in the series for $xp(x)$ and $x^2 q(x)$, provided that $F(r+1), F(r+2), \ldots, F(r+n), \ldots$ are not zero. The only values of r for which $F(r) = 0$ are $r = r_1$ and $r = r_2$; since $r_1 \geq r_2$, it follows that $r_1 + n$ is not equal to r_1 or r_2 for $n \geq 1$. Consequently, $F(r_1 + n) \neq 0$ for $n \geq 1$. Hence we can always determine one solution of equation (1) in the form (4), namely,

$$y_1(x) = x^{r_1} \left(1 + \sum_{n=1}^{\infty} a_n(r_1) x^n \right), \quad x > 0. \tag{9}$$

Here we have introduced the notation $a_n(r_1)$ to indicate that a_n has been determined from equation (8) with $r = r_1$. The solution involves an arbitrary constant; the solution in equation (9) is obtained by assigning a_0 the value 1.

If r_2 is not equal to r_1, and $r_1 - r_2$ is not a positive integer, then $r_2 + n$ is not equal to r_1 for any value of $n \geq 1$; hence $F(r_2 + n) \neq 0$, and we can also obtain a second solution

$$y_2(x) = x^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2)x^n\right), \quad x > 0. \tag{10}$$

Just as for the series solutions about ordinary points discussed in Section 5.3, the series in equations (9) and (10) converge at least in the interval $|x| < \rho$ where the series for both $xp(x)$ and $x^2q(x)$ converge. Within their radii of convergence, the power series $1 + \sum_{n=1}^{\infty} a_n(r_1)x^n$ and $1 + \sum_{n=1}^{\infty} a_n(r_2)x^n$ define functions that are analytic at $x = 0$. Thus the singular behavior, if there is any, of the solutions y_1 and y_2 is due to the factors x^{r_1} and x^{r_2} that multiply these two analytic functions.

Next, to obtain real-valued solutions for $x < 0$, we can make the substitution $x = -\xi$ with $\xi > 0$. As we might expect from our discussion of the Euler equation, it turns out that we need only replace x^{r_1} in equation (9) and x^{r_2} in equation (10) by $|x|^{r_1}$ and $|x|^{r_2}$, respectively.

Finally, note that if r_1 and r_2 are complex numbers, then they are necessarily complex conjugates and $r_2 \neq r_1 + N$ for any positive integer N. Thus, in this case we can always find two series solutions of the form (4); however, they are complex-valued functions of x. Real-valued solutions can be obtained by taking the real and imaginary parts of the complex-valued solutions.

The exceptional cases in which $r_1 = r_2$ or $r_1 - r_2 = N$, where N is a positive integer, require more discussion and will be considered later in this section.

It is important to realize that r_1 and r_2, the exponents at the singular point, are easy to find and that they determine the qualitative behavior of the solutions. To calculate r_1 and r_2, it is only necessary to solve the quadratic indicial equation

$$r(r-1) + p_0 r + q_0 = 0, \tag{11}$$

whose coefficients are given by

$$p_0 = \lim_{x\to 0} xp(x), \qquad q_0 = \lim_{x\to 0} x^2 q(x). \tag{12}$$

Note that these are exactly the limits that must be evaluated in order to classify the singularity as a regular singular point; thus they have usually been determined at an earlier stage of the investigation.

Further, if $x = 0$ is a regular singular point of the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{13}$$

where the functions P, Q, and R are polynomials, then $xp(x) = xQ(x)/P(x)$ and $x^2q(x) = x^2R(x)/P(x)$. Thus

$$p_0 = \lim_{x\to 0} x\frac{Q(x)}{P(x)}, \qquad q_0 = \lim_{x\to 0} x^2\frac{R(x)}{P(x)}. \tag{14}$$

Finally, the radii of convergence for the series in equations (9) and (10) are at least equal to the distance from the origin to the nearest zero of P other than the regular singular point $x = 0$ itself.

EXAMPLE 1

Discuss the nature of the solutions of the equation

$$2x(1+x)y'' + (3+x)y' - xy = 0$$

near the singular points.

Solution:

This equation is of the form (13) with $P(x) = 2x(1+x)$, $Q(x) = 3 + x$, and $R(x) = -x$. The points $x = 0$ and $x = -1$ are the only singular points. The point $x = 0$ is a regular singular point, since

$$\lim_{x\to 0} x\frac{Q(x)}{P(x)} = \lim_{x\to 0} x\frac{3+x}{2x(1+x)} = \frac{3}{2},$$

$$\lim_{x\to 0} x^2\frac{R(x)}{P(x)} = \lim_{x\to 0} x^2\frac{-x}{2x(1+x)} = 0.$$

Further, from equation (14), $p_0 = \frac{3}{2}$ and $q_0 = 0$. Thus the indicial equation is $r(r-1) + \frac{3}{2}r = 0$, and the roots are $r_1 = 0$, $r_2 = -\frac{1}{2}$. Since these roots are not equal and do not differ by an integer, there are two solutions of the form

$$y_1(x) = 1 + \sum_{n=1}^{\infty} a_n(0)x^n \quad\text{and}\quad y_2(x) = |x|^{-1/2}\left(1 + \sum_{n=1}^{\infty} a_n\left(-\tfrac{1}{2}\right)x^n\right)$$

for $0 < |x| < \rho$. A lower bound for the radius of convergence of each series is 1, the distance from $x = 0$ to $x = -1$, the other zero of $P(x)$. Note that the solution y_1 is bounded as $x \to 0$, indeed is analytic there, and that the second solution y_2 is unbounded as $x \to 0$.

The point $x = -1$ is also a regular singular point, since

$$\lim_{x\to -1} (x+1)\frac{Q(x)}{P(x)} = \lim_{x\to -1} \frac{(x+1)(3+x)}{2x(1+x)} = -1,$$

$$\lim_{x\to -1} (x+1)^2\frac{R(x)}{P(x)} = \lim_{x\to -1} \frac{(x+1)^2(-x)}{2x(1+x)} = 0.$$

In this case $p_0 = -1$, $q_0 = 0$, so the indicial equation is $r(r-1) - r = 0$. The roots of the indicial equation are $r_1 = 2$ and $r_2 = 0$. Corresponding to the larger root there is a solution of the form

$$y_1(x) = (x+1)^2\left(1 + \sum_{n=1}^{\infty} a_n(2)(x+1)^n\right).$$

The series converges at least for $|x+1| < 1$, and y_1 is an analytic function there. Since the two roots differ by a positive integer, there may or may not be a second solution of the form

$$y_2(x) = 1 + \sum_{n=1}^{\infty} a_n(0)(x+1)^n.$$

We cannot say more without further analysis.

Observe that no complicated calculations were required to discover the information about the solutions presented in this example. All that was needed was to evaluate a few limits and solve two quadratic equations.

