Elementary Differential Equations and Boundary Value Problems

Eleventh Edition

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Contents

Second-Order Linear Differential Equations

Linear differential equations of second order are of crucial importance in the study of differential equations for two main reasons. The first is that linear equations have a rich theoretical structure that underlies a number of systematic methods of solution. Further, a substantial portion of this structure and of these methods is understandable at a fairly elementary mathematical level. In order to present the key ideas in the simplest possible context, we describe them in this chapter for second-order equations. The second reason to study second-order linear differential equations is that they are vital to any serious investigation of the classical areas of mathematical physics. One cannot go very far in the development of fluid mechanics, heat conduction, wave motion, or electromagnetic phenomena without finding it necessary to solve second-order linear differential equations. We illustrate this at the end of this chapter with a discussion of the oscillations of some basic mechanical and electrical systems.

3.1 Homogeneous Differential Equations with Constant Coefficients

Many second-order ordinary differential equations have the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right), \tag{1}$$

where f is some given function. Usually, we will denote the independent variable by t since time is often the independent variable in physical problems, but sometimes we will use x instead. We will use y, or occasionally some other letter, to designate the dependent variable. Equation (1) is said to be **linear** if the function f has the form

$$f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)y, \tag{2}$$

that is, if f is linear in y and dy/dt. In equation (2) g, p, and q are specified functions of the independent variable t but do not depend on y. In this case we usually rewrite equation (1) as

$$y'' + p(t)y' + q(t)y = g(t), \tag{3}$$

where the primes denote differentiation with respect to t. Instead of equation (3), we sometimes see the equation

$$P(t)y'' + Q(t)y' + R(t)y = G(t). \tag{4}$$

Of course, if $P(t) \neq 0$, we can divide equation (4) by $P(t)$ and thereby obtain equation (3) with

$$p(t) = \frac{Q(t)}{P(t)}, \quad q(t) = \frac{R(t)}{P(t)}, \quad g(t) = \frac{G(t)}{P(t)}. \tag{5}$$

In discussing equation (3) and in trying to solve it, we will restrict ourselves to intervals in which p, q, and g are continuous functions.[1]

If equation (1) is not of the form (3) or (4), then it is called **nonlinear**. Analytical investigations of nonlinear equations are relatively difficult, so we will have little to say about them in this book. Numerical or geometical approaches are often more appropriate, and these are discussed in Chapters 8 and 9.

An initial value problem consists of a differential equation such as equations (1), (3), or (4) together with a pair of initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \tag{6}$$

where y_0 and y_0' are given numbers prescribing values for y and y' at the initial point t_0. Observe that the initial conditions for a second-order differential equation identify not only a particular point (t_0, y_0) through which the graph of the solution must pass, but also the slope y_0' of the graph at that point. It is reasonable to expect that two initial conditions are needed for a second-order differential equation because, roughly speaking, two integrations are required to find a solution and each integration introduces an arbitrary constant. Presumably, two initial conditions will suffice to determine values for these two constants.

A second-order linear differential equation is said to be **homogeneous** if the term $g(t)$ in equation (3), or the term $G(t)$ in equation (4), is zero for all t. Otherwise, the equation is called **nonhomogeneous**. Alternatively, the nonhomogeneous term $g(t)$, or $G(t)$, is sometimes called the forcing function because in many applications it describes an externally applied force. We begin our discussion with homogeneous equations, which we will write in the form

$$P(t)y'' + Q(t)y' + R(t)y = 0. \tag{7}$$

Later, in Sections 3.5 and 3.6, we will show that once the homogeneous equation has been solved, it is always possible to solve the corresponding nonhomogeneous equation (4), or at least to express the solution in terms of an integral. Thus the problem of solving the homogeneous equation is the more fundamental one.

In this chapter we will concentrate our attention on equations in which the functions P, Q, and R are constants. In this case, equation (7) becomes

$$ay'' + by' + cy = 0, \tag{8}$$

where a, b, and c are given constants. It turns out that equation (8) can always be solved easily in terms of the elementary functions of calculus. On the other hand, it is usually much more difficult to solve equation (7) if the coefficients are not constants, and a treatment of that case is deferred until Chapter 5. Before taking up equation (8), let us first gain some experience by looking at a simple example that in many ways is typical.

EXAMPLE 1

Solve the equation

$$y'' - y = 0. \tag{9}$$

Also find the solution that satisfies the initial conditions

$$y(0) = 2, \quad y'(0) = -1. \tag{10}$$

Solution:

Observe that equation (9) is just equation (8) with $a = 1$, $b = 0$, and $c = -1$. In words, equation (9) says that we seek a function with the property that the second derivative of the function is the same as the function itself. Do any of the functions that you studied in calculus have this property? A little thought will probably produce at least one such function, namely, $y_1(t) = e^t$, the exponential function. A little more thought may also produce a second function, $y_2(t) = e^{-t}$. Some further experimentation reveals that constant multiples of these two solutions are also solutions.

[1]There is a corresponding treatment of higher-order linear equations in Chapter 4. If you wish, you may read the appropriate parts of Chapter 4 in parallel with Chapter 3.

For example, the functions $2e^t$ and $5e^{-t}$ also satisfy equation (9), as you can verify by calculating their second derivatives. In the same way, the functions $c_1 y_1(t) = c_1 e^t$ and $c_2 y_2(t) = c_2 e^{-t}$ satisfy the differential equation (9) for all values of the constants c_1 and c_2.

Next, it is vital to notice that the sum of any two solutions of equation (9) is also a solution. In particular, since $c_1 y_1(t)$ and $c_2 y_2(t)$ are solutions of equation (9) for any values of c_1 and c_2, so is the function

$$y = c_1 y_1(t) + c_2 y_2(t) = c_1 e^t + c_2 e^{-t}. \tag{11}$$

Again, this can be verified by calculating the second derivative y'' from equation (11). We have $y' = c_1 e^t - c_2 e^{-t}$ and $y'' = c_1 e^t + c_2 e^{-t}$; thus y'' is the same as y, and equation (9) is satisfied.

Let us summarize what we have done so far in this example. Once we notice that the functions $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ are solutions of equation (9), it follows that the general linear combination (11) of these functions is also a solution. Since the coefficients c_1 and c_2 in equation (11) are arbitrary, this expression represents an infinite two-parameter family of solutions of the differential equation (9).

We now turn to the task of picking out a particular member of this infinite family of solutions that also satisfies the given pair of initial conditions (10). In other words, we seek the solution that passes through the point (0, 2) and at that point has the slope -1. First, to ensure the solution passes through the point $(0, 2)$, we set $t = 0$ and $y = 2$ in equation (11); this gives the equation

$$c_1 + c_2 = 2. \tag{12}$$

Next, we differentiate equation (11) with the result that

$$y' = c_1 e^t - c_2 e^{-t}. \tag{13}$$

Then, to enforce the condition that the slope at $(0, 2)$ is -1, we set $t = 0$ and $y' = -1$ in equation (13); this yields the equation

$$c_1 - c_2 = -1. \tag{14}$$

By solving equations (12) and (14) simultaneously for c_1 and c_2, we find that

$$c_1 = \frac{1}{2}, \quad c_2 = \frac{3}{2}. \tag{15}$$

Finally, inserting these values in equation (11), we obtain

$$y = \frac{1}{2}e^t + \frac{3}{2}e^{-t}, \tag{16}$$

the solution of the initial value problem consisting of the differential equation (9) and the initial conditions (10).

What conclusions can we draw from the preceding example that will help us to deal with the more general equation (8),

$$ay'' + by' + cy = 0,$$

whose coefficients a, b, and c are arbitrary (real) constants? In the first place, in the example the solutions were exponential functions. Further, once we had identified two solutions, we were able to use a linear combination of them to satisfy the given initial conditions as well as the differential equation itself.

It turns out that by exploiting these two ideas, we can solve equation (8) for any values of its coefficients and also satisfy any given set of initial conditions for y and y'.

We start by seeking exponential solutions of the form $y = e^{rt}$, where r is a parameter to be determined. Then it follows that $y' = re^{rt}$ and $y'' = r^2 e^{rt}$. By substituting these expressions for y, y', and y'' in equation (8), we obtain

$$(ar^2 + br + c)e^{rt} = 0.$$

Since $e^{rt} \neq 0$, this condition is satisfied only when the other factor is zero:

$$ar^2 + br + c = 0. \tag{17}$$

Equation (17) is called the **characteristic equation** for the differential equation (8). Its significance lies in the fact that if r is a root of the polynomial equation (17), then $y = e^{rt}$ is a solution of the differential equation (8). Since equation (17) is a quadratic equation with real coefficients, it has two roots, which may be real and different, complex conjugates, or real but repeated. We consider the first case here and the latter two cases in Sections 3.3 and 3.4, respectively.

Assuming that the roots of the characteristic equation (17) are real and different, let them be denoted by r_1 and r_2, where $r_1 \neq r_2$. Then $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are two solutions of equation (8). Just as in Example 1, it now follows that

$$y = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{18}$$

is also a solution of equation (8). To verify that this is so, we can differentiate the expression in equation (18); hence

$$y' = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t} \tag{19}$$

and

$$y'' = c_1 r_1^2 e^{r_1 t} + c_2 r_2^2 e^{r_2 t}. \tag{20}$$

Substituting these expressions for y, y', and y'' in equation (8) and rearranging terms, we obtain

$$ay'' + by' + cy = c_1\left(ar_1^2 + br_1 + c\right)e^{r_1 t} + c_2\left(ar_2^2 + br_2 + c\right)e^{r_2 t}. \tag{21}$$

The fact that r_1 is a root of equation (17) means that $ar_1^2 + br_1 + c = 0$. Since r_2 is also a root of the characteristic equation (17), it follows that $ar_2^2 + br_2 + c = 0$ as well. This completes the verification that y as given by equation (18) is indeed a solution of equation (8).

Now suppose that we want to find the particular member of the family of solutions (18) that satisfies the initial conditions (6)

$$y(t_0) = y_0, \quad y'(t_0) = y'_0.$$

By substituting $t = t_0$ and $y = y_0$ in equation (18), we obtain

$$c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y_0. \tag{22}$$

Similarly, setting $t = t_0$ and $y' = y'_0$ in equation (19) gives

$$c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} = y'_0. \tag{23}$$

On solving equations (22) and (23) simultaneously for c_1 and c_2, we find that

$$c_1 = \frac{y'_0 - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}, \quad c_2 = \frac{y_0 r_1 - y'_0}{r_1 - r_2} e^{-r_2 t_0}. \tag{24}$$

Since the roots of the characteristic equation (17) are assumed to be different, $r_1 - r_2 \neq 0$ so that the expressions in equation (24) always make sense. Thus, no matter what initial conditions are assigned—that is, regardless of the values of t_0, y_0, and y'_0 in equations (6)—it is always possible to determine c_1 and c_2 so that the initial conditions are satisfied. Moreover, there is only one possible choice of c_1 and c_2 for each set of initial conditions. With the values of c_1 and c_2 given by equation (24), the expression (18) is the solution of the initial value problem

$$ay'' + by' + cy = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y'_0. \tag{25}$$

It is possible to show, on the basis of the fundamental theorem cited in the next section, that all solutions of equation (8) are included in the expression (18), at least for the case in which the roots of equation (17) are real and different. Therefore, we call equation (18) the **general solution** of equation (8). The fact that any possible initial conditions can be satisfied by the proper choice of the constants in equation (18) makes more plausible the idea that this expression does include all solutions of equation (8).

Let us now look at some further examples.

EXAMPLE 2

Find the general solution of

$$y'' + 5y' + 6y = 0. \tag{26}$$

Solution:

We assume that $y = e^{rt}$, and it then follows that r must be a root of the characteristic equation

$$r^2 + 5r + 6 = (r + 2)(r + 3) = 0.$$

Thus the possible values of r are $r_1 = -2$ and $r_2 = -3$; the general solution of equation (26) is

$$y = c_1 e^{-2t} + c_2 e^{-3t}. \tag{27}$$

EXAMPLE 3

Find the solution of the initial value problem

$$y'' + 5y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = 3. \tag{28}$$

Solution:

The general solution of the differential equation was found in Example 2 and is given by equation (27). To satisfy the first initial condition, we set $t = 0$ and $y = 2$ in equation (27); thus c_1 and c_2 must satisfy

$$c_1 + c_2 = 2. \tag{29}$$

To use the second initial condition, we must first differentiate equation (27). This gives $y' = -2c_1 e^{-2t} - 3c_2 e^{-3t}$. Then, setting $t = 0$ and $y' = 3$, we obtain

$$-2c_1 - 3c_2 = 3. \tag{30}$$

By solving equations (29) and (30), we find that $c_1 = 9$ and $c_2 = -7$. Using these values in the expression (27), we obtain the solution

$$y = 9e^{-2t} - 7e^{-3t} \tag{31}$$

of the initial value problem (28). The graph of the solution is shown in Figure 3.1.1.

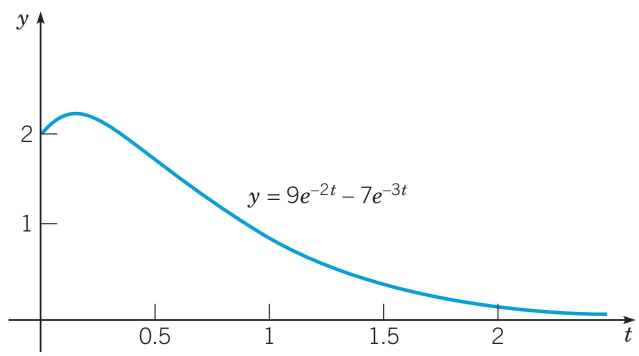

FIGURE 3.1.1 Solution of the initial value problem (28): $y'' + 5y' + 6y = 0,\ y(0) = 2,\ y'(0) = 3.$

EXAMPLE 4

Find the solution of the initial value problem

$$4y'' - 8y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}. \tag{32}$$

Solution:

If $y = e^{rt}$, then we obtain the characteristic equation

$$4r^2 - 8r + 3 = 0$$

whose roots are $r = \frac{3}{2}$ and $r = \frac{1}{2}$. Therefore, the general solution of the differential equation is

$$y = c_1 e^{3t/2} + c_2 e^{t/2}. \tag{33}$$

Applying the initial conditions, we obtain the following two equations for c_1 and c_2:

$$c_1 + c_2 = 2, \quad \frac{3}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2}.$$

The solution of these equations is $c_1 = -\frac{1}{2}$, $c_2 = \frac{5}{2}$, so the solution of the initial value problem (32) is

$$y = -\frac{1}{2}e^{3t/2} + \frac{5}{2}e^{t/2}. \tag{34}$$

Figure 3.1.2 shows the graph of the solution.

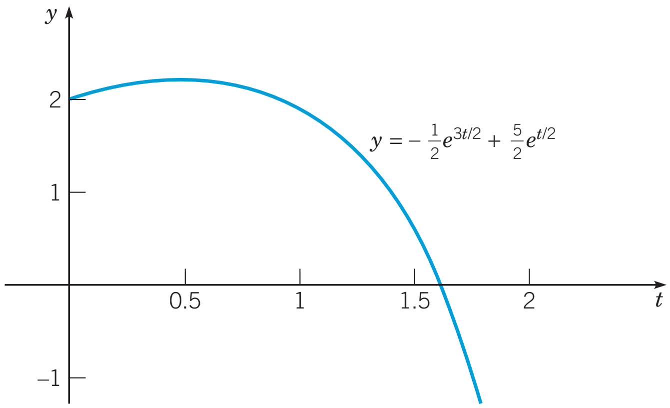

FIGURE 3.1.2 Solution of the initial value problem (32): $4y'' - 8y' + 3y = 0$, $y(0) = 2$, $y'(0) = 1/2$.

EXAMPLE 5

The solution (31) of the initial value problem (28) initially increases (because its initial slope is positive), but eventually approaches zero (because both terms involve negative exponential functions). Therefore, the solution must have a maximum point, and the graph in Figure 3.1.1 confirms this. Determine the location of this maximum point.

Solution:

The coordinates of the maximum point can be estimated from the graph, but to find them more precisely, we seek the point where the solution has a horizontal tangent line. By differentiating the solution (31), $y = 9e^{-2t} - 7e^{-3t}$, with respect to t, we obtain

$$y' = -18e^{-2t} + 21e^{-3t}. \tag{35}$$

Setting y' equal to zero and multiplying by e^{3t}, we find that the critical value t_m satisfies $e^t = 7/6$; hence

$$t_m = \ln(7/6) \cong 0.15415. \tag{36}$$

The corresponding maximum value y_m is given by

$$y_m = 9e^{-2t_m} - 7e^{-3t_m} = \frac{108}{49} \cong 2.20408. \tag{37}$$

In this example the initial slope is 3, but the solution of the given differential equation behaves in a similar way for any other positive initial slope. In Problem 19 you are asked to determine how the coordinates of the maximum point depend on the initial slope.