We now consider the cases in which the roots of the indicial equation are equal or differ by a positive integer, $r_1 - r_2 = N$. As we have shown earlier, there is always one solution of the form (9) corresponding to the larger root r_1 of the indicial equation. By analogy with the Euler equation, we might expect that if $r_1 = r_2$, then the second solution contains a logarithmic term. This may also be true if the roots differ by an integer.

Equal Roots. The method of finding the second solution is essentially the same as the one we used in finding the second solution of the Euler equation (see Section 5.4) when the roots of the indicial equation were equal. We consider r to be a continuous variable and determine a_n as a function of r by solving the recurrence relation (8). For this choice of $a_n(r)$ for $n \geq 1$, the terms in equation (6) involving $x^{r+1}, x^{r+2}, x^{r+3}, \ldots$ all have coefficients equal to zero. Therefore, since r_1 is a repeated root of $F(r)$, equation (6) reduces to

$$L[\phi](r,x) = a_0F(r)x^r = a_0(r-r_1)^2x^r. \tag{15}$$

Setting $r = r_1$ in equation (15), we find that $L[\phi](r_1, x) = 0$; hence, as we already know, $y_1(x)$ given by equation (9) is one solution of equation (1). But more important, it also follows from equation (15), just as for the Euler equation, that

$$L\left[\frac{\partial \phi}{\partial r}\right](r_1, x) = a_0 \frac{\partial}{\partial r}\left(x^r (r - r_1)^2\right)\Big|_{r=r_1}$$

$$= a_0 \left((r - r_1)^2 x^r \ln x + 2(r - r_1)x^r\right)\Big|_{r=r_1} = 0. \quad (16)$$

Hence, a second solution of equation (1) is

$$y_2(x) = \left.\frac{\partial \phi(r, x)}{\partial r}\right|_{r=r_1} = \frac{\partial}{\partial r}\left(x^r \left(a_0 + \sum_{n=1}^{\infty} a_n(r)x^n\right)\right)\Bigg|_{r=r_1}$$

$$= (x^{r_1} \ln x)\left(a_0 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right) + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1)x^n$$

$$= y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1)x^n, \quad x > 0, \quad (17)$$

where $a_n'(r_1)$ denotes $\dfrac{da_n}{dr}$ evaluated at $r = r_1$.

Although equation (17) provides an explicit expression for a second solution $y_2(x)$, it may turn out that it is difficult to determine $a_n(r)$ as a function of r from the recurrence relation (8) and then to differentiate the resulting expression with respect to r. An alternative is simply to assume that y has the *form* of equation (17). That is, assume that

$$y = y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} b_n x^n, \quad x > 0, \quad (18)$$

where $y_1(x)$ has already been found. The coefficients b_n are calculated, as usual, by substituting into the differential equation, collecting terms, and setting the coefficient of each power of x equal to zero. A third possibility is to use the method of reduction of order to find $y_2(x)$ once $y_1(x)$ is known.

Roots r_1 and r_2 Differing by an Integer N. For this case the derivation of the second solution is considerably more complicated and will not be given here. The form of this solution is stated in equation (24) in the following theorem. The coefficients $c_n(r_2)$ in equation (24) are given by

$$c_n(r_2) = \frac{d}{dr}[(r - r_2)a_n(r)]\Big|_{r=r_2}, \quad n = 1, 2, \ldots, \quad (19)$$

where $a_n(r)$ is determined from the recurrence relation (8) with $a_0 = 1$. Further, the coefficient a in equation (24) is

$$a = \lim_{r \to r_2} (r - r_2)a_N(r). \quad (20)$$

If $a_N(r_2)$ is finite, then $a = 0$ and there is no logarithmic term in y_2. A full derivation of formulas (19) and (20) may be found in Coddington (Chapter 4).

In practice, the best way to determine whether a is zero in the second solution is simply to try to compute the a_n corresponding to the root r_2 and to see whether it is possible to determine $a_N(r_2)$. If so, there is no further problem. If not, we must use the form (24) with $a \neq 0$.

When $r_1 - r_2 = N$, there are again three ways to find a second solution. First, we can calculate a and $c_n(r_2)$ directly by substituting the expression (24) for y in equation (1). Second, we can calculate $c_n(r_2)$ and a of equation (24) using the formulas (19) and (20). If this is the planned procedure, then in calculating the solution corresponding to $r = r_1$, be sure to obtain the general formula for $a_n(r)$ rather than just $a_n(r_1)$. The third alternative is to use the method of reduction of order.

The following theorem summarizes the results that we have obtained in this section.

Theorem 5.6.1

Consider the differential equation (1)

$$x^2y'' + x(xp(x))y' + \left(x^2q(x)\right)y = 0,$$

where $x = 0$ is a regular singular point. Then $xp(x)$ and $x^2q(x)$ are analytic at $x = 0$ with convergent power series expansions

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad x^2q(x) = \sum_{n=0}^{\infty} q_n x^n$$

for $|x| < \rho$, where $\rho > 0$ is the minimum of the radii of convergence of the power series for $xp(x)$ and $x^2q(x)$. Let r_1 and r_2 be the roots of the indicial equation

$$F(r) = r(r-1) + p_0 r + q_0 = 0,$$

with $r_1 \geq r_2$ if r_1 and r_2 are real. Then in either the interval $-\rho < x < 0$ or the interval $0 < x < \rho$, there exists a solution of the form

$$y_1(x) = |x|^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right), \tag{21}$$

where the $a_n(r_1)$ are given by the recurrence relation (8) with $a_0 = 1$ and $r = r_1$.

CASE 1 If $r_1 - r_2$ is not zero or a positive integer, then in either the interval $-\rho < x < 0$ or the interval $0 < x < \rho$, there exists a second solution of the form

$$y_2(x) = |x|^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2)x^n\right). \tag{22}$$

The $a_n(r_2)$ are also determined by the recurrence relation (8) with $a_0 = 1$ and $r = r_2$. The power series in equations (21) and (22) converge at least for $|x| < \rho$.

CASE 2 If $r_1 = r_2$, then the second solution is

$$y_2(x) = y_1(x)\ln|x| + |x|^{r_1}\sum_{n=1}^{\infty} b_n(r_1)x^n. \tag{23}$$

CASE 3 If $r_1 - r_2 = N$, a positive integer, then

$$y_2(x) = ay_1(x)\ln|x| + |x|^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n(r_2)x^n\right). \tag{24}$$

The coefficients $a_n(r_1)$, $b_n(r_1)$, and $c_n(r_2)$ and the constant a can be determined by substituting the form of the series solutions for y in equation (1). The constant a may turn out to be zero, in which case there is no logarithmic term in the solution (24). Each of the series in equations (23) and (24) converges at least for $|x| < \rho$ and defines a function that is analytic in some neighborhood of $x = 0$.

In all three cases, the two solutions $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of the given differential equation.

Problems

In each of Problems 1 through 8:

a. Find all the regular singular points of the given differential equation.

b. Determine the indicial equation and the exponents at the singularity for each regular singular point.

1. $xy'' + 2xy' + 6e^x y = 0$
2. $x^2y'' - x(2+x)y' + (2+x^2)y = 0$
3. $y'' + 4xy' + 6y = 0$
4. $2x(x+2)y'' + y' - xy = 0$
5. $x^2y'' + \frac{1}{2}(x + \sin x)y' + y = 0$
6. $x^2(1-x)y'' - (1+x)y' + 2xy = 0$
7. $(x-2)^2(x+2)y'' + 2xy' + 3(x-2)y = 0$
8. $(4-x^2)y'' + 2xy' + 3y = 0$

In each of Problems 9 through 12:

a. Show that $x = 0$ is a regular singular point of the given differential equation.

b. Find the exponents at the singular point $x = 0$.

c. Find the first three nonzero terms in each of two solutions (not multiples of each other) about $x = 0$.

9. $xy'' + y' - y = 0$
10. $xy'' + 2xy' + 6e^x y = 0$ (see Problem 1)

11. $xy'' + y = 0$

12. $x^2y'' + (\sin x)y' - (\cos x)y = 0$

13. a. Show that

$$(\ln x)y'' + \frac{1}{2}y' + y = 0$$

has a regular singular point at $x = 1$.

b. Determine the roots of the indicial equation at $x = 1$.

c. Determine the first three nonzero terms in the series $\sum_{n=0}^{\infty} a_n(x-1)^{r+n}$ corresponding to the larger root. You can assume $x - 1 > 0$.

d. What would you expect the radius of convergence of the series to be?

14. In several problems in mathematical physics, it is necessary to study the differential equation

$$x(1-x)y'' + (\gamma - (1+\alpha+\beta)x)y' - \alpha\beta y = 0, \qquad (25)$$

where α, β, and γ are constants. This equation is known as the **hypergeometric equation**.

a. Show that $x = 0$ is a regular singular point and that the roots of the indicial equation are 0 and $1 - \gamma$.

b. Show that $x = 1$ is a regular singular point and that the roots of the indicial equation are 0 and $\gamma - \alpha - \beta$.

c. Assuming that $1 - \gamma$ is not a positive integer, show that, in the neighborhood of $x = 0$, one solution of equation (25) is

$$y_1(x) = 1 + \frac{\alpha\beta}{\gamma \cdot 1!}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)2!}x^2 + \cdots.$$

What would you expect the radius of convergence of this series to be?

d. Assuming that $1 - \gamma$ is not an integer or zero, show that a second solution for $0 < x < 1$ is

$$y_2(x) = x^{1-\gamma}\left(1 + \frac{(\alpha-\gamma+1)(\beta-\gamma+1)}{(2-\gamma)1!}x + \frac{(\alpha-\gamma+1)(\alpha-\gamma+2)(\beta-\gamma+1)(\beta-\gamma+2)}{(2-\gamma)(3-\gamma)2!}x^2 + \cdots\right).$$

e. Show that the point at infinity is a regular singular point and that the roots of the indicial equation are α and β. See Problem 32 of Section 5.4.

15. Consider the differential equation

$$x^3y'' + \alpha xy' + \beta y = 0,$$

where α and β are real constants and $\alpha \neq 0$.

a. Show that $x = 0$ is an irregular singular point.

b. By attempting to determine a solution of the form $\sum_{n=0}^{\infty} a_n x^{r+n}$, show that the indicial equation for r is linear and that, consequently, there is only one formal solution of the assumed form.

c. Show that if $\beta/\alpha = -1, 0, 1, 2, \ldots$, then the formal series solution terminates and therefore is an actual solution. For other values of β/α, show that the formal series solution has a zero radius of convergence and so does not represent an actual solution in any interval.

16. Consider the differential equation

$$y'' + \frac{\alpha}{x^s}y' + \frac{\beta}{x^t}y = 0, \qquad (26)$$

where $\alpha \neq 0$ and $\beta \neq 0$ are real numbers, and s and t are positive integers that for the moment are arbitrary.

a. Show that if $s > 1$ or $t > 2$, then the point $x = 0$ is an irregular singular point.

b. Try to find a solution of equation (26) of the form

$$y = \sum_{n=0}^{\infty} a_n x^{r+n}, \quad x > 0. \qquad (27)$$

Show that if $s = 2$ and $t = 2$, then there is only one possible value of r for which there is a formal solution of equation (26) of the form (27).

c. Show that if $s = 1$ and $t = 3$, then there are no solutions of equation (26) of the form (27).

d. Show that the maximum values of s and t for which the indicial equation is quadratic in r [and hence we can hope to find two solutions of the form (27)] are $s = 1$ and $t = 2$. These are precisely the conditions that distinguish a "weak singularity," or a regular singular point, from an irregular singular point, as we defined them in Section 5.4.

As a note of caution, we point out that although it is sometimes possible to obtain a formal series solution of the form (27) at an irregular singular point, the series may not have a positive radius of convergence. See Problem 15 for an example.

5.7 Bessel's Equation

In this section we illustrate the discussion in Section 5.6 by considering three special cases of Bessel's[14] equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \qquad (1)$$

[14]Friedrich Wilhelm Bessel (1784–1846) left school at the age of 14 to embark on a career in the import-export business but soon became interested in astronomy and mathematics. He was appointed director of the observatory at Königsberg in 1810 and held this position until his death. His study of planetary perturbations led him in 1824 to make the first systematic analysis of the solutions, known as Bessel functions, of equation (1). He is also famous for making, in 1838, the first accurate determination of the distance from the earth to a star.

where ν is a constant. It is easy to show that $x = 0$ is a regular singular point of equation (1). We have

$$p_0 = \lim_{x\to 0} x\frac{Q(x)}{P(x)} = \lim_{x\to 0} x\frac{1}{x} = 1,$$

$$q_0 = \lim_{x\to 0} x^2\frac{R(x)}{P(x)} = \lim_{x\to 0} x^2\frac{x^2-\nu^2}{x^2} = -\nu^2.$$

Thus the indicial equation is

$$F(r) = r(r-1) + p_0 r + q_0 = r(r-1) + r - \nu^2 = r^2 - \nu^2 = 0,$$

with the roots $r = \pm\nu$. We will consider the three cases $\nu = 0$, $\nu = \frac{1}{2}$, and $\nu = 1$ for the interval $x > 0$. Bessel functions will reappear in Sections 11.4 and 11.5.