Returning to the equation $ay'' + by' + cy = 0$ with arbitrary coefficients, recall that when $r_1 \neq r_2$, its general solution (18) is the sum of two exponential functions. Therefore, the solution has a relatively simple geometrical behavior: as t increases, the magnitude of the solution either tends to zero (when both exponents are negative) or else exhibits unbounded growth (when at least one exponent is positive). These two cases are illustrated by the solutions of Examples 3 and 4, which are shown in Figures 3.1.1 and 3.1.2, respectively. Note that whether a growing solution approaches $+\infty$ or $-\infty$ as $t \to \infty$ is determined by the sign of the coefficient of the exponential for the larger root of the characteristic equation. (See Problem 21.) There is also a third case that occurs less often: the solution approaches a constant when one exponent is zero and the other is negative.

In Sections 3.3 and 3.4, respectively, we return to the problem of solving the equation $ay''+by'+cy = 0$ when the roots of the characteristic equation either are complex conjugates or are real and equal. In the meantime, in Section 3.2, we provide a systematic discussion of the mathematical structure of the solutions of all second-order linear homogeneous equations.

Problems

In each of Problems 1 through 6, find the general solution of the given differential equation.

1. $y'' + 2y' - 3y = 0$

2. $y'' + 3y' + 2y = 0$

3. $6y'' - y' - y = 0$

4. $y'' + 5y' = 0$

5. $4y'' - 9y = 0$

6. $y'' - 2y' - 2y = 0$

In each of Problems 7 through 12, find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as t increases.

G **7.** $y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$

G **8.** $y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1$

G **9.** $y'' + 3y' = 0, \quad y(0) = -2, \quad y'(0) = 3$

G **10.** $2y'' + y' - 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$

G **11.** $y'' + 8y' - 9y = 0, \quad y(1) = 1, \quad y'(1) = 0$

G **12.** $4y'' - y = 0, \quad y(-2) = 1, \quad y'(-2) = -1$

13. Find a differential equation whose general solution is $y = c_1e^{2t} + c_2e^{-3t}$.

G **14.** Find the solution of the initial value problem

$$y'' - y = 0, \quad y(0) = \frac{5}{4}, \quad y'(0) = -\frac{3}{4}.$$

Plot the solution for $0 \le t \le 2$ and determine its minimum value.

15. Find the solution of the initial value problem

$$2y'' - 3y' + y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}.$$

Then determine the maximum value of the solution and also find the point where the solution is zero.

16. Solve the initial value problem $y'' - y' - 2y = 0$, $y(0) = \alpha$, $y'(0) = 2$. Then find α so that the solution approaches zero as $t \to \infty$.

In each of Problems 17 and 18, determine the values of α, if any, for which all solutions tend to zero as $t \to \infty$; also determine the values of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.

17. $y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$

18. $y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0$

19. Consider the initial value problem (see Example 5)

$$y'' + 5y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = \beta,$$

where $\beta > 0$.

a. Solve the initial value problem.
b. Determine the coordinates t_m and y_m of the maximum point of the solution as functions of β.
c. Determine the smallest value of β for which $y_m \ge 4$.
d. Determine the behavior of t_m and y_m as $\beta \to \infty$.

20. Consider the equation $ay'' + by' + cy = d$, where a, b, c, and d are constants.

a. Find all equilibrium, or constant, solutions of this differential equation.
b. Let y_e denote an equilibrium solution, and let $Y = y - y_e$. Thus Y is the deviation of a solution y from an equilibrium solution. Find the differential equation satisfied by Y.

21. Consider the equation $ay'' + by' + cy = 0$, where a, b, and c are constants with $a > 0$. Find conditions on a, b, and c such that the roots of the characteristic equation are:

a. real, different, and negative.
b. real with opposite signs.
c. real, different, and positive.

In each case, determine the behavior of the solution as $t \to \infty$.

3.2 Solutions of Linear Homogeneous Equations; the Wronskian

In the preceding section we showed how to solve some differential equations of the form

$$ay'' + by' + cy = 0,$$

where a, b, and c are constants. Now we build on those results to provide a clearer picture of the structure of the solutions of all second-order linear homogeneous equations. In turn, this understanding will assist us in finding the solutions of other problems that we will encounter later.

To discuss general properties of linear differential equations, it is helpful to introduce a **differential operator** notation. Let p and q be continuous functions on an open interval I—that is, for $\alpha < t < \beta$. The cases for $\alpha = -\infty$, or $\beta = \infty$, or both, are included. Then, for any function ϕ that is twice differentiable on I, we define the differential operator L by the equation

$$L[\phi] = \phi'' + p\phi' + q\phi. \tag{1}$$

It is important to understand that the result of applying the operator L to a function ϕ is another function, which we refer to as $L[\phi]$. The value of $L[\phi]$ at a point t is

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t).$$

For example, if $p(t) = t^2$, $q(t) = 1 + t$, and $\phi(t) = \sin 3t$, then

$$\begin{aligned} L[\phi](t) &= (\sin 3t)'' + t^2(\sin 3t)' + (1+t)\sin 3t \\ &= -9\sin 3t + 3t^2\cos 3t + (1+t)\sin 3t. \end{aligned}$$

The operator L is often written as $L = D^2 + pD + q$, where D is the derivative operator, that is, $D[\phi] = \phi'$.

In this section we study the second-order linear homogeneous differential equation $L[\phi](t) = 0$. Since it is customary to use the symbol y to denote $\phi(t)$, we will usually write this equation in the form

$$L[y] = y'' + p(t)y' + q(t)y = 0. \tag{2}$$

With equation (2) we associate a set of initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \tag{3}$$

where t_0 is any point in the interval I, and y_0 and y_0' are given real numbers. We would like to know whether the initial value problem (2), (3) always has a solution, and whether it may have more than one solution. We would also like to know whether anything can be said about the form and structure of solutions that might be helpful in finding solutions of particular problems. Answers to these questions are contained in the theorems in this section.

The fundamental theoretical result for initial value problems for second-order linear equations is stated in Theorem 3.2.1, which is analogous to Theorem 2.4.1 for first-order linear equations. The result applies equally well to nonhomogeneous equations, so the theorem is stated in that form.

Theorem 3.2.1 | (Existence and Uniqueness Theorem)

Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0', \tag{4}$$

where p, q, and g are continuous on an open interval I that contains the point t_0. This problem has exactly one solution $y = \phi(t)$, and the solution exists throughout the interval I.

We emphasize that the theorem says three things:

1. The initial value problem *has* a solution; in other words, a solution *exists*.
2. The initial value problem has *only one* solution; that is, the solution is *unique*.
3. The solution ϕ is defined *throughout the interval* I where the coefficients are continuous and is at least twice differentiable there.

For some problems some of these assertions are easy to prove. For instance, we found in Example 1 of Section 3.1 that the initial value problem

$$y'' - y = 0, \quad y(0) = 2, \quad y'(0) = -1 \tag{5}$$

has the solution

$$y = \frac{1}{2}e^t + \frac{3}{2}e^{-t}. \tag{6}$$

The fact that we found a solution certainly establishes that a solution exists for this initial value problem. Further, the solution (6) is twice differentiable, indeed differentiable any number of times, throughout the interval $(-\infty, \infty)$ where the coefficients in the differential equation are continuous. On the other hand, it is not obvious, and is more difficult to show, that the initial value problem (5) has no solutions other than the one given by equation (6). Nevertheless, Theorem 3.2.1 states that this solution is indeed the only solution of the initial value problem (5).

For most problems of the form (4), it is not possible to write down a useful expression for the solution. This is a major difference between first-order and second-order linear differential equations. Therefore, all parts of the theorem must be proved by general methods that do not involve having such an expression. The proof of Theorem 3.2.1 is fairly difficult, and we do not discuss it here.[2] We will, however, accept Theorem 3.2.1 as true and make use of it whenever necessary.

EXAMPLE 1

Find the longest interval in which the solution of the initial value problem

$$(t^2 - 3t)y'' + ty' - (t + 3)y = 0, \quad y(1) = 2, \quad y'(1) = 1$$

is certain to exist.

Solution:

If the given differential equation is written in the form of equation (4), then

$$p(t) = \frac{1}{t-3}, \quad q(t) = -\frac{t+3}{t(t-3)}, \quad \text{and} \quad g(t) = 0.$$

The only points of discontinuity of the coefficients are $t = 0$ and $t = 3$. Therefore, the longest open interval, containing the initial point $t = 1$, in which all the coefficients are continuous is $0 < t < 3$. Thus this is the longest interval in which Theorem 3.2.1 guarantees that the solution exists.

EXAMPLE 2

Find the unique solution of the initial value problem

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = 0, \quad y'(t_0) = 0,$$

where p and q are continuous in an open interval I containing t_0.

Solution:

The function $y = \phi(t) = 0$ for all t in I certainly satisfies the differential equation and initial conditions. By the uniqueness part of Theorem 3.2.1, it is the only solution of the given problem.

[2] A proof of Theorem 3.2.1 can be found, for example, in Chapter 6, Section 8 of the book by Coddington listed in the references at the end of this chapter.

Let us now assume that y_1 and y_2 are two solutions of equation (2); in other words,

$$L[y_1] = y_1'' + py_1' + qy_1 = 0,$$

and similarly for y_2. Then, just as in the examples in Section 3.1, we can generate more solutions by forming linear combinations of y_1 and y_2. We state this result as a theorem.

Theorem 3.2.2 | (Principle of Superposition)

If y_1 and y_2 are two solutions of the differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 and c_2.

A special case of Theorem 3.2.2 occurs if either c_1 or c_2 is zero. Then we conclude that any constant multiple of a solution of equation (2) is also a solution.

To prove Theorem 3.2.2, we need only substitute

$$y = c_1y_1(t) + c_2y_2(t) \tag{7}$$

for y in equation (2). By calculating the indicated derivatives and rearranging terms, we obtain

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= [c_1y_1 + c_2y_2]'' + p[c_1y_1 + c_2y_2]' + q[c_1y_1 + c_2y_2] \\ &= c_1y_1'' + c_2y_2'' + c_1py_1' + c_2py_2' + c_1qy_1 + c_2qy_2 \\ &= c_1[y_1'' + py_1' + qy_1] + c_2[y_2'' + py_2' + qy_2] \\ &= c_1L[y_1] + c_2L[y_2]. \end{aligned}$$

Since $L[y_1] = 0$ and $L[y_2] = 0$, it follows that $L[c_1y_1 + c_2y_2] = 0$ also. Therefore, regardless of the values of c_1 and c_2, the function y as given by equation (7) satisfies the differential equation (2), and the proof of Theorem 3.2.2 is complete.

Theorem 3.2.2 states that, beginning with only two solutions of equation (2), we can construct an infinite family of solutions by means of equation (7). The next question is whether all solutions of equation (2) are included in equation (7) or whether there may be other solutions of a different form. We begin to address this question by examining whether the constants c_1 and c_2 in equation (7) can be chosen so as to satisfy the initial conditions (3). These initial conditions require c_1 and c_2 to satisfy the equations

$$\begin{aligned} c_1y_1(t_0) + c_2y_2(t_0) &= y_0, \\ c_1y_1'(t_0) + c_2y_2'(t_0) &= y_0'. \end{aligned} \tag{8}$$

The determinant of coefficients of the system (8) is

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0). \tag{9}$$

If $W \neq 0$, then equations (8) have a unique solution (c_1, c_2) regardless of the values of y_0 and y_0'. This solution is given by

$$c_1 = \frac{y_0y_2'(t_0) - y_0'y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}, \quad c_2 = \frac{-y_0y_1'(t_0) + y_0'y_1(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}, \tag{10}$$

or, in terms of determinants,

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}. \tag{11}$$

With these values for c_1 and c_2, the linear combination $y = c_1y_1(t) + c_2y_2(t)$ satisfies the initial conditions (3) as well as the differential equation (2). Note that the denominator in the expressions for c_1 and c_2 is the nonzero determinant W.

On the other hand, if $W = 0$, then the denominators appearing in equations (10) and (11) are zero. In this case equations (8) have no solution unless y_0 and y_0' have values that also make the numerators in equations (10) and (11) equal to zero. Thus, if $W = 0$, there are many initial conditions that cannot be satisfied no matter how c_1 and c_2 are chosen.

The determinant W is called the **Wronskian**[3] **determinant**, or simply the **Wronskian**, of the solutions y_1 and y_2. Sometimes we use the more extended notation $W[y_1, y_2](t_0)$ to stand for the expression on the right-hand side of equation (9), thereby emphasizing that the Wronskian depends on the functions y_1 and y_2, and that it is evaluated at the point t_0. The preceding argument establishes the following result.

Theorem 3.2.3

Suppose that y_1 and y_2 are two solutions of equation (2)

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and that the initial conditions (3)

$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$

are assigned. Then it is always possible to choose the constants c_1, c_2 so that

$$y = c_1 y_1(t) + c_2 y_2(t)$$

satisfies the differential equation (2) and the initial conditions (3) if and only if the Wronskian

$$W[y_1, y_2] = y_1 y_2' - y_1' y_2$$

is not zero at t_0.

EXAMPLE 3

In Example 2 of Section 3.1 we found that $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-3t}$ are solutions of the differential equation

$$y'' + 5y' + 6y = 0.$$

Find the Wronskian of y_1 and y_2.

Solution:

The Wronskian of these two functions is

$$W\left[e^{-2t}, e^{-3t}\right] = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}.$$

Since W is nonzero for all values of t, the functions y_1 and y_2 can be used to construct solutions of the given differential equation, together with initial conditions prescribed at any value of t. One such initial value problem was solved in Example 3 of Section 3.1.

The next theorem justifies the term "general solution" that we introduced in Section 3.1 for the linear combination $c_1 y_1 + c_2 y_2$.

[3] Wronskian determinants are named for Jósef Maria Hoëné-Wronski (1776–1853), who was born in Poland but spent most of his life in France. Wronski was a gifted but troubled man, and his life was marked by frequent heated disputes with other individuals and institutions.

Theorem 3.2.4

Suppose that y_1 and y_2 are two solutions of the second-order linear differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

Then the two-parameter family of solutions

$$y = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary coefficients c_1 and c_2 includes every solution of equation (2) if and only if there is a point t_0 where the Wronskian of y_1 and y_2 is not zero.

Let the function ϕ be any solution of equation (2). To prove the theorem, we must determine whether ϕ is included in the linear combinations $c_1 y_1 + c_2 y_2$. That is, we must determine whether there are values of the constants c_1 and c_2 that make the linear combination the same as ϕ. Let t_0 be a point where the Wronskian of y_1 and y_2 is nonzero. Then evaluate ϕ and ϕ' at this point and call these values y_0 and y_0', respectively; that is,

$$y_0 = \phi(t_0), \quad y_0' = \phi'(t_0).$$

Next, consider the initial value problem

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'. \tag{12}$$

The function ϕ is certainly a solution of this initial value problem. Further, because we are assuming that $W[y_1, y_2](t_0)$ is nonzero, it is possible (by Theorem 3.2.3) to choose c_1 and c_2 such that $y = c_1 y_1(t) + c_2 y_2(t)$ is also a solution of the initial value problem (9). In fact, the proper values of c_1 and c_2 are given by equations (10) or (11). The uniqueness part of Theorem 3.2.1 guarantees that these two solutions of the same initial value problem are actually the same function; thus, for the proper choice of c_1 and c_2,

$$\phi(t) = c_1 y_1(t) + c_2 y_2(t), \tag{13}$$

and therefore ϕ is included in the family of functions $c_1 y_1 + c_2 y_2$. Finally, since ϕ is an *arbitrary* solution of equation (2), it follows that *every* solution of this equation is included in this family.

Now suppose that there is no point t_0 where the Wronskian is nonzero. Thus $W[y_1, y_2](t_0) = 0$ for every point t_0. Then (by Theorem 3.2.3) there are values of y_0 and y_0' such that no values of c_1 and c_2 satisfy the system (8). Select a pair of such values for y_0 and y_0' and choose the solution $\phi(t)$ of equation (2) that satisfies the initial condition (3). Observe that this initial value problem is guaranteed to have a solution by Theorem 3.2.1. However, this solution is not included in the family $y = c_1 y_1 + c_2 y_2$. Thus, in cases where $W[y_1, y_2](t_0) = 0$ for every t_0, the linear combinations of y_1 and y_2 do not include all solutions of equation (2). This completes the proof of Theorem 3.2.4.

Theorem 3.2.4 states that the Wronskian of y_1 and y_2 is not everywhere zero if and only if the linear combination $c_1 y_1 + c_2 y_2$ contains all solutions of equation (2). It is therefore natural (and we have already done this in the preceding section) to call the expression

$$y = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary constant coefficients the **general solution** of equation (2). The solutions y_1 and y_2 are said to form a **fundamental set of solutions** of equation (2) if and only if their Wronskian is nonzero.

We can restate the result of Theorem 3.2.4 in slightly different language: to find the general solution, and therefore all solutions, of an equation of the form (2), we need only find two solutions of the given equation whose Wronskian is nonzero. We did precisely this in several examples in Section 3.1, although there we did not calculate the Wronskians. You should now go back and do that, thereby verifying that all the solutions we called "general solutions" in Section 3.1 do satisfy the necessary Wronskian condition.

Now that you have a little experience verifying the nonzero Wronskian condition for the examples from Section 3.1, the following example handles all second-order linear differential equations whose characteristic polynomial has two distinct real roots.

EXAMPLE 4

Suppose that $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are two solutions of an equation of the form (2). Show that if $r_1 \neq r_2$, then y_1 and y_2 form a fundamental set of solutions of equation (2).

Solution:

We calculate the Wronskian of y_1 and y_2:

$$W = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1)\exp[(r_1 + r_2)t].$$

Since the exponential function is never zero, and since we are assuming that $r_2 - r_1 \neq 0$, it follows that W is nonzero for every value of t. Consequently, y_1 and y_2 form a fundamental set of solutions of equation (2).

EXAMPLE 5

Show that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ form a fundamental set of solutions of

$$2t^2 y'' + 3ty' - y = 0, \qquad t > 0. \tag{14}$$

Solution:

We will show how to solve equation (14) later (see Problem 25 in Section 3.3). However, at this stage we can verify by direct substitution that y_1 and y_2 are solutions of the differential equation (14). Since $y_1'(t) = \frac{1}{2}t^{-1/2}$ and $y_1''(t) = -\frac{1}{4}t^{-3/2}$, we have

$$2t^2\left(-\frac{1}{4}t^{-3/2}\right) + 3t\left(\frac{1}{2}t^{-1/2}\right) - t^{1/2} = \left(-\frac{1}{2} + \frac{3}{2} - 1\right)t^{1/2} = 0.$$

Similarly, $y_2'(t) = -t^{-2}$ and $y_2''(t) = 2t^{-3}$, so

$$2t^2\left(2t^{-3}\right) + 3t\left(-t^{-2}\right) - t^{-1} = (4 - 3 - 1)t^{-1} = 0.$$

Next we calculate the Wronskian W of y_1 and y_2:

$$W = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -\frac{3}{2}t^{-3/2}. \tag{15}$$

Since $W \neq 0$ for $t > 0$, we conclude that y_1 and y_2 form a fundamental set of solutions there. Thus the general solution of differential equation (14) is $y(t) = c_1 t^{1/2} + c_2 t^{-1}$ for $t > 0$.

In several cases we have been able to find a fundamental set of solutions, and therefore the general solution, of a given differential equation. However, this is often a difficult task, and the question arises as to whether a differential equation of the form (2) always has a fundamental set of solutions. The following theorem provides an affirmative answer to this question.

Theorem 3.2.5

Consider the differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

whose coefficients p and q are continuous on some open interval I. Choose some point t_0 in I. Let y_1 be the solution of equation (2) that also satisfies the initial conditions

$$y(t_0) = 1, \quad y'(t_0) = 0,$$

and let y_2 be the solution of equation (2) that satisfies the initial conditions

$$y(t_0) = 0, \quad y'(t_0) = 1.$$

Then y_1 and y_2 form a fundamental set of solutions of equation (2).

First observe that the *existence* of the functions y_1 and y_2 is ensured by the existence part of Theorem 3.2.1. To show that they form a fundamental set of solutions, we need only calculate their Wronskian at t_0:

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Since their Wronskian is not zero at the point t_0, the functions y_1 and y_2 do form a fundamental set of solutions, thus completing the proof of Theorem 3.2.5.

Note that the potentially difficult part of this proof, demonstrating the existence of a pair of solutions, is taken care of by reference to Theorem 3.2.1. Note also that Theorem 3.2.5 does not address the question of how to find the solutions y_1 and y_2 by solving the specified initial value problems. Nevertheless, it may be reassuring to know that a fundamental set of solutions always exists.

EXAMPLE 6

Find the fundamental set of solutions y_1 and y_2 specified by Theorem 3.2.5 for the differential equation

$$y'' - y = 0, \tag{16}$$

using the initial point $t_0 = 0$.

Solution:

In Section 3.1 we noted that two solutions of equation (16) are $y_1(t) = e^t$ and $y_2(t) = e^{-t}$. The Wronskian of these solutions is $W[y_1, y_2](t) = -2 \neq 0$, so they form a fundamental set of solutions. However, they are not the fundamental solutions indicated by Theorem 3.2.5 because they do not satisfy the initial conditions mentioned in that theorem at the point $t = 0$.

To find the fundamental solutions specified by the theorem, we need to find the solutions satisfying the proper initial conditions. Let us denote by $y_3(t)$ the solution of equation (16) that satisfies the initial conditions

$$y(0) = 1, \quad y'(0) = 0. \tag{17}$$

The general solution of equation (16) is

$$y = c_1 e^t + c_2 e^{-t}, \tag{18}$$

and the initial conditions (17) are satisfied if $c_1 = 1/2$ and $c_2 = 1/2$. Thus

$$y_3(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh t.$$

Similarly, if $y_4(t)$ satisfies the initial conditions

$$y(0) = 0, \quad y'(0) = 1, \tag{19}$$

then

$$y_4(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh t.$$

Since the Wronskian of y_3 and y_4 is

$$W[y_3, y_4](t) = \cosh^2 t - \sinh^2 t = 1,$$

these functions also form a fundamental set of solutions, as stated by Theorem 3.2.5. Therefore, the general solution of equation (16) can be written as

$$y = k_1 \cosh t + k_2 \sinh t, \tag{20}$$

as well as in the form (18). We have used k_1 and k_2 for the arbitrary constants in equation (20) because they are not the same as the constants c_1 and c_2 in equation (18). One purpose of this example is to make it clear that a given differential equation has more than one fundamental set of solutions; indeed, it has infinitely many (see Problem 16). As a rule, you should choose the set that is most convenient.

In the next section we will encounter equations that have complex-valued solutions. The following theorem is fundamental in dealing with such equations and their solutions.

Theorem 3.2.6

Consider again the second-order linear differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous real-valued functions. If $y = u(t) + iv(t)$ is a complex-valued solution of differential equation (2), then its real part u and its imaginary part v are also solutions of this equation.

To prove this theorem, we substitute $u(t) + iv(t)$ for y in $L[y]$, obtaining

$$L[y](t) = u''(t) + iv''(t) + p(t)(u'(t) + iv'(t)) + q(t)(u(t) + iv(t)). \tag{21}$$

Then, by separating equation (21) into its real and imaginary parts—and this is where we need to know that $p(t)$ and $q(t)$ are real-valued—we find that

$$\begin{aligned} L[y](t) &= (u''(t) + p(t)u'(t) + q(t)u(t)) + i(v''(t) + p(t)v'(t) + q(t)v(t)) \\ &= L[u](t) + iL[v](t). \end{aligned}$$

Recall that a complex number is zero if and only if its real and imaginary parts are both zero. We know that $L[y] = 0$ because y is a solution of equation (2). Therefore, both $L[u] = 0$ and $L[v] = 0$; consequently, the two real-valued functions u and v are also solutions of equation (2), so the theorem is established. We will see examples of the use of Theorem 3.2.6 in Section 3.3.

Incidentally, the complex conjugate $\overline{y}$ of a solution y is also a solution. While this can be proved by an argument similar to the one just used to prove Theorem 3.2.6, it is also a consequence of Theorem 3.2.2 since $\overline{y} = u(t) - iv(t)$ is a linear combination of two solutions.

Now let us examine further the properties of the Wronskian of two solutions of a second-order linear homogeneous differential equation. The following theorem, perhaps surprisingly, gives a simple explicit formula for the Wronskian of any two solutions of any such equation, even if the solutions themselves are not known.

Theorem 3.2.7 | (Abel's Theorem)[4]

If y_1 and y_2 are solutions of the second-order linear differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \tag{22}$$

where p and q are continuous on an open interval I, then the Wronskian $W[y_1, y_2](t)$ is given by

$$W[y_1, y_2](t) = c\exp\left(-\int p(t)\,dt\right), \tag{23}$$

where c is a certain constant that depends on y_1 and y_2, but not on t. Further, $W[y_1, y_2](t)$ either is zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

To prove Abel's theorem, we start by noting that y_1 and y_2 satisfy

$$\begin{aligned} y_1'' + p(t)y_1' + q(t)y_1 &= 0, \\ y_2'' + p(t)y_2' + q(t)y_2 &= 0. \end{aligned} \tag{24}$$

If we multiply the first equation by $-y_2$, multiply the second by y_1, and add the resulting equations, we obtain

$$(y_1y_2'' - y_1''y_2) + p(t)(y_1y_2' - y_1'y_2) = 0. \tag{25}$$

[4]The result in Theorem 3.2.7 was derived by the Norwegian mathematician Niels Henrik Abel (1802–1829) in 1827 and is known as **Abel's formula**. Abel also showed that there is no general formula for solving a quintic, or fifth degree, polynomial equation in terms of explicit algebraic operations on the coefficients, thereby resolving a question that had remained unanswered since the sixteenth century. His greatest contributions, however, were in analysis, particularly in the study of elliptic functions. Unfortunately, his work was not widely noticed until after his death. The distinguished French mathematician Legendre called it a "monument more lasting than bronze."

Next, we let $W(t) = W[y_1, y_2](t)$ and observe that

$$W' = y_1 y_2'' - y_1'' y_2. \tag{26}$$

Then we can write equation (25) in the form

$$W' + p(t)W = 0. \tag{27}$$

Equation (27) can be solved immediately since it is both a first-order linear differential equation (Section 2.1) and a separable differential equation (Section 2.2). Thus

$$W(t) = c\exp\left(-\int p(t)\,dt\right), \tag{28}$$

where c is a constant.

The value of c depends on which pair of solutions of equation (22) is involved. However, since the exponential function is never zero, $W(t)$ is not zero unless $c = 0$, in which case $W(t)$ is zero for all t. This completes the proof of Theorem 3.2.7.

Note that the Wronskians of any two fundamental sets of solutions of the same differential equation can differ only by a multiplicative constant, and that the Wronskian of any fundamental set of solutions can be determined, up to a multiplicative constant, without solving the differential equation. Further, since under the conditions of Theorem 3.2.7 the Wronskian W is either always zero or never zero, you can determine which case actually occurs by evaluating W at any single convenient value of t.

EXAMPLE 7

In Example 5 we verified that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ are solutions of the equation

$$2t^2y'' + 3ty' - y = 0, \quad t > 0. \tag{29}$$

Verify that the Wronskian of y_1 and y_2 is given by Abel's formula (23).

Solution:

From the example just cited we know that $W[y_1, y_2](t) = -\frac{3}{2}t^{-3/2}$. To use equation (23), we must write the differential equation (29) in the standard form with the coefficient of y'' equal to 1. Thus we obtain

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0,$$

so $p(t) = \dfrac{3}{2t}$. Hence

$$\begin{aligned} W[y_1, y_2](t) &= c\,\exp\left(-\int \frac{3}{2t}\,dt\right) = c\exp\left(-\frac{3}{2}\ln t\right) \\ &= c\,t^{-3/2}. \end{aligned} \tag{30}$$

Equation (30) gives the Wronskian of any pair of solutions of equation (29). For the particular solutions given in this example, we must choose $c = -\frac{3}{2}$.

Summary. We can summarize the discussion in this section as follows: to find the general solution of the differential equation

$$y'' + p(t)y' + q(t)y = 0, \quad \alpha < t < \beta,$$

we must first find two functions y_1 and y_2 that satisfy the differential equation in $\alpha < t < \beta$. Then we must make sure that there is a point in the interval where the Wronskian W of y_1 and y_2 is nonzero. Under these circumstances y_1 and y_2 form a fundamental set of solutions, and the general solution is

$$y = c_1y_1(t) + c_2y_2(t),$$

where c_1 and c_2 are arbitrary constants. If initial conditions are prescribed at a given point in $\alpha < t < \beta$, then c_1 and c_2 can be chosen so as to satisfy these conditions.

Problems

In each of Problems 1 through 5, find the Wronskian of the given pair of functions.

1. $e^{2t}, \quad e^{-3t/2}$

2. $\cos t, \quad \sin t$

3. $e^{-2t}, \quad te^{-2t}$

4. $e^t \sin t, \quad e^t \cos t$

5. $\cos^2\theta, \quad 1+\cos(2\theta)$

In each of Problems 6 through 9, determine the longest interval in which the given initial value problem is certain to have a unique twice-differentiable solution. Do not attempt to find the solution.

6. $ty''+3y=t, \quad y(1)=1, \quad y'(1)=2$

7. $t(t-4)y''+3ty'+4y=2, \quad y(3)=0, \quad y'(3)=-1$

8. $y''+(\cos t)y'+3(\ln|t|)y=0, \quad y(2)=3, \quad y'(2)=1$

9. $(x-2)y''+y'+(x-2)(\tan x)y=0, \quad y(3)=1, \quad y'(3)=2$

10. Verify that $y_1(t)=t^2$ and $y_2(t)=t^{-1}$ are two solutions of the differential equation $t^2y''-2y=0$ for $t>0$. Then show that $y=c_1t^2+c_2t^{-1}$ is also a solution of this equation for any c_1 and c_2.

11. Verify that $y_1(t)=1$ and $y_2(t)=t^{1/2}$ are solutions of the differential equation $yy''+(y')^2=0$ for $t>0$. Then show that $y=c_1+c_2t^{1/2}$ is not, in general, a solution of this equation. Explain why this result does not contradict Theorem 3.2.2.

12. Show that if $y=\phi(t)$ is a solution of the differential equation $y''+p(t)y'+q(t)y=g(t)$, where $g(t)$ is not always zero, then $y=c\phi(t)$, where c is any constant other than 1, is not a solution. Explain why this result does not contradict the remark following Theorem 3.2.2.

13. Can $y=\sin(t^2)$ be a solution on an interval containing $t=0$ of an equation $y''+p(t)y'+q(t)y=0$ with continuous coefficients? Explain your answer.

14. If the Wronskian W of f and g is $3e^{4t}$, and if $f(t)=e^{2t}$, find $g(t)$.

15. If the Wronskian of f and g is $t\cos t-\sin t$, and if $u=f+3g$, $v=f-g$, find the Wronskian of u and v.

16. Assume that y_1 and y_2 are a fundamental set of solutions of $y''+p(t)y'+q(t)y=0$ and let $y_3=a_1y_1+a_2y_2$ and $y_4=b_1y_1+b_2y_2$, where a_1, a_2, b_1, and b_2 are any constants. Show that

$$W[y_3,y_4]=(a_1b_2-a_2b_1)W[y_1,y_2].$$

Are y_3 and y_4 also a fundamental set of solutions? Why or why not?

In each of Problems 17 and 18, find the fundamental set of solutions specified by Theorem 3.2.5 for the given differential equation and initial point.

17. $y''+y'-2y=0, \quad t_0=0$

18. $y''+4y'+3y=0, \quad t_0=1$

In each of Problems 19 through 21, verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions?

19. $y''+4y=0; \quad y_1(t)=\cos(2t), \quad y_2(t)=\sin(2t)$

20. $y''-2y'+y=0; \quad y_1(t)=e^t, \quad y_2(t)=te^t$

21. $x^2y''-x(x+2)y'+(x+2)y=0, \quad x>0;$
$y_1(x)=x, \quad y_2(x)=xe^x$

22. Consider the equation $y''-y'-2y=0$.

a. Show that $y_1(t)=e^{-t}$ and $y_2(t)=e^{2t}$ form a fundamental set of solutions.

b. Let $y_3(t)=-2e^{2t}$, $y_4(t)=y_1(t)+2y_2(t)$, and $y_5(t)=2y_1(t)-2y_3(t)$. Are $y_3(t)$, $y_4(t)$, and $y_5(t)$ also solutions of the given differential equation?

c. Determine whether each of the following pairs forms a fundamental set of solutions: $\{y_1(t),y_3(t)\}$; $\{y_2(t),y_3(t)\}$; $\{y_1(t),y_4(t)\}$; $\{y_4(t),y_5(t)\}$.

In each of Problems 23 through 25, find the Wronskian of two solutions of the given differential equation without solving the equation.

23. $t^2y''-t(t+2)y'+(t+2)y=0$

24. $(\cos t)y''+(\sin t)y'-ty=0$

25. $(1-x^2)y''-2xy'+\alpha(\alpha+1)y=0, \quad$ Legendre's equation

26. Show that if p is differentiable and $p(t)>0$, then the Wronskian $W(t)$ of two solutions of $[p(t)y']'+q(t)y=0$ is $W(t)=c/p(t)$, where c is a constant.