Bessel Equation of Order Zero. In this case $\nu = 0$, so differential equation (1) reduces to

$$L[y] = x^2y'' + xy' + x^2y = 0, \tag{2}$$

and the roots of the indicial equation are equal: $r_1 = r_2 = 0$. Substituting

$$y = \phi(r, x) = a_0x^r + \sum_{n=1}^{\infty} a_nx^{r+n} \tag{3}$$

in equation (2), we obtain

$$\begin{aligned} L[\phi](r,x) &= \sum_{n=0}^{\infty} a_n((r+n)(r+n-1) + (r+n))x^{r+n} + \sum_{n=0}^{\infty} a_nx^{r+n+2} \\ &= a_0(r(r-1)+r)x^r + a_1((r+1)r + (r+1))x^{r+1} \\ &\quad + \sum_{n=2}^{\infty}\big(a_n((r+n)(r+n-1)+(r+n)) + a_{n-2}\big)x^{r+n} = 0. \end{aligned} \tag{4}$$

As we have already noted, the roots of the indicial equation $F(r) = r(r-1) + r = 0$ are $r_1 = 0$ and $r_2 = 0$. The recurrence relation is

$$a_n(r) = -\frac{a_{n-2}(r)}{(r+n)(r+n-1)+(r+n)} = -\frac{a_{n-2}(r)}{(r+n)^2}, \quad n \geq 2. \tag{5}$$

To determine $y_1(x)$, we set r equal to 0. Then, from equation (4), it follows that for the coefficient of x^{r+1} to be zero we must choose $a_1 = 0$. Hence, from equation (5), $a_3 = a_5 = a_7 = \cdots = 0$. Further,

$$a_n(0) = -\frac{a_{n-2}(0)}{n^2}, \quad n = 2, 4, 6, 8, \ldots,$$

or, letting $n = 2m$, we obtain

$$a_{2m}(0) = -\frac{a_{2m-2}(0)}{(2m)^2}, \quad m = 1, 2, 3, \ldots.$$

Thus

$$a_2(0) = -\frac{a_0}{2^2}, \quad a_4(0) = \frac{a_0}{2^4 2^2}, \quad a_6(0) = -\frac{a_0}{2^6(3\cdot 2)^2},$$

and, in general,

$$a_{2m}(0) = \frac{(-1)^m a_0}{2^{2m}(m!)^2}, \quad m = 1, 2, 3, \ldots. \tag{6}$$

Hence

$$y_1(x) = a_0\left(1 + \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right), \quad x > 0. \tag{7}$$

The function in brackets is known as the **Bessel function of the first kind of order zero** and is denoted by $J_0(x)$. It follows from Theorem 5.6.1 that the series converges for all x and that J_0 is analytic at $x = 0$. Some of the important properties of J_0 are discussed in the problems. Figure 5.7.1 shows the graphs of $y = J_0(x)$ and some of the partial sums of the series (7).

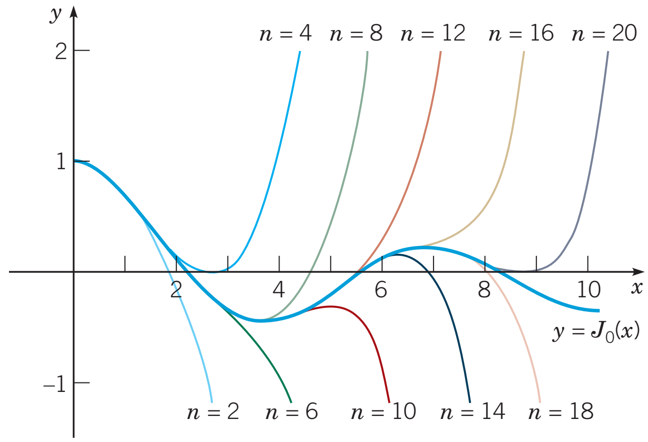

FIGURE 5.7.1 Polynomial approximations to $J_0(x)$, the Bessel function of the first kind of order zero. The value of n is the degree of the approximating polynomial.

To determine $y_2(x)$ we will use equation (17) in Section 5.6. This requires that we calculate[15] $a_n'(0)$. First we note from the coefficient of x^{r+1} in differential equation (4) that $(r+1)^2 a_1(r) = 0$. Thus $a_1(r) = 0$ for all r near $r = 0$. So not only does $a_1(0) = 0$ but also $a_1'(0) = 0$. From the recurrence relation (5) it follows that

$$a_3'(0) = a_5'(0) = \cdots = a_{2n+1}'(0) = \cdots = 0;$$

hence we need only compute $a_{2m}'(0)$, $m = 1, 2, 3, \ldots$. From equation (5) we have

$$a_{2m}(r) = -\frac{a_{2m-2}(r)}{(r+2m)^2}, \qquad m = 1, 2, 3, \ldots.$$

By solving this recurrence relation, we obtain

$$a_2(r) = -\frac{a_0}{(r+2)^2}, \qquad a_4(r) = \frac{a_0}{(r+2)^2(r+4)^2},$$

and, in general,

$$a_{2m}(r) = \frac{(-1)^m a_0}{(r+2)^2 \cdots (r+2m)^2}, \qquad m \ge 3. \tag{8}$$

The computation of $a_{2m}'(r)$ can be carried out most conveniently by noting that if

$$f(x) = (x-\alpha_1)^{\beta_1}(x-\alpha_2)^{\beta_2}(x-\alpha_3)^{\beta_3}\cdots(x-\alpha_n)^{\beta_n},$$

and if x is not equal to $\alpha_1, \alpha_2, \ldots, \alpha_n$, then

$$\frac{f'(x)}{f(x)} = \frac{\beta_1}{x-\alpha_1} + \frac{\beta_2}{x-\alpha_2} + \cdots + \frac{\beta_n}{x-\alpha_n}.$$

Applying this result to $a_{2m}(r)$ from equation (8), we find that

$$\frac{a_{2m}'(r)}{a_{2m}(r)} = -2\left(\frac{1}{r+2} + \frac{1}{r+4} + \cdots + \frac{1}{r+2m}\right),$$

and setting r equal to 0, we obtain

$$a_{2m}'(0) = -2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2m}\right) a_{2m}(0).$$

[15]Problem 9 outlines an alternative procedure, in which we simply substitute the form (23) of Section 5.6 in equation (2) and then determine the b_n.

Substituting for $a_{2m}(0)$ from equation (6), and letting

$$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}, \tag{9}$$

we obtain, finally,

$$a'_{2m}(0) = -H_m \frac{(-1)^m a_0}{2^{2m}(m!)^2}, \qquad m = 1, 2, 3, \ldots .$$

The second solution of the Bessel equation of order zero is found by setting $a_0 = 1$ and substituting for $y_1(x)$ and $a'_{2m}(0) = b_{2m}(0)$ in equation (23) of Section 5.6. We obtain

$$y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}, \qquad x > 0. \tag{10}$$

Instead of y_2, the second solution is usually taken to be a certain linear combination of J_0 and y_2. It is known as the **Bessel function of the second kind of order zero** and is denoted by Y_0. Following Copson (Chapter 12), we define[16]

$$Y_0(x) = \frac{2}{\pi}[y_2(x) + (\gamma - \ln 2) J_0(x)]. \tag{11}$$

Here γ is a constant known as the Euler–Máscheroni[17] constant; it is defined by the equation

$$\gamma = \lim_{n\to\infty} (H_n - \ln n) \cong 0.5772. \tag{12}$$

Substituting for $y_2(x)$ in equation (11), we obtain

$$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{x}{2}\right) J_0(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}\right], \qquad x > 0. \tag{13}$$

The general solution of the Bessel equation of order zero for $x > 0$ is

$$y = c_1 J_0(x) + c_2 Y_0(x).$$

Note that $J_0(x) \to 1$ as $x \to 0$ and that $Y_0(x)$ has a logarithmic singularity at $x = 0$; that is, $Y_0(x)$ behaves as $(2/\pi) \ln x$ when $x \to 0$ through positive values. Thus, if we are interested in solutions of Bessel's equation of order zero that are finite at the origin, which is often the case, we must discard Y_0. The graphs of the functions J_0 and Y_0 are shown in Figure 5.7.2.