27. If the differential equation $ty''+2y'+te^ty=0$ has y_1 and y_2 as a fundamental set of solutions and if $W[y_1,y_2](1)=2$, find the value of $W[y_1,y_2](5)$.

28. If the Wronskian of any two solutions of $y''+p(t)y'+q(t)y=0$ is constant, what does this imply about the coefficients p and q?

In Problems 29 and 30, assume that p and q are continuous and that the functions y_1 and y_2 are solutions of the differential equation $y''+p(t)y'+q(t)y=0$ on an open interval I.

29. Prove that if y_1 and y_2 are zero at the same point in I, then they cannot be a fundamental set of solutions on that interval.

30. Prove that if y_1 and y_2 have a common point of inflection t_0 in I, then they cannot be a fundamental set of solutions on I unless both p and q are zero at t_0.

31. **Exact Equations.** The equation

$$P(x)y''+Q(x)y'+R(x)y=0$$

is said to be exact if it can be written in the form

$$(P(x)y')'+(f(x)y)'=0,$$

where $f(x)$ is to be determined in terms of $P(x)$, $Q(x)$, and $R(x)$. The latter equation can be integrated once immediately, resulting in a first-order linear equation for y that can be solved as in Section 2.1. By equating the coefficients of the preceding equations and then eliminating $f(x)$, show that a necessary condition for exactness is

$$P''(x)-Q'(x)+R(x)=0.$$

It can be shown that this is also a sufficient condition.

In each of Problems 32 through 34, use the result of Problem 31 to determine whether the given equation is exact. If it is, then solve the equation.

32. $y''+xy'+y=0$

33. $xy''-(\cos x)y'+(\sin x)y=0, \quad x>0$

34. $x^2y''+xy'-y=0, \quad x>0$

35. **The Adjoint Equation.** If a second-order linear homogeneous equation is not exact, it can be made exact by multiplying by an appropriate integrating factor $\mu(x)$. Thus we require that $\mu(x)$ be such that

$$\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = 0$$

can be written in the form

$$(\mu(x)P(x)y')' + (f(x)y)' = 0.$$

By equating coefficients in these two equations and eliminating $f(x)$, show that the function μ must satisfy

$$P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0.$$

This equation is known as the **adjoint** of the original equation and is important in the advanced theory of differential equations. In general, the problem of solving the adjoint differential equation is as difficult as that of solving the original equation, so only occasionally is it possible to find an integrating factor for a second-order equation.

In each of Problems 36 and 37, use the result of Problem 35 to find the adjoint of the given differential equation.

36. $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, Bessel's equation

37. $y'' - xy = 0$, Airy's equation

38. A second-order linear equation $P(x)y'' + Q(x)y' + R(x)y = 0$ is said to be **self-adjoint** if its adjoint is the same as the original equation. Show that a necessary condition for this equation to be self-adjoint is that $P'(x) = Q(x)$. Determine whether each of the equations in Problems 36 and 37 is self-adjoint.

3.3 Complex Roots of the Characteristic Equation

We continue our discussion of the second-order linear differential equation

$$ay'' + by' + cy = 0, \tag{1}$$

where a, b, and c are given real numbers. In Section 3.1 we found that if we seek solutions of the form $y = e^{rt}$, then r must be a root of the characteristic equation

$$ar^2 + br + c = 0. \tag{2}$$

We showed in Section 3.1 that if the roots r_1 and r_2 are real and different, which occurs whenever the discriminant $b^2 - 4ac$ is positive, then the general solution of equation (1) is

$$y = c_1e^{r_1t} + c_2e^{r_2t}. \tag{3}$$

Suppose now that $b^2 - 4ac$ is negative. Then the roots of equation (2) are conjugate complex numbers; we denote them by

$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu, \tag{4}$$

where λ and μ are real. The corresponding expressions for y are

$$y_1(t) = \exp((\lambda + i\mu)t), \quad y_2(t) = \exp((\lambda - i\mu)t). \tag{5}$$

Our first task is to explore what is meant by these expressions, which involve evaluating the exponential function for a complex exponent. For example, if $\lambda = -1$, $\mu = 2$, and $t = 3$, then from equation (5),

$$y_1(3) = e^{-3+6i}. \tag{6}$$

What does it mean to raise the number e to a complex power? The answer is provided by an important relation known as Euler's formula.

Euler's Formula. To assign a meaning to the expressions in equations (5), we need to give a definition of the complex exponential function. Of course, we want the definition to reduce to the familiar real exponential function when the exponent is real. There are several ways to discover how this extension of the exponential function should be defined. Here we use a method based on infinite series; an alternative is outlined in Problem 20.

Recall from calculus that the Taylor series for e^t about $t = 0$ is

$$e^t = 1 + t + \frac{t^2}{2} + \cdots + \frac{t^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \quad -\infty < t < \infty. \tag{7}$$

If we now assume that we can substitute it for t in equation (7), then we have

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}. \tag{8}$$

To simplify this series, we write $(it)^n = i^n t^n$ and make use of the facts that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and so forth. When n is even, there is an integer k with $n = 2k$; in this case $i^n = i^{2k} = (-1)^k$. And when n is odd, $n = 2k + 1$, so $i^n = i^{2k+1} = i(-1)^k$. This suggests separating the terms in the right-hand side of (8) into its real and imaginary parts. The result is[5]

$$e^{it} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}. \tag{9}$$

The first series in equation (9) is precisely the Taylor series for $\cos t$ about $t = 0$, and the second is the Taylor series for $\sin t$ about $t = 0$. Thus we have

$$e^{it} = \cos t + i \sin t. \tag{10}$$

Equation (10) is known as **Euler's formula** and is an extremely important mathematical relationship.

Although our derivation of equation (10) is based on the unverified assumption that the series (7) can be used for complex as well as real values of the independent variable, our intention is to use this derivation only to make equation (10) seem plausible. We now put matters on a firm foundation by adopting equation (10) as the *definition* of e^{it}. In other words, whenever we write e^{it}, we mean the expression on the right-hand side of equation (10).

There are some variations of Euler's formula that are also worth noting. If we replace t by $-t$ in equation (10) and recall that $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$, then we have

$$e^{-it} = \cos t - i \sin t. \tag{11}$$

Further, if t is replaced by μt in equation (10), then we obtain a generalized version of Euler's formula, namely,

$$e^{i\mu t} = \cos(\mu t) + i \sin(\mu t). \tag{12}$$

Next, we want to extend the definition of the exponential function to arbitrary complex exponents of the form $(\lambda + i\mu)t$. Since we want the usual properties of the exponential function to hold for complex exponents, we certainly want $\exp((\lambda + i\mu)t)$ to satisfy

$$e^{(\lambda+i\mu)t} = e^{\lambda t} e^{i\mu t}. \tag{13}$$

Then, substituting for $e^{i\mu t}$ from equation (12), we obtain

$$\begin{aligned} e^{(\lambda+i\mu)t} &= e^{\lambda t}(\cos(\mu t) + i \sin(\mu t)) \\ &= e^{\lambda t} \cos(\mu t) + i e^{\lambda t} \sin(\mu t). \end{aligned} \tag{14}$$

We now take equation (14) as the definition of $\exp[(\lambda + i\mu)t]$. The value of the exponential function with a complex exponent is a complex number whose real and imaginary parts are given by the terms on the right-hand side of equation (14). Observe that the real and imaginary parts of $\exp((\lambda + i\mu)t)$ are expressed entirely in terms of elementary real-valued functions. For example, the quantity in equation (6) has the value

$$e^{-3+6i} = e^{-3} \cos 6 + i e^{-3} \sin 6 \cong 0.0478041 - 0.0139113i.$$

With the definitions (10) and (14), it is straightforward to show that the usual laws of exponents are valid for the complex exponential function. You can also use equation (14) to verify that the differentiation formula

$$\frac{d}{dt}(e^{rt}) = r e^{rt} \tag{15}$$

holds for complex values of r.

[5]Recall from calculus that the reordering of terms in the right-hand side of equation (9) is allowed because the series converges absolutely for all $-\infty < t < \infty$.

EXAMPLE 1

Find the general solution of the differential equation

$$y'' + y' + 9.25y = 0. \tag{16}$$

Also find the solution that satisfies the initial conditions

$$y(0) = 2, \quad y'(0) = 8, \tag{17}$$

and draw its graph for $0 < t < 10$.

Solution:

The characteristic equation for equation (16) is

$$r^2 + r + 9.25 = 0$$

so its roots are

$$r_1 = -\frac{1}{2} + 3i, \quad r_2 = -\frac{1}{2} - 3i.$$

Therefore, two solutions of equation (16) are

$$y_1(t) = \exp\left(\left(-\frac{1}{2} + 3i\right)t\right) = e^{-t/2}(\cos(3t) + i\sin(3t)) \tag{18}$$

and

$$y_2(t) = \exp\left(\left(-\frac{1}{2} - 3i\right)t\right) = e^{-t/2}(\cos(3t) - i\sin(3t)). \tag{19}$$

You can verify that the Wronskian $W[y_1, y_2](t) = -6ie^{-t}$, which is not zero, so the general solution of equation (15) can be expressed as a linear combination of $y_1(t)$ and $y_2(t)$ with arbitrary coefficients.

However, the initial value problem (16), (17) has only real coefficients, and it is often desirable to express the solution of such a problem in terms of real-valued functions. To do this we can make use of Theorem 3.2.6, which states that the real and imaginary parts of a complex-valued solution of equation (16) are also solutions of the same differential equation. Thus, starting from $y_1(t)$, we obtain

$$u(t) = e^{-t/2}\cos(3t), \quad v(t) = e^{-t/2}\sin(3t) \tag{20}$$

as real-valued solutions[6] of equation (16). On calculating the Wronskian of $u(t)$ and $v(t)$, we find that $W[u, v](t) = 3e^{-t}$, which is not zero; thus $u(t)$ and $v(t)$ form a fundamental set of solutions, and the general solution of equation (16) can be written as

$$y = c_1 u(t) + c_2 v(t) = e^{-t/2}\big(c_1\cos(3t) + c_2\sin(3t)\big), \tag{21}$$

where c_1 and c_2 are arbitrary constants.

To satisfy the initial conditions (17), we first substitute $t = 0$ and $y = 2$ in the solution (20) with the result that $c_1 = 2$. Then, by differentiating equation (21), setting $t = 0$, and setting $y' = 8$, we obtain $-\frac{1}{2}c_1 + 3c_2 = 8$ so that $c_2 = 3$. Thus the solution of the initial value problem (16), (17) is

$$y = e^{-t/2}(2\cos(3t) + 3\sin(3t)). \tag{22}$$

The graph of this solution is shown in Figure 3.3.1.

From the graph we see that the solution of this problem oscillates, with period $2\pi/3$ and a decaying amplitude. The sine and cosine factors control the oscillatory nature of the solution, and the negative exponential factor in each term causes the magnitude of the oscillations to decrease toward zero as time increases.

[6] If you are not completely sure that $u(t)$ and $v(t)$ are solutions of the given differential equation, you should substitute these functions into equation (16) and confirm that they satisfy it. (See Problem 23.)

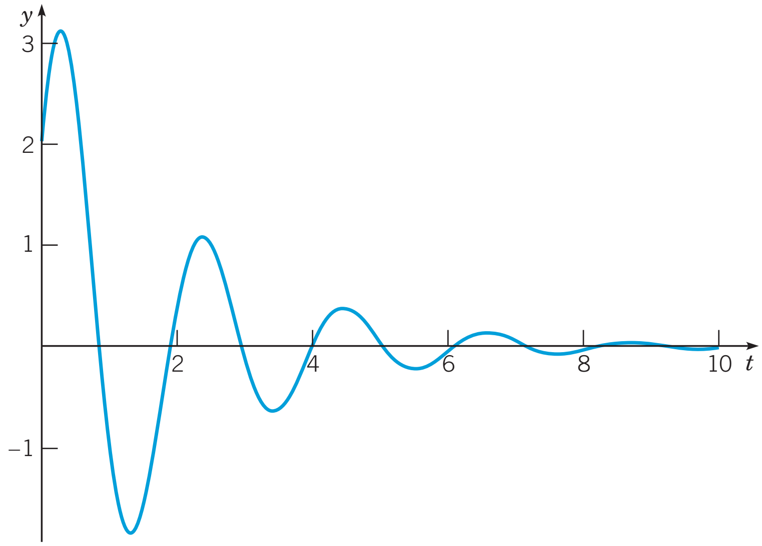

FIGURE 3.3.1 Solution of the initial value problem (16), (17): $y'' + y' + 9.25y = 0,\ \ y(0) = 2,\ \ y'(0) = 8.$

Complex Roots; The General Case. The functions $y_1(t)$ and $y_2(t)$, given by equations (5) and with the meaning expressed by equation (14), are solutions of equation (1) when the roots of the characteristic equation (2) are complex numbers $\lambda \pm i\mu$. However, the solutions y_1 and y_2 are complex-valued functions, whereas in general we would prefer to have real-valued solutions because the differential equation itself has real coefficients. Just as in Example 1, we can use Theorem 3.2.6 to find a fundamental set of real-valued solutions by choosing the real and imaginary parts of either $y_1(t)$ or $y_2(t)$. In this way we obtain the solutions

$$u(t) = e^{\lambda t}\cos(\mu t), \qquad v(t) = e^{\lambda t}\sin(\mu t). \tag{23}$$

By direct computation (see Problem 19), you can show that the Wronskian of u and v is

$$W[u, v](t) = \mu e^{2\lambda t}. \tag{24}$$

Thus, as long as $\mu \neq 0$, the Wronskian W is not zero, so u and v form a fundamental set of solutions. (Of course, if $\mu = 0$, then the roots are real and equal and the discussions in this section, and in Section 3.1, are not applicable.) Consequently, if the roots of the characteristic equation are complex numbers $\lambda \pm i\mu$, with $\mu \neq 0$, then the general solution of equation (1) is

$$y = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t), \tag{25}$$

where c_1 and c_2 are arbitrary constants. Note that the solution (25) can be written down as soon as the values of λ and μ are known. Let us now consider some further examples.

EXAMPLE 2

Find the solution of the initial value problem

$$16y'' - 8y' + 145y = 0, \qquad y(0) = -2, \quad y'(0) = 1. \tag{26}$$

Solution:

The characteristic equation is $16r^2 - 8r + 145 = 0$ and its roots are $r = \frac{1}{4} \pm 3i$. Thus the general solution of the differential equation is

$$y(t) = c_1 e^{t/4}\cos(3t) + c_2 e^{t/4}\sin(3t). \tag{27}$$

To apply the first initial condition, we set $t = 0$ in equation (27); this gives

$$y(0) = c_1 = -2.$$

For the second initial condition, we must differentiate equation (27) before substituting $t = 0$. In this way we find that

$$y'(0) = \frac{1}{4}c_1 + 3c_2 = 1,$$

from which we determine that $c_2 = \frac{1}{2}$. Using these values of c_1 and c_2 in the general solution (27), we obtain

$$y = -2e^{t/4}\cos(3t) + \frac{1}{2}e^{t/4}\sin(3t) \tag{28}$$

as the solution of the initial value problem (26). The graph of this solution is shown in Figure 3.3.2.

In this case we observe that the solution is a growing oscillation. Again the trigonometric factors in equation (28) determine the oscillatory part of the solution (again with period $2\pi/3$), while the exponential factor (with a positive exponent this time) causes the magnitude of the oscillation to increase with time.