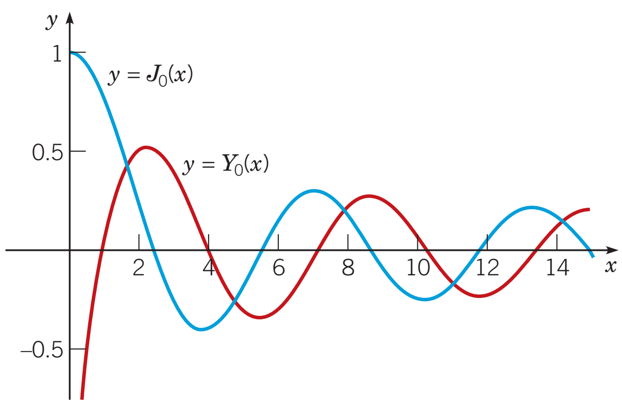

FIGURE 5.7.2 The Bessel functions of order zero: $y = J_0(x)$ (blue) and $y = Y_0(x)$ (red).

[16]Other authors use other definitions for Y_0. The present choice for Y_0 is also known as the Weber function, after Heinrich Weber (1842–1913), who taught at several German universities.

[17]The Euler–Máscheroni constant first appeared in 1734 in a paper by Euler. Lorenzo Máscheroni (1750–1800) was an Italian priest and professor at the University of Pavia. He correctly calculated the first 19 decimal places of γ in 1790.

It is interesting to note from Figure 5.7.2 that for x large, both $J_0(x)$ and $Y_0(x)$ are oscillatory. Such a behavior might be anticipated from the original equation; indeed it is true for the solutions of the Bessel equation of order ν. If we divide equation (1) by x^2, we obtain

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0.$$

For x very large, it is reasonable to conjecture that the terms $(1/x)y'$ and $(\nu^2/x^2)y$ are small and hence can be neglected. If this is true, then the Bessel equation of order ν can be approximated by

$$y'' + y = 0.$$

The solutions of this equation are $\sin x$ and $\cos x$; thus we might anticipate that the solutions of Bessel's equation for large x are similar to linear combinations of $\sin x$ and $\cos x$. This is correct insofar as the Bessel functions are oscillatory; however, it is only partly correct. For x large the functions J_0 and Y_0 also decay as x increases; thus the equation $y'' + y = 0$ does not provide an adequate approximation to the Bessel equation for large x, and a more delicate analysis is required. In fact, it is possible to show that

$$J_0(x) \cong \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\pi}{4}\right) \quad \text{as } x \to \infty \tag{14}$$

and that

$$Y_0(x) \cong \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{\pi}{4}\right) \quad \text{as } x \to \infty. \tag{15}$$

These asymptotic approximations, as $x \to \infty$, are actually very good. For example, Figure 5.7.3 shows that the asymptotic approximation (14) to $J_0(x)$ is reasonably accurate for all $x \geq 1$. Thus to approximate $J_0(x)$ over the entire range from zero to infinity, you can use two or three terms of the series (7) for $x \leq 1$ and the asymptotic approximation (14) for $x \geq 1$.

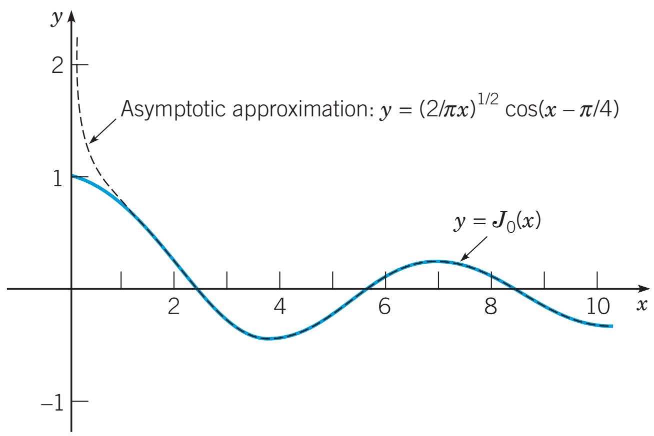

FIGURE 5.7.3 Asymptotic approximation to $J_0(x)$.

Bessel Equation of Order One-Half. This case illustrates the situation in which the roots of the indicial equation differ by a positive integer but there is no logarithmic term in the second solution. Setting $\nu = \frac{1}{2}$ in equation (1) gives

$$L[y] = x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0. \tag{16}$$

When we substitute the series (3) for $y = \phi(r, x)$, we obtain

$$\begin{aligned} L[\phi](r,x) &= \sum_{n=0}^{\infty}\left((r+n)(r+n-1)+(r+n)-\frac{1}{4}\right)a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} \\ &= \left(r^2-\frac{1}{4}\right)a_0x^r + \left((r+1)^2-\frac{1}{4}\right)a_1 x^{r+1} \\ &\quad + \sum_{n=2}^{\infty}\left(\left((r+n)^2-\frac{1}{4}\right)a_n + a_{n-2}\right)x^{r+n} = 0. \end{aligned} \tag{17}$$

The roots of the indicial equation $r^2 - \frac{1}{4} = 0$ are $r_1 = \frac{1}{2}$ and $r_2 = -\frac{1}{2}$; hence the roots differ by an integer. The recurrence relation is

$$\left((r+n)^2-\frac{1}{4}\right)a_n = -a_{n-2}, \quad n \geq 2. \tag{18}$$

Corresponding to the larger root $r_1 = \frac{1}{2}$, we find, from the coefficient of x^{r+1} in equation (17), that $a_1 = 0$. Hence, from equation (18), $a_3 = a_5 = \cdots = a_{2n+1} = \cdots = 0$. Further, for $r = \frac{1}{2}$,

$$a_n = -\frac{a_{n-2}}{n(n+1)}, \quad n = 2, 4, 6 \ldots,$$

or, letting $n = 2m$, we obtain

$$a_{2m} = -\frac{a_{2m-2}}{2m(2m+1)}, \quad m = 1, 2, 3, \ldots.$$

By solving this recurrence relation, we find that

$$a_2 = -\frac{a_0}{3!}, \quad a_4 = \frac{a_0}{5!}, \ldots$$

and, in general,

$$a_{2m} = \frac{(-1)^m a_0}{(2m+1)!}, \quad m = 1, 2, 3, \ldots.$$

Hence, taking $a_0 = 1$, we obtain

$$y_1(x) = x^{1/2}\left(1 + \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{(2m+1)!}\right) = x^{-1/2}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!}, \quad x > 0. \tag{19}$$