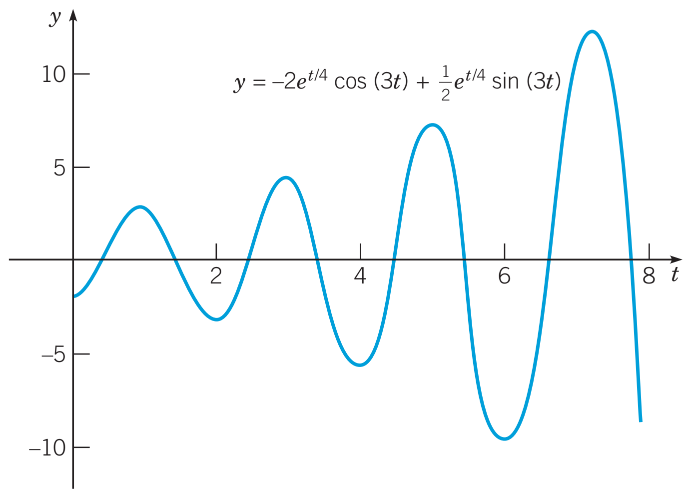

FIGURE 3.3.2 Solution of the initial value problem (26): $16y'' - 8y' + 145y = 0,\ y(0) = -2,\ y'(0) = 1.$

EXAMPLE 3

Find the general solution of

$$y'' + 9y = 0. \tag{29}$$

Solution:

The characteristic equation is $r^2 + 9 = 0$ with the roots $r = \pm 3i$; thus $\lambda = 0$ and $\mu = 3$. The general solution is

$$y = c_1\cos(3t) + c_2\sin(3t). \tag{30}$$

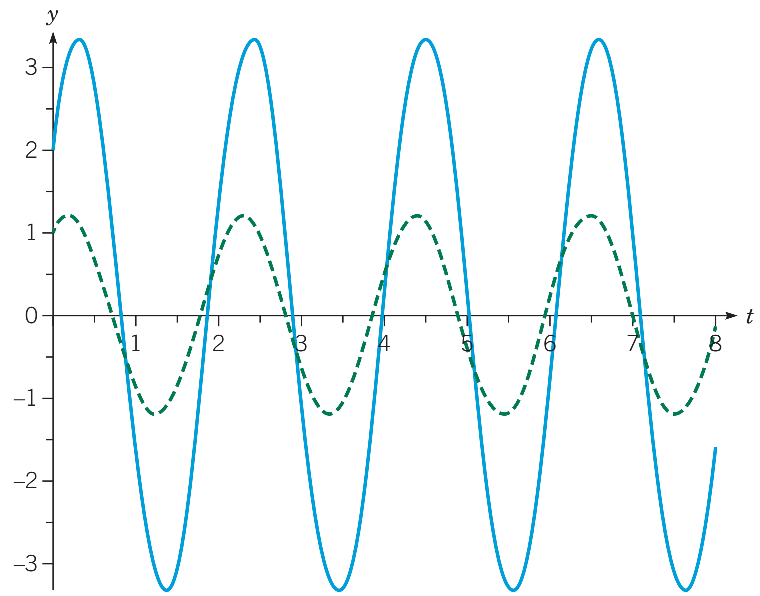

FIGURE 3.3.3 Solutions of equation (29): $y'' + 9y = 0$, with two sets of initial conditions: $y(0) = 1$, $y'(0) = 2$ (dashed, green) and $y(0) = 2$, $y'(0) = 8$ (solid, blue). Both solutions have the same period, but different amplitudes and phase shifts.

Note that if the real part of the roots is zero, as in this example, then there is no exponential factor in the solution. Figure 3.3.3 shows the graph of two solutions of equation (28) with different initial conditions. In each case the solution is a pure oscillation with period $2\pi/3$ but whose amplitude and phase shift are determined by the initial conditions. Since there is no exponential factor in the solution (30), the amplitude of each oscillation remains constant in time.

Problems

In each of Problems 1 through 4, use Euler's formula to write the given expression in the form $a + ib$.

1. $\exp(2 - 3i)$

2. $e^{i\pi}$

3. $e^{2-(\pi/2)i}$

4. 2^{1-i}

In each of Problems 5 through 11, find the general solution of the given differential equation.

5. $y'' - 2y' + 2y = 0$

6. $y'' - 2y' + 6y = 0$

7. $y'' + 2y' + 2y = 0$

8. $y'' + 6y' + 13y = 0$

9. $y'' + 2y' + 1.25y = 0$

10. $9y'' + 9y' - 4y = 0$

11. $y'' + 4y' + 6.25y = 0$

In each of Problems 12 through 15, find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

G **12.** $y'' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$

G **13.** $y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2$

G **14.** $y'' + y = 0, \quad y(\pi/3) = 2, \quad y'(\pi/3) = -4$

G **15.** $y'' + 2y' + 2y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2$

N **16.** Consider the initial value problem

$$3u'' - u' + 2u = 0, \quad u(0) = 2, \quad u'(0) = 0.$$

a. Find the solution $u(t)$ of this problem.
b. For $t > 0$, find the first time at which $|u(t)| = 10$.

N **17.** Consider the initial value problem

$$5u'' + 2u' + 7u = 0, \quad u(0) = 2, \quad u'(0) = 1.$$

a. Find the solution $u(t)$ of this problem.
b. Find the smallest T such that $|u(t)| \leq 0.1$ for all $t > T$.

N **18.** Consider the initial value problem

$$y'' + 2y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = \alpha \geq 0.$$

a. Find the solution $y(t)$ of this problem.
b. Find α such that $y = 0$ when $t = 1$.
c. Find, as a function of α, the smallest positive value of t for which $y = 0$.
d. Determine the limit of the expression found in part c as $\alpha \to \infty$.

19. Show that $W\left[e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t)\right] = \mu e^{2\lambda t}$.

20. In this problem we outline a different derivation of Euler's formula.

a. Show that $y_1(t) = \cos t$ and $y_2(t) = \sin t$ are a fundamental set of solutions of $y'' + y = 0$; that is, show that they are solutions and that their Wronskian is not zero.

b. Show (formally) that $y = e^{it}$ is also a solution of $y'' + y = 0$. Therefore,

$$e^{it} = c_1 \cos t + c_2 \sin t \tag{31}$$

for some constants c_1 and c_2. Why is this so?

c. Set $t = 0$ in equation (31) to show that $c_1 = 1$.

d. Assuming that equation (15) is true, differentiate equation (31) and then set $t = 0$ to conclude that $c_2 = i$. Use the values of c_1 and c_2 in equation (31) to arrive at Euler's formula.

21. Using Euler's formula, show that

$$\frac{e^{it} + e^{-it}}{2} = \cos t, \qquad \frac{e^{it} - e^{-it}}{2i} = \sin t.$$

22. If e^{rt} is given by equation (14), show that $e^{(r_1+r_2)t} = e^{r_1 t}e^{r_2 t}$ for any complex numbers r_1 and r_2.

23. Consider the differential equation

$$ay'' + by' + cy = 0,$$

where $b^2 - 4ac < 0$ and the characteristic equation has complex roots $\lambda \pm i\mu$. Substitute the functions

$$u(t) = e^{\lambda t}\cos(\mu t) \quad \text{and} \quad v(t) = e^{\lambda t}\sin(\mu t)$$

for y in the differential equation and thereby confirm that they are solutions.

24. If the functions y_1 and y_2 are a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$, show that between consecutive zeros of y_1 there is one and only one zero of y_2. Note that this result is illustrated by the solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ of the equation $y'' + y = 0$.
Hint: Suppose that t_1 and t_2 are two zeros of y_1 between which there are no zeros of y_2. Apply Rolle's theorem to y_1/y_2 to reach a contradiction.

Change of Variables. Sometimes a differential equation with variable coefficients,

$$y'' + p(t)y' + q(t)y = 0, \tag{32}$$

can be put in a more suitable form for finding a solution by making a change of the independent variable. We explore these ideas in Problems 25 through 36. In particular, in Problem 25 we show that a class of equations known as Euler equations can be transformed into equations with constant coefficients by a simple change of the independent variable. Problems 26 through 31 are examples of this type of equation. Problem 32 determines conditions under which the more general equation (32) can be transformed into a differential equation with constant coefficients. Problems 33 through 36 give specific applications of this procedure.

25. **Euler Equations.** An equation of the form

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0, \quad t > 0, \tag{33}$$

where α and β are real constants, is called an **Euler equation**.

a. Let $x = \ln t$ and calculate dy/dt and d^2y/dt^2 in terms of dy/dx and d^2y/dx^2.

b. Use the results of part a to transform equation (33) into

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0. \tag{34}$$

Observe that differential equation (34) has constant coefficients. If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of equation (34), then $y_1(\ln t)$ and $y_2(\ln t)$ form a fundamental set of solutions of equation (33).

In each of Problems 26 through 31, use the method of Problem 25 to solve the given equation for $t > 0$.

26. $t^2y'' + ty' + y = 0$

27. $t^2y'' + 4ty' + 2y = 0$

28. $t^2y'' - 4ty' - 6y = 0$

29. $t^2y'' - 4ty' + 6y = 0$

30. $t^2y'' + 3ty' - 3y = 0$

31. $t^2y'' + 7ty' + 10y = 0$

32. In this problem we determine conditions on p and q that enable equation (32) to be transformed into an equation with constant coefficients by a change of the independent variable. Let $x = u(t)$ be the new independent variable, with the relation between x and t to be specified later.

a. Show that

$$\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx}, \qquad \frac{d^2y}{dt^2} = \left(\frac{dx}{dt}\right)^2\frac{d^2y}{dx^2} + \frac{d^2x}{dt^2}\frac{dy}{dx}.$$

b. Show that the differential equation (32) becomes

$$\left(\frac{dx}{dt}\right)^2\frac{d^2y}{dx^2} + \left(\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right)\frac{dy}{dx} + q(t)y = 0. \tag{35}$$

c. In order for equation (35) to have constant coefficients, the coefficients of d^2y/dx^2, dy/dx, and y must all be proportional. If $q(t) > 0$, then we can choose the constant of proportionality to be 1; hence, after integrating with respect to t,

$$x = u(t) = \int (q(t))^{1/2}\,dt. \tag{36}$$

d. With x chosen as in part c, show that the coefficient of dy/dx in equation (35) is also a constant, provided that the expression

$$\frac{q'(t) + 2p(t)q(t)}{2(q(t))^{3/2}} \tag{37}$$

is a constant. Thus equation (32) can be transformed into an equation with constant coefficients by a change of the independent variable, provided that the function $(q' + 2pq)/q^{3/2}$ is a constant.

e. How must the analysis and results in d be modified if $q(t) < 0$?

In each of Problems 33 through 36, try to transform the given equation into one with constant coefficients by the method of Problem 32. If this is possible, find the general solution of the given equation.

33. $y'' + ty' + e^{-t^2}y = 0, \quad -\infty < t < \infty$

34. $y'' + 3ty' + t^2y = 0, \quad -\infty < t < \infty$

35. $ty'' + (t^2 - 1)y' + t^3y = 0, \quad 0 < t < \infty$

36. $y'' + ty' - e^{-t^2}y = 0$

3.4 Repeated Roots; Reduction of Order

In Sections 3.1 and 3.3 we showed how to solve the equation

$$ay'' + by' + cy = 0 \tag{1}$$

when the roots of the characteristic equation

$$ar^2 + br + c = 0 \tag{2}$$

either are real and different or are complex conjugates. Now we consider the third possibility, namely, that the two roots r_1 and r_2 are equal. This case is transitional between the other two and occurs when the discriminant $b^2 - 4ac$ is zero. Then it follows from the quadratic formula that

$$r_1 = r_2 = -\frac{b}{2a}. \tag{3}$$

The difficulty is immediately apparent; both roots yield the same solution

$$y_1(t) = e^{-bt/(2a)} \tag{4}$$

of the differential equation (1), and it is not obvious how to find a second solution.

EXAMPLE 1

Solve the differential equation

$$y'' + 4y' + 4y = 0. \tag{5}$$

Solution:

The characteristic equation is

$$r^2 + 4r + 4 = (r+2)^2 = 0,$$

so $r_1 = r_2 = -2$. Therefore, one solution of equation (5) is $y_1(t) = e^{-2t}$. To find the general solution of equation (5), we need a second solution that is not a constant multiple of y_1. This second solution can be found in several ways (see Problems 15 through 17); here we use a method originated by d'Alembert[7] in the eighteenth century. Recall that since $y_1(t)$ is a solution of equation (1), so is $cy_1(t)$ for any constant c. The basic idea is to generalize this observation by replacing c by a function $v(t)$ and then trying to determine $v(t)$ so that the product $v(t)y_1(t)$ is also a solution of equation (1).

To carry out this program, we substitute $y = v(t)y_1(t)$ in equation (5) and use the resulting equation to find $v(t)$. Starting with

$$y = v(t)y_1(t) = v(t)e^{-2t}, \tag{6}$$

we differentiate once to find

$$y' = v'(t)e^{-2t} - 2v(t)e^{-2t} \tag{7}$$

and a second differentiation yields

$$y'' = v''(t)e^{-2t} - 4v'(t)e^{-2t} + 4v(t)e^{-2t}. \tag{8}$$

By substituting the expressions in equations (6), (7), and (8) in equation (5) and collecting terms, we obtain

$$(v''(t) - 4v'(t) + 4v(t) + 4v'(t) - 8v(t) + 4v(t))e^{-2t} = 0,$$

[7] Jean d'Alembert (1717–1783), a French mathematician, was a contemporary of Euler and Daniel Bernoulli and is known primarily for his work in mechanics and differential equations. D'Alembert's principle in mechanics and d'Alembert's paradox in hydrodynamics are named for him, and the wave equation first appeared in his paper on vibrating strings in 1747. In his later years he devoted himself primarily to philosophy and to his duties as science editor of Diderot's *Encyclopédie*.

which simplifies to

$$v''(t) = 0. \tag{9}$$

Therefore,

$$v'(t) = c_1$$

and

$$v(t) = c_1 t + c_2, \tag{10}$$

where c_1 and c_2 are arbitrary constants. Finally, substituting for $v(t)$ in equation (6), we obtain

$$y = c_1 t e^{-2t} + c_2 e^{-2t}. \tag{11}$$

The second term on the right-hand side of equation (11) corresponds to the original solution $y_1(t) = \exp(-2t)$, but the first term arises from a second solution, namely, $y_2(t) = t\exp(-2t)$. We can verify that these two solutions form a fundamental set by calculating their Wronskian:

$$\begin{aligned} W[y_1, y_2](t) &= \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t} - 2te^{-4t} + 2te^{-4t} \\ &= e^{-4t} \neq 0. \end{aligned}$$

Therefore,

$$y_1(t) = e^{-2t}, \quad y_2(t) = te^{-2t} \tag{12}$$

form a fundamental set of solutions of equation (5), and the general solution of that equation is given by equation (11). Note that both $y_1(t)$ and $y_2(t)$ tend to zero as $t \to \infty$; consequently, all solutions of equation (5) behave in this way. The graphs of typical solutions are shown in Figure 3.4.1.

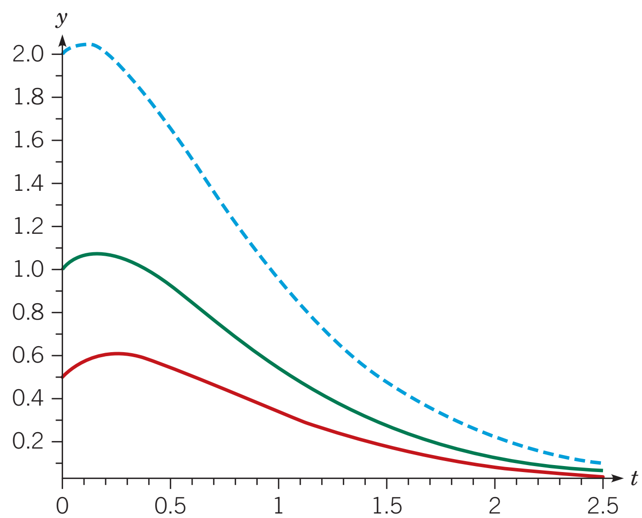

FIGURE 3.4.1 Three solutions of equation (5): $y'' + 4y' + 4y = 0$, with different sets of initial conditions: $y(0) = 2$, $y'(0) = 1$ (blue, dashed); $y(0) = 1$, $y'(0) = 1$ (green, solid); $y(0) = 1/2$, $y'(0) = 1$ (red).

The procedure used in Example 1 can be extended to a general equation whose characteristic equation has repeated roots. That is, we assume that the coefficients in equation (1) satisfy $b^2 - 4ac = 0$, in which case

$$y_1(t) = e^{-bt/(2a)}$$

is a solution. To find a second solution, we assume that

$$y = v(t)y_1(t) = v(t)e^{-bt/(2a)} \tag{13}$$

and substitute for y in equation (1) to determine $v(t)$. We have

$$y' = v'(t)e^{-bt/(2a)} - \frac{b}{2a}v(t)e^{-bt/(2a)} \tag{14}$$

and

$$y'' = v''(t)e^{-bt/(2a)} - \frac{b}{a}v'(t)e^{-bt/(2a)} + \frac{b^2}{4a^2}v(t)e^{-bt/(2a)}. \tag{15}$$

Then, by substituting in equation (1), we obtain

$$\left(a\left(v''(t) - \frac{b}{a}v'(t) + \frac{b^2}{4a^2}v(t)\right) + b\left(v'(t) - \frac{b}{2a}v(t)\right) + cv(t)\right)e^{-bt/(2a)} = 0. \tag{16}$$

Canceling the factor $e^{-b/(2a)}$, which is nonzero, and rearranging the remaining terms, we find that

$$av''(t) + (-b+b)v'(t) + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)v(t) = 0. \tag{17}$$

The term involving $v'(t)$ is obviously zero. Further, the coefficient of $v(t)$ is $c - b^2/(4a)$, which is also zero because $b^2 - 4ac = 0$ in the problem that we are considering. Thus, just as in Example 1, equation (17) reduces to

$$v''(t) = 0,$$

so

$$v(t) = c_1 + c_2 t.$$

Hence, from equation (13), we have

$$y = c_1 e^{-bt/(2a)} + c_2 t e^{-bt/(2a)}. \tag{18}$$

Thus y is a linear combination of the two solutions

$$y_1(t) = e^{-bt/(2a)}, \quad y_2(t) = te^{-bt/(2a)}. \tag{19}$$

The Wronskian of these two solutions is

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-bt/(2a)} & te^{-bt/(2a)} \\ -\frac{b}{2a}e^{-bt/(2a)} & \left(1 - \frac{bt}{2a}\right)e^{-bt/(2a)} \end{vmatrix} = e^{-bt/a}. \tag{20}$$

Since $W[y_1, y_2](t)$ is never zero, the solutions y_1 and y_2 given by equation (19) are a fundamental set of solutions. Further, equation (18) is the general solution of equation (1) when the roots of the characteristic equation are equal. In other words, in this case there is one exponential solution corresponding to the repeated root and a second solution that is obtained by multiplying the exponential solution by t.