The second power series in equation (19) is precisely the Taylor series for $\sin x$; hence one solution of the Bessel equation of order one-half is $x^{-1/2}\sin x$. The **Bessel function of the first kind of order one-half**, $J_{1/2}$, is defined as $(2/\pi)^{1/2}y_1$. Thus

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\sin x, \quad x > 0. \tag{20}$$

Corresponding to the root $r_2 = -\frac{1}{2}$, it is possible that we may have difficulty in computing a_1 since $N = r_1 - r_2 = 1$. However, from equation (17) for $r = -\frac{1}{2}$, the coefficients of x^r and x^{r+1} are both zero regardless of the choice of a_0 and a_1. Hence a_0 and a_1 can be chosen arbitrarily. From the recurrence relation (18), we obtain a set of even-numbered coefficients corresponding to a_0 and a set of odd-numbered coefficients corresponding to a_1. Thus no logarithmic term is needed to obtain a second solution in this case. It is left as an exercise to show that, for $r = -\frac{1}{2}$,

$$a_{2n} = \frac{(-1)^n a_0}{(2n)!}, \quad a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}, \quad n = 1, 2, \ldots.$$

Hence

$$y_2(x) = x^{-1/2}\left(a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)$$

$$= a_0 \frac{\cos x}{x^{1/2}} + a_1 \frac{\sin x}{x^{1/2}}, \qquad x > 0. \tag{21}$$

The constant a_1 simply introduces a multiple of $y_1(x)$. The second solution of the Bessel equation of order one-half is usually taken to be the solution for which $a_0 = (2/\pi)^{1/2}$ and $a_1 = 0$. It is denoted by $J_{-1/2}$. Then

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \qquad x > 0. \tag{22}$$

The general solution of equation (16) is $y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$.

By comparing equations (20) and (22) with equations (14) and (15), we see that, except for a phase shift of $\pi/4$, the functions $J_{-1/2}$ and $J_{1/2}$ resemble J_0 and Y_0, respectively, for large x. The graphs of $J_{1/2}$ and $J_{-1/2}$ are shown in Figure 5.7.4.

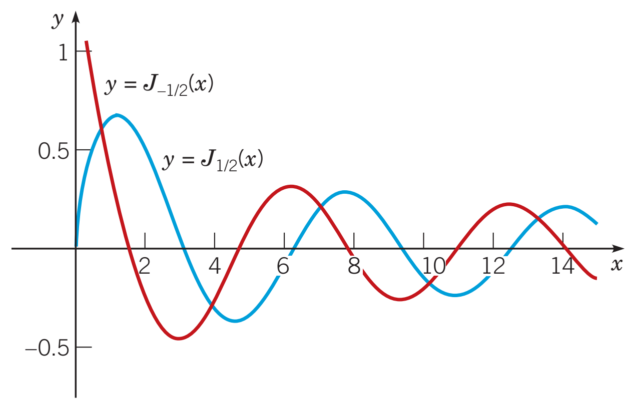

FIGURE 5.7.4 The Bessel functions of order one-half: $y = J_{1/2}(x)$ (blue) and $y = J_{-1/2}(x)$ (red).

Bessel Equation of Order One. This case illustrates the situation in which the roots of the indicial equation differ by a positive integer and the second solution involves a logarithmic term. Setting $\nu = 1$ in equation (1) gives

$$L[y] = x^2 y'' + xy' + (x^2 - 1)y = 0. \tag{23}$$

If we substitute the series (3) for $y = \phi(r, x)$ and collect terms as in the preceding cases, we obtain

$$L[\phi](r, x) = a_0(r^2 - 1)x^r + a_1\Big((r+1)^2 - 1\Big)x^{r+1}$$

$$+ \sum_{n=2}^{\infty}\Big(\Big((r+n)^2 - 1\Big)a_n + a_{n-2}\Big)x^{r+n} = 0. \tag{24}$$

The roots of the indicial equation $r^2 - 1 = 0$ are $r_1 = 1$ and $r_2 = -1$. The recurrence relation is

$$\Big((r+n)^2 - 1\Big)a_n(r) = -a_{n-2}(r), \qquad n \geq 2. \tag{25}$$

Corresponding to the larger root $r = 1$, the recurrence relation becomes

$$a_n = -\frac{a_{n-2}}{(n+2)n}, \qquad n = 2, 3, 4, \ldots .$$

We also find, from the coefficient of x^{r+1} in equation (24), that $a_1 = 0$; hence, from the recurrence relation, $a_3 = a_5 = \cdots = 0$. For even values of n, we can write $n = 2m$, where m

is a positive integer; then

$$a_{2m} = -\frac{a_{2m-2}}{(2m+2)(2m)} = -\frac{a_{2m-2}}{2^2(m+1)m}, \qquad m = 1, 2, 3, \ldots .$$

By solving this recurrence relation, we obtain

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m+1)!m!}, \qquad m = 1, 2, 3, \ldots . \tag{26}$$

The Bessel function of the first kind of order one, denoted by J_1, is obtained by choosing $a_0 = 1/2$. Hence

$$J_1(x) = \frac{x}{2}\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m+1)!m!}. \tag{27}$$

The series converges absolutely for all x, so the function J_1 is analytic everywhere.

In determining a second solution of Bessel's equation of order one, we illustrate the method of direct substitution. The calculation of the general term in equation (28) below is rather complicated, but the first few coefficients can be found fairly easily. According to Theorem 5.6.1, we assume that

$$y_2(x) = aJ_1(x)\ln x + x^{-1}\left(1 + \sum_{n=1}^{\infty} c_n x^n\right), \qquad x > 0. \tag{28}$$

Computing $y_2'(x)$ and $y_2''(x)$, substituting in equation (23), and making use of the fact that J_1 is a solution of equation (23), we obtain

$$2axJ_1'(x) + \sum_{n=0}^{\infty}((n-1)(n-2)c_n + (n-1)c_n - c_n)x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0, \tag{29}$$

where $c_0 = 1$. Substituting for $J_1(x)$ from equation (27), shifting the indices of summation in the two series, and carrying out several steps of algebra, we arrive at

$$-c_1 + \big(0\cdot c_2 + c_0\big)x + \sum_{n=2}^{\infty}\big((n^2-1)c_{n+1} + c_{n-1}\big)x^n$$
$$= -a\left(x + \sum_{m=1}^{\infty} \frac{(-1)^m(2m+1)x^{2m+1}}{2^{2m}(m+1)!\,m!}\right). \tag{30}$$