EXAMPLE 2

Find the solution of the initial value problem

$$y'' - y' + \frac{y}{4} = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{3}. \tag{21}$$

Solution:

The characteristic equation is

$$r^2 - r + \frac{1}{4} = 0,$$

so the roots are $r_1 = r_2 = 1/2$. Thus the general solution of the differential equation is

$$y = c_1 e^{t/2} + c_2 t e^{t/2}. \tag{22}$$

The first initial condition requires that

$$y(0) = c_1 = 2.$$

To satisfy the second initial condition, we first differentiate equation (22) and then set $t = 0$. This gives

$$y'(0) = \frac{1}{2}c_1 + c_2 = \frac{1}{3},$$

so $c_2 = -2/3$. Thus the solution of the initial value problem is

$$y = 2e^{t/2} - \frac{2}{3}te^{t/2}. \tag{23}$$

The graph of this solution is shown by the blue curve in Figure 3.4.2.

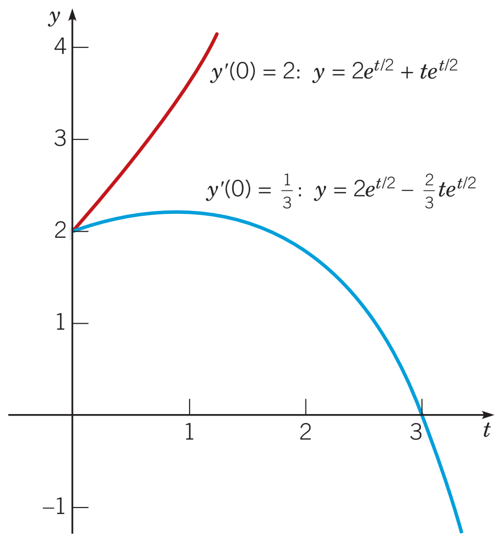

FIGURE 3.4.2 Solutions of $y'' - y' + y/4 = 0$, $y(0) = 2$, with $y'(0) = 1/3$ (blue) and with $y'(0) = 2$ (red).

Let us now modify the initial value problem (16) by changing the initial slope; to be specific, let the second initial condition be $y'(0) = 2$. The solution of this modified problem is

$$y = 2e^{t/2} + te^{t/2},$$

and its graph is shown by the red curve in Figure 3.4.2. The graphs shown in this figure suggest that there is a critical initial slope, with a value between $1/3$ and 2, that separates solutions that increase as $t \to \infty$ from those that ultimately decrease as $t \to \infty$. In Problem 12 you are asked to determine this critical initial slope.

The asymptotic behavior of solutions is similar in this case to that when the roots are real and different. If the exponents are either positive or negative, then the magnitude of the solution grows or decays accordingly; the linear factor t has little influence. A decaying solution is shown in Figure 3.4.1 and growing solutions in Figure 3.4.2. However, if the repeated root is zero, then the differential equation is $y'' = 0$ and the general solution is a linear function of t.

Summary. We can now summarize the results that we have obtained for second-order linear homogeneous equations with constant coefficients

$$ay'' + by' + cy = 0. \tag{24}$$

Let r_1 and r_2 be the roots of the corresponding characteristic equation

$$ar^2 + br + c = 0. \tag{25}$$

If r_1 and r_2 are real but not equal, then the general solution of differential equation (24) is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \tag{26}$$

If r_1 and r_2 are complex conjugates $\lambda \pm i\mu$, then the general solution is

$$y = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t). \tag{27}$$

If $r_1 = r_2$, then the general solution is

$$y = c_1 e^{r_1 t} + c_2 t e^{r_1 t}. \tag{28}$$

Reduction of Order. It is worth noting that the procedure used in this section for equations with constant coefficients is more generally applicable. Suppose that we know one solution $y_1(t)$, not everywhere zero, of

$$y'' + p(t)y' + q(t)y = 0. \tag{29}$$

To find a second solution, let

$$y = v(t)y_1(t); \tag{30}$$

then

$$y' = v'(t)y_1(t) + v(t)y_1'(t)$$

and

$$y'' = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t).$$

Substituting for y, y', and y'' in equation (29) and collecting terms, we find that

$$y_1 v'' + (2y_1' + py_1)v' + (y_1'' + py_1' + qy_1)v = 0. \tag{31}$$

Since y_1 is a solution of equation (29), the coefficient of v in equation (31) is zero so that equation (31) becomes

$$y_1 v'' + (2y_1' + py_1)v' = 0. \tag{32}$$

Despite its appearance, equation (32) is actually a first-order differential equation for the function v' and can be solved either as a first-order linear equation or as a separable equation. Once v' has been found, then v is obtained by an integration. Finally, y is determined from equation (30). This procedure is called the method of **reduction of order**, because the crucial step is the solution of a first-order differential equation for v' rather than the original second-order differential equation for y. Although it is possible to write down a formula for $v(t)$, we will instead illustrate how this method works by an example.

EXAMPLE 3

Given that $y_1(t) = t^{-1}$ is a solution of

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0, \tag{33}$$

find a fundamental set of solutions.

Solution:

We set $y = v(t)t^{-1}$; then

$$y' = v't^{-1} - vt^{-2}, \quad y'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}.$$

Substituting for y, y', and y'' in equation (33) and collecting terms, we obtain

$$\begin{aligned} 2t^2\left(v''t^{-1} - 2v't^{-2} + 2vt^{-3}\right) + 3t\left(v't^{-1} - vt^{-2}\right) - vt^{-1} \\ = 2tv'' + (-4+3)v' + \left(4t^{-1} - 3t^{-1} - t^{-1}\right)v \\ = 2tv'' - v' = 0. \end{aligned} \tag{34}$$

Note that the coefficient of v is zero, as it should be; this provides a useful check on our algebraic calculations.

If we let $w = v'$, then the second-order linear differential equation (34) reduces to the separable first-order differential equation

$$2tw' - w = 0.$$

Separating the variables and solving for $w(t)$, we find that

$$w(t) = v'(t) = ct^{1/2};$$

then, one final integration yields

$$v(t) = \frac{2}{3}ct^{3/2} + k.$$

It follows that

$$y = v(t)t^{-1} = \frac{2}{3}ct^{1/2} + kt^{-1}, \tag{35}$$

where c and k are arbitrary constants. The second term on the right-hand side of equation (35) is a multiple of $y_1(t)$ and can be dropped, but the first term provides a new solution $y_2(t) = t^{1/2}$. You can verify that the Wronskian of y_1 and y_2 is

$$W[y_1, y_2](t) = \frac{3}{2}t^{-3/2} \neq 0 \text{ for } t > 0. \tag{36}$$

Consequently, y_1 and y_2 form a fundamental set of solutions of equation (33) for $t > 0$.

Problems

In each of Problems 1 through 8, find the general solution of the given differential equation.

1. $y'' - 2y' + y = 0$
2. $9y'' + 6y' + y = 0$
3. $4y'' - 4y' - 3y = 0$
4. $y'' - 2y' + 10y = 0$
5. $y'' - 6y' + 9y = 0$
6. $4y'' + 17y' + 4y = 0$
7. $16y'' + 24y' + 9y = 0$
8. $2y'' + 2y' + y = 0$

In each of Problems 9 through 11, solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t.

9. $9y'' - 12y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = -1$
10. $y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2$
11. $y'' + 4y' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 1$
12. Consider the following modification of the initial value problem in Example 2:

$$y'' - y' + \frac{y}{4} = 0, \quad y(0) = 2, \quad y'(0) = b.$$

Find the solution as a function of b, and then determine the critical value of b that separates solutions that remain positive for all $t > 0$ from those that eventually become negative.

N 13. Consider the initial value problem

$$4y'' + 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

a. Solve the initial value problem and plot the solution.
b. Determine the coordinates (t_M, y_M) of the maximum point.
c. Change the second initial condition to $y'(0) = b > 0$ and find the solution as a function of b.
d. Find the coordinates (t_M, y_M) of the maximum point in terms of b. Describe the dependence of t_M and y_M on b as b increases.

14. Consider the equation $ay'' + by' + cy = 0$. If the roots of the corresponding characteristic equation are real, show that a solution to the differential equation either is everywhere zero or else can take on the value zero at most once.

Problems 15 through 17 indicate other ways of finding the second solution when the characteristic equation has repeated roots.

15. **a.** Consider the equation $y'' + 2ay' + a^2y = 0$. Show that the roots of the characteristic equation are $r_1 = r_2 = -a$ so that one solution of the equation is e^{-at}.

b. Use Abel's formula [equation (23) of Section 3.2] to show that the Wronskian of any two solutions of the given equation is

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = c_1e^{-2at},$$

where c_1 is a constant.

c. Let $y_1(t) = e^{-at}$ and use the result of part b to obtain a differential equation satisfied by a second solution $y_2(t)$. By solving this equation, show that $y_2(t) = te^{-at}$.

16. Suppose that r_1 and r_2 are roots of $ar^2 + br + c = 0$ and that $r_1 \neq r_2$; then $\exp(r_1 t)$ and $\exp(r_2 t)$ are solutions of the differential equation $ay'' + by' + cy = 0$. Show that

$$\phi(t; r_1, r_2) = \frac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1}$$

is also a solution of the equation for $r_2 \neq r_1$. Then think of r_1 as fixed, and use l'Hôpital's rule to evaluate the limit of $\phi(t; r_1, r_2)$ as $r_2 \to r_1$, thereby obtaining the second solution in the case of equal roots.

17. a. If $ar^2 + br + c = 0$ has equal roots r_1, show that

$$L\left[e^{rt}\right] = a(e^{rt})'' + b(e^{rt})' + ce^{rt} = a(r - r_1)^2 e^{rt}. \qquad (37)$$

Since the right-hand side of equation (37) is zero when $r = r_1$, it follows that $\exp(r_1 t)$ is a solution of $L[y] = ay'' + by' + cy = 0$.

b. Differentiate equation (37) with respect to r, and interchange differentiation with respect to r and with respect to t, thus showing that

$$\begin{aligned} \frac{\partial}{\partial r} L\left[e^{rt}\right] &= L\left[\frac{\partial}{\partial r} e^{rt}\right] = L\left[te^{rt}\right] \\ &= ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1). \qquad (38) \end{aligned}$$

Since the right-hand side of equation (36) is zero when $r = r_1$, conclude that $t\exp(r_1 t)$ is also a solution of $L[y] = 0$.

In each of Problems 18 through 22, use the method of reduction of order to find a second solution of the given differential equation.

18. $t^2 y'' - 4ty' + 6y = 0, \quad t > 0; \quad y_1(t) = t^2$

19. $t^2 y'' + 2ty' - 2y = 0, \quad t > 0; \quad y_1(t) = t$

20. $t^2 y'' + 3ty' + y = 0, \quad t > 0; \quad y_1(t) = t^{-1}$

21. $xy'' - y' + 4x^3 y = 0, \quad x > 0; \quad y_1(x) = \sin(x^2)$

22. $x^2 y'' + xy' + (x^2 - 0.25)y = 0, \quad x > 0; \quad y_1(x) = x^{-1/2}\sin x$

23. The differential equation

$$y'' + \delta(xy' + y) = 0$$

arises in the study of the turbulent flow of a uniform stream past a circular cylinder. Verify that $y_1(x) = \exp(-\delta x^2/2)$ is one solution, and then find the general solution in the form of an integral.

24. The method of Problem 15 can be extended to second-order equations with variable coefficients. If y_1 is a known nonvanishing solution of $y'' + p(t)y' + q(t)y = 0$, show that a second solution y_2 satisfies $(y_2/y_1)' = W[y_1, y_2]/y_1^2$, where $W[y_1, y_2]$ is the Wronskian of y_1 and y_2. Then use Abel's formula (equation (23) of Section 3.2) to determine y_2.

In each of Problems 25 through 27, use the method of Problem 24 to find a second independent solution of the given equation.

25. $t^2 y'' + 3ty' + y = 0, \quad t > 0; \quad y_1(t) = t^{-1}$

26. $ty'' - y' + 4t^3 y = 0, \quad t > 0; \quad y_1(t) = \sin(t^2)$

27. $x^2 y'' + xy' + (x^2 - 0.25)y = 0, \quad x > 0; \quad y_1(x) = x^{-1/2}\sin x$

Behavior of Solutions as $t \to \infty$. Problems 28 through 30 are concerned with the behavior of solutions as $t \to \infty$.

28. If a, b, and c are positive constants, show that all solutions of $ay'' + by' + cy = 0$ approach zero as $t \to \infty$.

29. a. If $a > 0$ and $c > 0$, but $b = 0$, show that the result of Problem 28 is no longer true, but that all solutions are bounded as $t \to \infty$.

b. If $a > 0$ and $b > 0$, but $c = 0$, show that the result of Problem 28 is no longer true, but that all solutions approach a constant that depends on the initial conditions as $t \to \infty$. Determine this constant for the initial conditions $y(0) = y_0$, $y'(0) = y_0'$.

30. Show that $y = \sin t$ is a solution of

$$y'' + (k\sin^2 t)y' + (1 - k\cos t\sin t)y = 0$$

for any value of the constant k. If $0 < k < 2$, show that $1 - k\cos t\sin t > 0$ and $k\sin^2 t \geq 0$. Thus observe that even though the coefficients of this variable-coefficient differential equation are nonnegative (and the coefficient of y' is zero only at the points $t = 0, \pi, 2\pi, \ldots$), it has a solution that does not approach zero as $t \to \infty$. Compare this situation with the result of Problem 28. Thus we observe a not unusual situation in the study of differential equations: equations that are apparently very similar can have quite different properties.

Euler Equations. In each of Problems 31 through 34, use the substitution introduced in Problem 25 in Section 3.3 to solve the given differential equation.

31. $t^2 y'' - 3ty' + 4y = 0, \quad t > 0$

32. $t^2 y'' + 2ty' + 0.25y = 0, \quad t > 0$

33. $t^2 y'' + 3ty' + y = 0, \quad t > 0$

34. $4t^2 y'' - 8ty' + 9y = 0, \quad t > 0$

3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

We now turn our attention to the nonhomogeneous second-order linear differential equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \qquad (1)$$

where p, q, and g are given (continuous) functions on the open interval I. The equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \qquad (2)$$

in which $g(t) = 0$ and p and q are the same as in equation (1), is called the homogeneous differential equation corresponding to equation (1). The following two results describe the structure of solutions of the nonhomogeneous equation (1) and provide a foundation for constructing its general solution.

Theorem 3.5.1

If Y_1 and Y_2 are two solutions of the nonhomogeneous linear differential equation (1), then their difference $Y_1 - Y_2$ is a solution of the corresponding homogeneous differential equation (2). If, in addition, y_1 and y_2 form a fundamental set of solutions of equation (2), then

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t), \tag{3}$$

where c_1 and c_2 are certain constants.

To prove this result, note that Y_1 and Y_2 satisfy the equations

$$L[Y_1](t) = g(t), \quad L[Y_2](t) = g(t). \tag{4}$$

Subtracting the second of these equations from the first, we have

$$L[Y_1](t) - L[Y_2](t) = g(t) - g(t) = 0. \tag{5}$$

However,

$$L[Y_1] - L[Y_2] = L[Y_1 - Y_2],$$

so equation (5) becomes

$$L[Y_1 - Y_2](t) = 0. \tag{6}$$

Equation (6) states that $Y_1 - Y_2$ is a solution of equation (2). Finally, since by Theorem 3.2.4 all solutions of equation (2) can be expressed as linear combinations of a fundamental set of solutions, it follows that the solution $Y_1 - Y_2$ can be so written. Hence equation (3) holds and the proof is complete.