From equation (30) we observe first that $c_1 = 0$, and $a = -c_0 = -1$. Further, since there are only odd powers of x on the right, the coefficient of each even power of x on the left must be zero. Thus, since $c_1 = 0$, we have $c_3 = c_5 = \cdots = 0$. Corresponding to the odd powers of x, writing $n = 2m+1$ on the left-hand side of equation (30), we obtain the following recurrence relation:

$$\Big((2m+1)^2 - 1\Big)c_{2m+2} + c_{2m} = \frac{(-1)^m(2m+1)}{2^{2m}(m+1)!\,m!}, \qquad m = 1, 2, 3, \ldots . \tag{31}$$

When we set $m = 1$ in equation (31), we obtain

$$(3^2-1)c_4 + c_2 = \frac{(-1)3}{2^2\cdot 2!}.$$

Notice that c_2 can be selected *arbitrarily*, and then this equation determines c_4. Also notice that in the equation for the coefficient of x, c_2 appeared multiplied by 0, and that equation was used to determine a. That c_2 is arbitrary is not surprising, since c_2 is the coefficient of x in the expression $x^{-1}\left(1 + \sum\limits_{n=1}^{\infty} c_n x^n\right)$. Consequently, c_2 simply generates a multiple of J_1, and y_2 is determined only up to an additive multiple of J_1. In accordance with the usual practice, we choose $c_2 = 1/2^2$. Then we obtain

$$c_4 = \frac{-1}{2^4 \cdot 2}\left(\frac{3}{2} + 1\right) = \frac{-1}{2^4 2!}\left(\left(1 + \frac{1}{2}\right) + 1\right)$$
$$= \frac{(-1)}{2^4 \cdot 2!}(H_2 + H_1).$$

It is possible to show that the solution of the recurrence relation (31) is

$$c_{2m} = \frac{(-1)^{m+1}(H_m + H_{m-1})}{2^{2m} m!(m-1)!}, \qquad m = 1, 2, \ldots$$

with the understanding that $H_0 = 0$. Thus

$$y_2(x) = -J_1(x) \ln x + \frac{1}{x}\left(1 - \sum_{m=1}^{\infty} \frac{(-1)^m (H_m + H_{m-1})}{2^{2m} m!(m-1)!} x^{2m}\right), \qquad x > 0. \tag{32}$$

The calculation of $y_2(x)$ using the alternative procedure (see equations (19) and (20) of Section 5.6) in which we determine the $c_n(r_2)$ is slightly easier. In particular, the latter procedure yields the general formula for c_{2m} without the necessity of solving a recurrence relation of the form (31) (see Problem 10). In this regard, you may also wish to compare the calculations of the second solution of Bessel's equation of order zero in the text and in Problem 9.

The second solution of equation (23), the Bessel function of the second kind of order one, Y_1, is usually taken to be a certain linear combination of J_1 and y_2. Following Copson (Chapter 12), Y_1 is defined as

$$Y_1(x) = \frac{2}{\pi}\big(-y_2(x) + (\gamma - \ln 2) J_1(x)\big), \tag{33}$$

where γ is defined in equation (12). The general solution of equation (23) for $x > 0$ is

$$y = c_1 J_1(x) + c_2 Y_1(x).$$

Notice that although J_1 is analytic at $x = 0$, the second solution Y_1 becomes unbounded in the same manner as $1/x$ as $x \to 0$. The graphs of J_1 and Y_1 are shown in Figure 5.7.5.

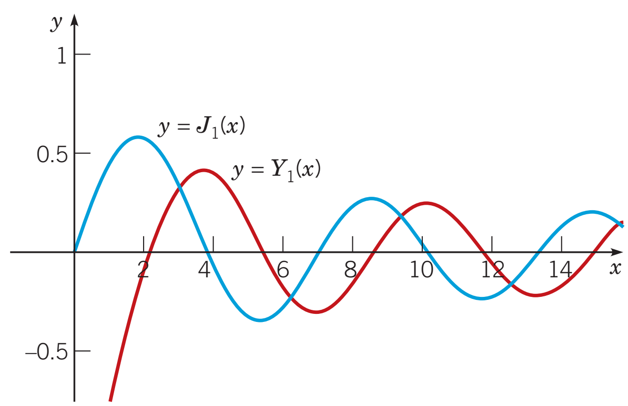

FIGURE 5.7.5 The Bessel functions of order one: $y = J_1(x)$ (blue) and $y = Y_1(x)$ (red).

Problems

In each of Problems 1 through 3, show that the given differential equation has a regular singular point at $x = 0$, and determine two solutions for $x > 0$.

1. $x^2y'' + 2xy' + xy = 0$

2. $x^2y'' + 3xy' + (1 + x)y = 0$

3. $x^2y'' + xy' + 2xy = 0$

4. Find two solutions (not multiples of each other) of the Bessel equation of order $\frac{3}{2}$

$$x^2y'' + xy' + \left(x^2 - \frac{9}{4}\right)y = 0, \quad x > 0.$$

5. Show that the Bessel equation of order one-half

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0, \quad x > 0$$

can be reduced to the equation

$$v'' + v = 0$$

by the change of dependent variable $y = x^{-1/2}v(x)$. From this, conclude that $y_1(x) = x^{-1/2}\cos x$ and $y_2(x) = x^{-1/2}\sin x$ are solutions of the Bessel equation of order one-half.

6. Show directly that the series for $J_0(x)$, equation (7), converges absolutely for all x.

7. Show directly that the series for $J_1(x)$, equation (27), converges absolutely for all x and that $J_0'(x) = -J_1(x)$.

8. Consider the Bessel equation of order ν

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad x > 0,$$

where ν is real and positive.

a. Show that $x = 0$ is a regular singular point and that the roots of the indicial equation are ν and $-\nu$.

b. Corresponding to the larger root ν, show that one solution is

$$y_1(x) = x^{\nu}\left(1 - \frac{1}{1!(1+\nu)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(1+\nu)(2+\nu)}\left(\frac{x}{2}\right)^4 + \sum_{m=3}^{\infty}\frac{(-1)^m}{m!(1+\nu)\cdots(m+\nu)}\left(\frac{x}{2}\right)^{2m}\right).$$

c. If 2ν is not an integer, show that a second solution is

$$y_2(x) = x^{-\nu}\left(1 - \frac{1}{1!(1-\nu)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(1-\nu)(2-\nu)}\left(\frac{x}{2}\right)^4 + \sum_{m=3}^{\infty}\frac{(-1)^m}{m!(1-\nu)\cdots(m-\nu)}\left(\frac{x}{2}\right)^{2m}\right).$$

Note that $y_1(x) \to 0$ as $x \to 0$, and that $y_2(x)$ is unbounded as $x \to 0$.

d. Verify by direct methods that the power series in the expressions for $y_1(x)$ and $y_2(x)$ converge absolutely for all x. Also verify that y_2 is a solution, provided only that ν is not an integer.