Theorem 3.5.2

The general solution of the nonhomogeneous equation (1) can be written in the form

$$y = \phi(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t), \tag{7}$$

where y_1 and y_2 form a fundamental set of solutions of the corresponding homogeneous equation (2), c_1 and c_2 are arbitrary constants, and Y is any solution of the nonhomogeneous equation (1).

The proof of Theorem 3.5.2 follows quickly from Theorem 3.5.1. Note that equation (3) holds if we identify Y_1 with an arbitrary solution ϕ of equation (1) and Y_2 with the specific solution Y. From equation (3) we thereby obtain

$$\phi(t) - Y(t) = c_1 y_1(t) + c_2 y_2(t), \tag{8}$$

which is equivalent to equation (7). Since ϕ is an arbitrary solution of equation (1), the expression on the right-hand side of equation (7) includes all solutions of equation (1); thus it is natural to call it the general solution of equation (1).

In somewhat different words, Theorem 3.5.2 states that to solve the nonhomogeneous equation (1), we must do three things:

1. Find the general solution $c_1 y_1(t) + c_2 y_2(t)$ of the corresponding homogeneous equation. This solution is frequently called the **complementary solution** and may be denoted by $y_c(t)$.
2. Find any solution $Y(t)$ of the nonhomogeneous equation. Often this solution is referred to as a **particular solution**.
3. Form the sum of the functions found in steps 1 and 2.

We have already discussed how to find $y_c(t)$, at least when the homogeneous equation (2) has constant coefficients. Therefore, in the remainder of this section and Section 3.6, we will focus on finding a particular solution $Y(t)$ of the nonhomogeneous linear differential equation (1). There are two methods that we wish to discuss. They are known as the method of undetermined coefficients (discussed here) and the method of variation of parameters (see Section 3.6). Each has some advantages and some possible shortcomings.

Method of Undetermined Coefficients. The method of undetermined coefficients requires us to make an initial assumption about the form of the particular solution $Y(t)$, but with the coefficients left unspecified. We then substitute the assumed expression into the nonhomogeneous differential equation (1) and attempt to determine the coefficients so as to satisfy that equation. If we are successful, then we have found a solution of the differential equation (1) and can use it for the particular solution $Y(t)$. If we cannot determine the coefficients, then this means that there is no solution of the form that we assumed. In this case we may modify the initial assumption and try again.

The main advantage of the method of undetermined coefficients is that it is straightforward to execute once the assumption is made about the form of $Y(t)$. Its major limitation is that it is useful primarily for equations for which we can easily write down the correct form of the particular solution in advance. For this reason, this method is usually used only for problems in which the homogeneous equation has constant coefficients and the nonhomogeneous term is restricted to a relatively small class of functions. In particular, we consider only nonhomogeneous terms that consist of polynomials, exponential functions, sines, and cosines. Despite this limitation, the method of undetermined coefficients is useful for solving many problems that have important applications. However, the algebraic details may become tedious, and a computer algebra system can be very helpful in practical applications. We will illustrate the method of undetermined coefficients by several simple examples and then summarize some rules for using it.

EXAMPLE 1

Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t}. \tag{9}$$

Solution:

We seek a function Y such that the combination $Y''(t) - 3Y'(t) - 4Y(t)$ is equal to $3e^{2t}$. Since the exponential function reproduces itself through differentiation, the most plausible way to achieve the desired result is to assume that $Y(t)$ is some multiple of e^{2t},

$$Y(t) = Ae^{2t},$$

where the coefficient A is yet to be determined. To find A, we calculate the first two derivatives of Y:

$$Y'(t) = 2Ae^{2t}, \quad Y''(t) = 4Ae^{2t},$$

and substitute for y, y', and y'' in the nonhomogeneous differential equation (9). We obtain

$$Y'' - 3Y' - 4Y = (4A - 6A - 4A)e^{2t} = 3e^{2t}.$$

Hence $-6Ae^{2t}$ must equal $3e^{2t}$, so $-6A = 3$ and we conclude that $A = -\frac{1}{2}$. Thus a particular solution is

$$Y(t) = -\frac{1}{2}e^{2t}. \tag{10}$$

EXAMPLE 2

Find a particular solution of

$$y'' - 3y' - 4y = 2\sin t. \tag{11}$$

Solution:

By analogy with Example 1, let us assume that $Y(t) = A\sin t$, where A is a constant to be determined. On substituting this guess in equation (11) we obtain

$$Y'' - 3Y' - 4Y = -A\sin t - 3A\cos t - 4A\sin t = 2\sin t,$$

or, moving all terms to the left-hand side and collecting the terms involving $\sin t$ and $\cos t$, we arrive at,

$$(2 + 5A)\sin t + 3A\cos t = 0. \tag{12}$$

We want equation (12) to hold for all t. Thus it must hold for two specific points, such as $t = 0$ and $t = \frac{\pi}{2}$. At these points equation (12) reduces to $3A = 0$ and $2 + 5A = 0$, respectively. These contradictory requirements mean that there is no choice of the constant A that makes equation (12) true for $t = 0$ and $t = \frac{\pi}{2}$, much less for all t. Thus we conclude that our assumption concerning $Y(t)$ is inadequate.

The appearance of the cosine term in equation (12) suggests that we modify our original assumption to include a cosine term in $Y(t)$; that is,

$$Y(t) = A\sin t + B\cos t,$$

where A and B are the undetermined coefficients. Then

$$Y'(t) = A\cos t - B\sin t, \quad Y''(t) = -A\sin t - B\cos t.$$

By substituting these expressions for y, y', and y'' in equation (11) and collecting terms, we obtain

$$Y'' - 3Y' - 4Y = (-A + 3B - 4A)\sin t + (-B - 3A - 4B)\cos t = 2\sin t. \tag{13}$$

Now, working exactly as with the first guess, move all terms to the left-hand side and evaluate $t = 0$ and $t = \frac{\pi}{2}$ to find that A and B must satisfy the equations

$$-5A + 3B - 2 = 0, \quad -3A - 5B = 0.$$

Solving these algebraic equations for A and B, we obtain $A = -\frac{5}{17}$ and $B = \frac{3}{17}$; hence a particular solution of equation (11) is

$$Y(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t.$$

The method illustrated in the preceding examples can also be used when the right-hand side of the equation is a polynomial. Thus, to find a particular solution of

$$y'' - 3y' - 4y = 4t^2 - 1, \tag{14}$$

we initially assume that $Y(t)$ is a polynomial of the same degree as the nonhomogeneous term; that is, $Y(t) = At^2 + Bt + C$.

To summarize our conclusions up to this point: if the nonhomogeneous term $g(t)$ in differential equation (1) is an exponential function $e^{\alpha t}$, then assume that $Y(t)$ is proportional to the same exponential function; if $g(t)$ is $\sin(\beta t)$ or $\cos(\beta t)$, then assume that $Y(t)$ is a linear combination of $\sin(\beta t)$ and $\cos(\beta t)$; if $g(t)$ is a polynomial of degree n, then assume that $Y(t)$ is a polynomial of degree n. The same principle extends to the case where $g(t)$ is a product of any two, or all three, of these types of functions, as the next example illustrates.

EXAMPLE 3

Find a particular solution of

$$y'' - 3y' - 4y = -8e^t\cos(2t). \tag{15}$$

Solution:

In this case we assume that $Y(t)$ is the product of e^t and a linear combination of $\cos(2t)$ and $\sin(2t)$; that is,

$$Y(t) = Ae^t \cos(2t) + Be^t \sin(2t).$$

The algebra is more tedious in this example, but it follows that

$$Y'(t) = (A + 2B)e^t \cos(2t) + (-2A + B)e^t \sin(2t)$$

and

$$Y''(t) = (-3A + 4B)e^t \cos(2t) + (-4A - 3B)e^t \sin(2t).$$

By substituting these expressions in equation (15), we find that A and B must satisfy

$$10A + 2B = 8, \quad 2A - 10B = 0.$$

Hence $A = \frac{10}{13}$ and $B = \frac{2}{13}$; therefore, a particular solution of equation (15) is

$$Y(t) = \frac{10}{13}e^t \cos(2t) + \frac{2}{13}e^t \sin(2t).$$

Now suppose that $g(t)$ is the sum of two terms, $g(t) = g_1(t) + g_2(t)$, and suppose that Y_1 and Y_2 are solutions of the equations

$$ay'' + by' + cy = g_1(t) \tag{16}$$

and

$$ay'' + by' + cy = g_2(t), \tag{17}$$

respectively. Then $Y_1 + Y_2$ is a solution of the equation

$$ay'' + by' + cy = g(t). \tag{18}$$

To prove this statement, substitute $Y_1(t) + Y_2(t)$ for y in equation (18) and make use of equations (16) and (17). A similar conclusion holds if $g(t)$ is the sum of any finite number of terms. The practical significance of this result is that for an equation whose nonhomogeneous function $g(t)$ can be expressed as a sum, you can consider instead several simpler equations and then add together the results. The following example is an illustration of this procedure.

EXAMPLE 4

Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos(2t). \tag{19}$$

Solution:

By splitting up the right-hand side of equation (19), we obtain the three equations

$$y'' - 3y' - 4y = 3e^{2t},$$
$$y'' - 3y' - 4y = 2\sin t,$$

and

$$y'' - 3y' - 4y = -8e^t \cos(2t).$$

Solutions of these three equations have been found in Examples 1, 2, and 3, respectively. Therefore, a particular solution of equation (19) is their sum, namely,

$$Y(t) = -\frac{1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^t \cos(2t) + \frac{2}{13}e^t \sin(2t).$$

The procedure illustrated in these examples enables us to solve a fairly large class of problems in a reasonably efficient manner. However, there is one difficulty that sometimes occurs. The next example illustrates how it arises.

EXAMPLE 5

Find a particular solution of

$$y'' - 3y' - 4y = 2e^{-t}. \tag{20}$$

Solution:

Proceeding as in Example 1, we assume that $Y(t) = Ae^{-t}$. By substituting in equation (20), we obtain

$$Y'' - 3Y' - 4Y = (A + 3A - 4A)e^{-t} = 2e^{-t}. \tag{21}$$

Since the left-hand side of equation (21) is zero, there is no choice of A for which $0 = 2e^{-t}$. Therefore, there is no particular solution of equation (20) of the assumed form. The reason for this possibly unexpected result becomes clear if we solve the homogeneous equation

$$y'' - 3y' - 4y = 0 \tag{22}$$

that corresponds to equation (20). The two functions in a fundamental set of solutions of equation (22) are $y_1(t) = e^{-t}$ and $y_2(t) = e^{4t}$. Thus our assumed particular solution of equation (20) is actually a solution of the homogeneous equation (22); consequently, it cannot possibly be a solution of the nonhomogeneous equation (20). To find a solution of equation (20), we must therefore consider functions of a somewhat different form.

At this stage, we have several possible alternatives. One is simply to try to guess the proper form of the particular solution of equation (20). Another is to solve this equation in some different way and then to use the result to guide our assumptions if this situation arises again in the future; see Problems 22 and 27 for other solution methods. Still another possibility is to seek a simpler equation where this difficulty occurs and to use its solution to suggest how we might proceed with equation (20). Adopting the latter approach, we look for a first-order equation analogous to equation (20). One possibility is the linear equation

$$y' + y = 2e^{-t}. \tag{23}$$

If we try to find a particular solution of equation (23) of the form Ae^{-t}, we will fail because e^{-t} is a solution of the homogeneous equation $y' + y = 0$. However, from Section 2.1 we already know how to solve equation (23). An integrating factor is $\mu(t) = e^t$, and by multiplying by $\mu(t)$ and then integrating both sides, we obtain the solution

$$y = 2te^{-t} + ce^{-t}. \tag{24}$$

The second term on the right-hand side of equation (24) is the general solution of the homogeneous equation $y' + y = 0$, but the first term is a solution of the full nonhomogeneous equation (23). Observe that it involves the exponential factor e^{-t} multiplied by the factor t. This is the clue that we were looking for.

We now return to equation (20) and assume a particular solution of the form $Y(t) = Ate^{-t}$. Then

$$Y'(t) = Ae^{-t} - Ate^{-t}, \quad Y''(t) = -2Ae^{-t} + Ate^{-t}. \tag{25}$$

Substituting these expressions for y, y', and y'' in equation (20), we obtain

$$Y'' - 3Y' - 4Y = (-2A - 3A)e^{-t} + (A + 3A - 4A)te^{-t} = 2e^{-t}.$$

The coefficient of te^{-t} is zero, and from the terms involving e^t we have $-5A = 2$, so $A = -\frac{2}{5}$. Thus a particular solution of equation (20) is

$$Y(t) = -\frac{2}{5}te^{-t}. \tag{26}$$

The outcome of Example 5 suggests a modification of the principle stated previously: if the assumed form of the particular solution duplicates a solution of the corresponding homogeneous equation, then modify the assumed particular solution by multiplying it by t. Occasionally, this modification will be insufficient to remove all duplication with the solutions of the homogeneous equation, in which case it is necessary to multiply by t a second time. For a second-order equation, it will never be necessary to carry the process further than this.

Summary. We now summarize the steps involved in finding the solution of an initial value problem consisting of a nonhomogeneous linear differential equation of the form

$$ay'' + by' + cy = g(t), \tag{27}$$

where the coefficients a, b, and c are constants, together with a given set of initial conditions.

1. Find the general solution of the corresponding homogeneous equation.
2. Make sure that the function $g(t)$ in equation (27) belongs to the class of functions discussed in this section; that is, be sure it involves nothing more than exponential functions, sines, cosines, polynomials, or sums or products of such functions. If this is not the case, use the method of variation of parameters (discussed in Section 3.6).
3. If $g(t) = g_1(t) + \cdots + g_n(t)$—that is, if $g(t)$ is a sum of n terms—then form n subproblems, each of which contains only one of the terms $g_1(t), \ldots, g_n(t)$. The i^{th} subproblem consists of the equation
$$ay'' + by' + cy = g_i(t),$$
where i runs from 1 to n.
4. For the i^{th} subproblem, assume a particular solution $Y_i(t)$ consisting of the appropriate exponential function, sine, cosine, polynomial, or combination thereof. If there is any duplication in the assumed form of $Y_i(t)$ with the solutions of the homogeneous equation (found in step 1), then multiply $Y_i(t)$ by t, or (if necessary) by t^2, so as to remove the duplication. See Table 3.5.1.

TABLE 3.5.1 The Particular Solution of $ay'' + by' + cy = g_i(t)$

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\sin\beta t\\ \cos\beta t\end{cases}$	$t^s\big((A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}\cos(\beta t) + (B_0t^n + B_1t^{n-1} + \cdots + B_n)e^{\alpha t}\sin(\beta t)\big)$

Notes: Here, s is the smallest nonnegative integer ($s = 0$, 1, or 2) that will ensure that no term in $Y_i(t)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, s is the number of times 0 is a root of the characteristic equation, α is a root of the characteristic equation, and $\alpha + i\beta$ is a root of the characteristic equation, respectively.

5. Find a particular solution $Y_i(t)$ for each of the subproblems. Then $Y_1(t) + \cdots + Y_n(t)$ is a particular solution of the full nonhomogeneous equation (27).
6. Form the sum of the general solution of the homogeneous equation (step 1) and the particular solution of the nonhomogeneous equation (step 5). This is the general solution of the nonhomogeneous equation.
7. When initial conditions are provided, use them to determine the values of the arbitrary constants remaining in the general solution.

For some problems this entire procedure is easy to carry out by hand, but often the algebraic calculations are lengthy. Once you understand clearly how the method works, a computer algebra system can be of great assistance in executing the details.

The method of undetermined coefficients is self-correcting in the sense that if you assume too little for $Y(t)$, then a contradiction is soon reached that usually points the way to the modification that is needed in the assumed form. On the other hand, if you assume too many terms, then some unnecessary work is done and some coefficients turn out to be zero, but at least the correct answer is obtained.

Proof of the Method of Undetermined Coefficients. In the preceding discussion we have described the method of undetermined coefficients on the basis of several examples. To prove that the procedure always works as stated, we now give a general argument, in which we consider three cases corresponding to different forms for the nonhomogeneous term $g(t)$.

Case 1: $g(t) = P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$. In this case equation (27) becomes
$$ay'' + by' + cy = a_0t^n + a_1t^{n-1} + \cdots + a_n. \tag{28}$$

To obtain a particular solution, we assume that

$$Y(t) = A_0 t^n + A_1 t^{n-1} + \cdots + A_{n-2} t^2 + A_{n-1} t + A_n. \tag{29}$$

Substituting in equation (28), we obtain

$$\begin{aligned} a\Big(n(n-1)A_0 t^{n-2} + \cdots + 2A_{n-2}\Big) &+ b(nA_0 t^{n-1} + \cdots + A_{n-1}) \\ &+ c(A_0 t^n + A_1 t^{n-1} + \cdots + A_n) = a_0 t^n + \cdots + a_n. \end{aligned} \tag{30}$$

Equating the coefficients of like powers of t, beginning with t^n, leads to the following sequence of equations:

$$\begin{aligned} cA_0 &= a_0, \\ cA_1 + nbA_0 &= a_1, \\ &\vdots \\ cA_n + bA_{n-1} + 2aA_{n-2} &= a_n. \end{aligned}$$

Provided that $c \neq 0$, the solution of the first equation is $A_0 = a_0/c$, and the remaining equations determine $A_1, \ldots, A_n$ successively.