9. In this section we showed that one solution of Bessel's equation of order zero

$$L[y] = x^2y'' + xy' + x^2y = 0$$

is J_0, where $J_0(x)$ is given by equation (7) with $a_0 = 1$. According to Theorem 5.6.1, a second solution has the form $(x > 0)$

$$y_2(x) = J_0(x)\ln x + \sum_{n=1}^{\infty} b_n x^n.$$

a. Show that

$$L[y_2](x) = \sum_{n=2}^{\infty} n(n-1)b_n x^n + \sum_{n=1}^{\infty} nb_n x^n + \sum_{n=1}^{\infty} b_n x^{n+2} + 2xJ_0'(x). \tag{34}$$

b. Substituting the series representation for $J_0(x)$ in equation (34), show that

$$b_1 x + 2^2 b_2 x^2 + \sum_{n=3}^{\infty}(n^2 b_n + b_{n-2})x^n = -2\sum_{n=1}^{\infty}\frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2}. \tag{35}$$

c. Note that only even powers of x appear on the right-hand side of equation (35). Show that $b_1 = b_3 = b_5 = \cdots = 0$, $b_2 = \frac{1}{2^2(1!)^2}$, and that

$$(2n)^2 b_{2n} + b_{2n-2} = -2\frac{(-1)^n(2n)}{2^{2n}(n!)^2}, \quad n = 2, 3, 4, \ldots .$$

Deduce that

$$b_4 = -\frac{1}{2^2 4^2}\left(1 + \frac{1}{2}\right) \quad \text{and} \quad b_6 = \frac{1}{2^2 4^2 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right).$$

The general solution of the recurrence relation is $b_{2n} = \frac{(-1)^{n+1}H_n}{2^{2n}(n!)^2}$. Substituting for b_n in the expression for $y_2(x)$, we obtain the solution given in equation (10).

10. Find a second solution of Bessel's equation of order one by computing the $c_n(r_2)$ and a of equation (24) of Section 5.6 according to the formulas (19) and (20) of that section. Some guidelines along the way of this calculation are the following. First, use equation (24) of this section to show that $a_1(-1)$ and $a_1'(-1)$ are 0. Then show that $c_1(-1) = 0$ and, from the recurrence relation, that $c_n(-1) = 0$ for $n = 3, 5, \ldots$. Finally, use equation (25) to show that

$$a_2(r) = -\frac{a_0}{(r+1)(r+3)},$$

$$a_4(r) = \frac{a_0}{(r+1)(r+3)(r+3)(r+5)},$$

and that

$$a_{2m}(r) = \frac{(-1)^m a_0}{(r+1)\cdots(r+2m-1)(r+3)\cdots(r+2m+1)}, \ m \ge 3.$$

Then show that

$$c_{2m}(-1) = \frac{(-1)^{m+1}(H_m + H_{m-1})}{2^{2m}m!(m-1)!}, \quad m \ge 1.$$

11. By a suitable change of variables it is sometimes possible to transform another differential equation into a Bessel equation. For example, show that a solution of

$$x^2y'' + \left(\alpha^2\beta^2x^{2\beta} + \frac{1}{4} - \nu^2\beta^2\right)y = 0, \quad x > 0$$

is given by $y = x^{1/2}f\left(\alpha x^{\beta}\right)$, where $f(\xi)$ is a solution of the Bessel equation of order ν.

12. Using the result of Problem 11, show that the general solution of the Airy equation

$$y'' - xy = 0, \quad x > 0$$

is $y = x^{1/2}\left(c_1 f_1\left(\frac{2}{3}ix^{3/2}\right) + c_2 f_2\left(\frac{2}{3}ix^{3/2}\right)\right)$, where $f_1(\xi)$ and $f_2(\xi)$ are a fundamental set of solutions of the Bessel equation of order one-third.

13. It can be shown that J_0 has infinitely many zeros for $x > 0$. In particular, the first three zeros are approximately 2.405, 5.520, and 8.653 (see Figure 5.7.1). Let λ_j, $j = 1, 2, 3, \ldots$, denote the zeros of J_0; it follows that

$$J_0(\lambda_j x) = \begin{cases} 1, & x = 0, \\ 0, & x = 1. \end{cases}$$

Verify that $y = J_0(\lambda_j x)$ satisfies the differential equation

$$y'' + \frac{1}{x}y' + \lambda_j^2 y = 0, \quad x > 0.$$

Hence show that

$$\int_0^1 xJ_0(\lambda_i x)J_0(\lambda_j x)\,dx = 0 \quad \text{if } \lambda_i \neq \lambda_j.$$

This important property of $J_0(\lambda_i x)$, which is known as the **orthogonality property**, is useful in solving boundary value problems.

Hint: Write the differential equation for $J_0(\lambda_i x)$. Multiply it by $xJ_0(\lambda_j x)$ and subtract that result from $xJ_0(\lambda_i x)$ times the differential equation for $J_0(\lambda_j x)$. Then integrate from 0 to 1.

References

Coddington, E. A., *An Introduction to Ordinary Differential Equations* (Englewood Cliffs, NJ: Prentice-Hall, 1961; New York: Dover, 1989).

Coddington, E. A., and Carlson, R., *Linear Ordinary Differential Equations* (Philadelphia, PA: Society for Industrial and Applied Mathematics, 1997).

Copson, E. T., *An Introduction to the Theory of Functions of a Complex Variable* (Oxford: Oxford University Press, 1935).

K. Knopp, *Theory and Applications of Infinite Series* (NewYork: Hafner, 1951).

Proofs of Theorems 5.3.1 and 5.6.1 can be found in intermediate or advanced books; for example, see Chapters 3 and 4 of Coddington, Chapters 5 and 6 of Coddington and Carlson, or Chapters 3 and 4 of

Rainville, E. D., *Intermediate Differential Equations* (2nd ed.) (New York: Macmillan, 1964).

Also see these texts for a discussion of the point at infinity, which was mentioned in Problem 32 of Section 5.4. The behavior of solutions near an irregular singular point is an even more advanced topic; a brief discussion can be found in Chapter 5 of

Coddington, E. A., and Levinson, N., *Theory of Ordinary Differential Equations* (New York: McGraw-Hill, 1955; Malabar, FL: Krieger, 1984).

Fuller discussions of the Bessel equation, the Legendre equation, and many of the other named equations can be found in advanced books on differential equations, methods of applied mathematics, and special functions. One text dealing with special functions such as the Legendre polynomials and the Bessel functions is

Hochstadt, H., *Special Functions of Mathematical Physics* (New York: Holt, 1961).

An excellent compilation of formulas, graphs, and tables of Bessel functions, Legendre functions, and other special functions of mathematical physics may be found in

Abramowitz, M., and Stegun, I. A. (eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (New York: Dover, 1965); originally published by the National Bureau of Standards, Washington, DC, 1964.

The digital successor to Abramowitz and Stegun is

Digital Library of Mathematical Functions. Released August 29, 2011. National Institute of Standards and Technology from http://dlmf.nist.gov/.