If $c = 0$ but $b \neq 0$, then the polynomial on the left-hand side of equation (30) is of degree $n-1$, and we cannot satisfy equation (30). To be sure that $aY''(t) + bY'(t)$ is a polynomial of degree n, we must choose $Y(t)$ to be a polynomial of degree $n+1$. Hence we assume that

$$Y(t) = t(A_0 t^n + \cdots + A_n).$$

Substituting this guess into equation (28), with $c = 0$, and simplifying yields

$$\begin{aligned} aY'' + bY' &= bA_0(n+1)t^n + (aA_0(n+1)n + bA_1 n)t^{n-1} + \cdots \\ &= a_0 t^n + a_1 t^{n-1} + \cdots + a_n. \end{aligned}$$

There is no constant term in this expression for $Y(t)$, but there is no need to include such a term since a constant is a solution of the homogeneous equation when $c = 0$. Since $b \neq 0$, we find $A_0 = a_0/(b(n+1))$, and the other coefficients $A_1, \ldots, A_n$ can be determined similarly.

If both c and b are zero, then the characteristic equation is $ar^2 = 0$ and $r = 0$ is a repeated root. Thus $y_1 = e^{0t} = 1$ and $y_2 = te^{0t} = t$ form a fundamental set of solutions of the corresponding homogeneous equation. This leads us to assume that

$$Y(t) = t^2(A_0 t^n + \cdots + A_n).$$

The term $aY''(t)$ gives rise to a term of degree n, and we can proceed as before. Again the constant and linear terms in $Y(t)$ are omitted since, in this case, they are both solutions of the homogeneous equation.

Case 2: $g(t) = e^{\alpha t} P_n(t)$. The problem of determining a particular solution of

$$ay'' + by' + cy = e^{\alpha t} P_n(t) \tag{31}$$

can be reduced to the preceding case by a substitution. Let

$$Y(t) = e^{\alpha t} u(t);$$

then

$$Y'(t) = e^{\alpha t}(u'(t) + \alpha u(t))$$

and

$$Y''(t) = e^{\alpha t}(u''(t) + 2\alpha u'(t) + \alpha^2 u(t)).$$

Substituting for y, y', and y'' in equation (31), canceling the factor $e^{\alpha t}$, and collecting terms, we obtain

$$au''(t) + (2a\alpha + b)u'(t) + (a\alpha^2 + b\alpha + c)u(t) = P_n(t). \tag{32}$$

The determination of a particular solution of equation (32) is precisely the same problem, except for the names of the constants, as solving equation (28). Therefore, if $a\alpha^2 + b\alpha + c$ is not zero, we assume that $u(t) = A_0 t^n + \cdots + A_n$; hence a particular solution of equation (31) is of the form

$$Y(t) = e^{\alpha t}(A_0 t^n + A_1 t^{n-1} + \cdots + A_n). \tag{33}$$

On the other hand, if $a\alpha^2 + b\alpha + c$ is zero but $2a\alpha + b$ is not, we must take $u(t)$ to be of the form $t(A_0t^n + \cdots + A_n)$. The corresponding form for $Y(t)$ is t times the expression on the right-hand side of equation (33). Note that if $a\alpha^2 + b\alpha + c$ is zero, then $e^{\alpha t}$ is a solution of the homogeneous equation.

If both $a\alpha^2 + b\alpha + c$ and $2a\alpha + b$ are zero (and this implies that both $e^{\alpha t}$ and $te^{\alpha t}$ are solutions of the homogeneous equation), then the correct form for $u(t)$ is $t^2(A_0t^n + \cdots + A_n)$. Hence $Y(t)$ is t^2 times the expression on the right-hand side of equation (33).

Case 3: $g(t) = e^{\alpha t}P_n(t)\cos(\beta t)$ or $e^{\alpha t}P_n(t)\sin(\beta t)$. These two cases are similar, so we consider only the latter. We can reduce this problem to the preceding one by noting that, as a consequence of Euler's formula, $\sin(\beta t) = (e^{i\beta t} - e^{-i\beta t})/(2i)$. Hence $g(t)$ is of the form

$$g(t) = P_n(t)\frac{e^{(\alpha+i\beta)t} - e^{(\alpha-i\beta)t}}{2i},$$

and we should choose

$$Y(t) = e^{(\alpha+i\beta)t}(A_0t^n + \cdots + A_n) + e^{(\alpha-i\beta)t}(B_0t^n + \cdots + B_n),$$

or, equivalently,

$$Y(t) = e^{\alpha t}(A_0t^n + \cdots + A_n)\cos(\beta t) + e^{\alpha t}(B_0t^n + \cdots + B_n)\sin(\beta t).$$

Usually, the latter form is preferred because it does not involve the use of complex-valued coefficients. If $\alpha \pm i\beta$ satisfy the characteristic equation corresponding to the homogeneous equation, we must, of course, multiply each of the polynomials by t to increase their degrees by 1.

If the nonhomogeneous function involves both $\cos(\beta t)$ and $\sin(\beta t)$, it is usually convenient to treat these terms together, since each one individually may give rise to the same form for a particular solution. For example, if $g(t) = t\sin t + 2\cos t$, the form for $Y(t)$ would be

$$Y(t) = (A_0t + A_1)\sin t + (B_0t + B_1)\cos t,$$

provided that $\sin t$ and $\cos t$ are not solutions of the homogeneous equation.

Problems

In each of Problems 1 through 10, find the general solution of the given differential equation.

1. $y'' - 2y' - 3y = 3e^{2t}$
2. $y'' - y' - 2y = -2t + 4t^2$
3. $y'' + y' - 6y = 12e^{3t} + 12e^{-2t}$
4. $y'' - 2y' - 3y = -3te^{-t}$
5. $y'' + 2y' = 3 + 4\sin(2t)$
6. $y'' + 2y' + y = 2e^{-t}$
7. $y'' + y = 3\sin(2t) + t\cos(2t)$
8. $u'' + \omega_0^2 u = \cos(\omega t), \quad \omega^2 \neq \omega_0^2$
9. $u'' + \omega_0^2 u = \cos(\omega_0 t)$
10. $y'' + y' + 4y = 2\sinh t \qquad$ *Hint:* $\sinh t = (e^t - e^{-t})/2$

In each of Problems 11 through 15, find the solution of the given initial value problem.

11. $y'' + y' - 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1$
12. $y'' + 4y = t^2 + 3e^t, \quad y(0) = 0, \quad y'(0) = 2$
13. $y'' - 2y' + y = te^t + 4, \quad y(0) = 1, \quad y'(0) = 1$
14. $y'' + 4y = 3\sin(2t), \quad y(0) = 2, \quad y'(0) = -1$
15. $y'' + 2y' + 5y = 4e^{-t}\cos(2t), \quad y(0) = 1, \quad y'(0) = 0$

In each of Problems 16 through 21:

a. Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.

N **b.** Use a computer algebra system to find a particular solution of the given equation.

16. $y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin(3t)$
17. $y'' - 5y' + 6y = e^t\cos(2t) + e^{2t}(3t + 4)\sin t$
18. $y'' + 2y' + 2y = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2\sin t$
19. $y'' + 4y = t^2\sin(2t) + (6t + 7)\cos(2t)$
20. $y'' + 3y' + 2y = e^t(t^2 + 1)\sin(2t) + 3e^{-t}\cos t + 4e^t$
21. $y'' + 2y' + 5y = 3te^{-t}\cos(2t) - 2te^{-2t}\cos t$
22. Consider the equation

$$y'' - 3y' - 4y = 2e^{-t} \qquad (34)$$

from Example 5. Recall that $y_1(t) = e^{-t}$ and $y_2(t) = e^{4t}$ are solutions of the corresponding homogeneous equation. Adapting the method of reduction of order (Section 3.4), seek a solution of the nonhomogeneous equation of the form $Y(t) = v(t)y_1(t) = v(t)e^{-t}$, where $v(t)$ is to be determined.

a. Substitute $Y(t)$, $Y'(t)$, and $Y''(t)$ into equation (34) and show that $v(t)$ must satisfy $v'' - 5v' = 2$.

b. Let $w(t) = v'(t)$ and show that $w(t)$ satisfies $w' - 5w = 2$. Solve this equation for $w(t)$.

c. Integrate $w(t)$ to find $v(t)$ and then show that

$$Y(t) = -\frac{2}{5}te^{-t} + \frac{1}{5}c_1e^{4t} + c_2e^{-t}.$$

The first term on the right-hand side is the desired particular solution of the nonhomogeneous equation. Note that it is a product of t and e^{-t}.

23. Determine the general solution of

$$y'' + \lambda^2 y = \sum_{m=1}^{N} a_m \sin(m\pi t),$$

where $\lambda > 0$ and $\lambda \neq m\pi$ for $m = 1, \ldots, N$.

N 24. In many physical problems the nonhomogeneous term may be specified by different formulas in different time periods. As an example, determine the solution $y = \phi(t)$ of

$$y'' + y = \begin{cases} t, & 0 \le t \le \pi, \\ \pi e^{\pi - t}, & t > \pi, \end{cases}$$

satisfying the initial conditions $y(0) = 0$ and $y'(0) = 1$. Assume that y and y' are also continuous at $t = \pi$. Plot the nonhomogeneous term and the solution as functions of time. *Hint:* First solve the initial value problem for $t \le \pi$; then solve for $t > \pi$, determining the constants in the latter solution from the continuity conditions at $t = \pi$.

Behavior of Solutions as $t \to \infty$. In Problems 25 and 26, we continue the discussion started with Problems 28 through 30 of Section 3.4. Consider the differential equation

$$ay'' + by' + cy = g(t), \tag{35}$$

where a, b, and c are positive.

25. If $Y_1(t)$ and $Y_2(t)$ are solutions of equation (35), show that $Y_1(t) - Y_2(t) \to 0$ as $t \to \infty$. Is this result true if $b = 0$?

26. If $g(t) = d$, a constant, show that every solution of equation (35) approaches d/c as $t \to \infty$. What happens if $c = 0$? What if $b = 0$ also?

27. In this problem we indicate an alternative procedure[8] for solving the differential equation

$$y'' + by' + cy = (D^2 + bD + c)y = g(t), \tag{36}$$

where b and c are constants, and D denotes differentiation with respect to t. Let r_1 and r_2 be the zeros of the characteristic polynomial of the corresponding homogeneous equation. These roots may be real and different, real and equal, or conjugate complex numbers.

a. Verify that equation (36) can be written in the factored form

$$(D - r_1)(D - r_2)y = g(t),$$

where $r_1 + r_2 = -b$ and $r_1r_2 = c$.

b. Let $u = (D - r_2)y$. Then show that the solution of equation (36) can be found by solving the following two first-order equations:

$$(D - r_1)u = g(t), \quad (D - r_2)y = u(t).$$

In each of Problems 28 through 30, use the method of Problem 27 to solve the given differential equation.

28. $y'' - 3y' - 4y = 3e^{2t}$ (see Example 1)

29. $y'' + 2y' + y = 2e^{-t}$ (see Problem 6)

30. $y'' + 2y' = 3 + 4\sin(2t)$ (see Problem 5)

[8] R. S. Luthar, "Another Approach to a Standard Differential Equation," *Two Year College Mathematics Journal 10* (1979), pp. 200–201. Also see D. C. Sandell and F. M. Stein, "Factorization of Operators of Second-Order Linear Homogeneous Ordinary Differential Equations," *Two Year College Mathematics Journal 8* (1977), pp. 132–141, for a more general discussion of factoring operators.

3.6 Variation of Parameters

In this section we describe a second method of finding a particular solution of a nonhomogeneous equation. This method, **variation of parameters**, is due to Lagrange and complements the method of undetermined coefficients rather well. The main advantage of variation of parameters is that it is a *general method*; in principle at least, it can be applied to any equation, and it requires no detailed assumptions about the form of the solution. In fact, later in this section we use this method to derive a formula for a particular solution of an arbitrary second-order linear nonhomogeneous differential equation. On the other hand, the method of variation of parameters eventually requires us to evaluate certain integrals involving the nonhomogeneous term in the differential equation, and this may present difficulties. Before looking at this method in the general case, we illustrate its use in an example.

EXAMPLE 1

Find the general solution of

$$y'' + 4y = 8\tan t \quad -\pi/2 < t < \pi/2. \tag{1}$$

Solution:

Observe that this problem is not a good candidate for the method of undetermined coefficients, as described in Section 3.5, because the nonhomogeneous term $g(t) = 8\tan t$ involves a quotient (rather than a sum or a product) of $\sin t$ and $\cos t$. Therefore, the method of undetermined coefficients cannot be applied; we need a different approach.

Observe also that the homogeneous equation corresponding to equation (1) is

$$y'' + 4y = 0, \tag{2}$$

and that the general solution of equation (2) is

$$y_c(t) = c_1\cos(2t) + c_2\sin(2t). \tag{3}$$

The basic idea in the method of variation of parameters is similar to the method of reduction of order introduced at the end of Section 3.4. In the general solution (3), replace the constants c_1 and c_2 by functions $u_1(t)$ and $u_2(t)$, respectively, and then determine these functions so that the resulting expression

$$y = u_1(t)\cos(2t) + u_2(t)\sin(2t) \tag{4}$$

is a solution of the nonhomogeneous equation (1).

To determine u_1 and u_2, we need to substitute for y from equation (4) in differential equation (1). However, even without carrying out this substitution, we can anticipate that the result will be a single equation involving some combination of u_1, u_2, and their first two derivatives. Since there is only one equation and two unknown functions, we can expect that there are many possible choices of u_1 and u_2 that will meet our needs. Alternatively, we may be able to impose a second condition of our own choosing, thereby obtaining two equations for the two unknown functions u_1 and u_2. We will soon show (following Lagrange) that it is possible to choose this second condition in a way that makes the computation markedly more efficient.[9]

Returning now to equation (4), we differentiate it and rearrange the terms, thereby obtaining

$$y' = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t) + u_1'(t)\cos(2t) + u_2'(t)\sin(2t). \tag{5}$$

Keeping in mind the possibility of choosing a second condition on u_1 and u_2, let us require the sum of the last two terms on the right-hand side of equation (5) to be zero; that is, we require that

$$u_1'(t)\cos(2t) + u_2'(t)\sin(2t) = 0. \tag{6}$$

It then follows from equation (5) that

$$y' = -2u_1(t)\sin(2t) + 2u_2(t)\cos(2t). \tag{7}$$

Although the ultimate effect of the condition (6) is not yet clear, the removal of the terms involving u_1' and u_2' has simplified the expression for y'. Further, by differentiating equation (7), we obtain

$$y'' = -4u_1(t)\cos(2t) - 4u_2(t)\sin(2t) - 2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t). \tag{8}$$

Then, substituting for y and y'' in equation (1) from equations (4) and (8), respectively, we find that

$$\begin{aligned} y'' + 4y = &- 4u_1(t)\cos(2t) - 4u_2(t)\sin(2t) - 2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t) \\ &+ 4u_1(t)\cos(2t) + 4u_2(t)\sin(2t) = 8\tan t. \end{aligned}$$

Hence u_1 and u_2 must satisfy

$$-2u_1'(t)\sin(2t) + 2u_2'(t)\cos(2t) = 8\tan t. \tag{9}$$

Summarizing our results to this point, we want to choose u_1 and u_2 so as to satisfy equations (6) and (9). These equations can be viewed as a pair of linear *algebraic* equations for the two unknown quantities $u_1'(t)$ and $u_2'(t)$. Equations (6) and (9) can be solved in various ways. For example, solving equation (6) for $u_2'(t)$, we have

$$u_2'(t) = -u_1'(t)\frac{\cos(2t)}{\sin(2t)}. \tag{10}$$

Then, substituting for $u_2'(t)$ in equation (9) and simplifying, we obtain

$$u_1'(t) = -\frac{8\tan t\sin(2t)}{2} = -8\sin^2 t. \tag{11}$$

[9] An alternate, and more mathematically appealing, derivation of the second condition can be found in Problems 17 to 19 in Section 7.9.

Further, putting this expression for $u_1'(t)$ back in equation (10) and using the double-angle formulas, we find that

$$u_2'(t) = \frac{8\sin^2 t\cos(2t)}{\sin(2t)} = 4\frac{\sin t(2\cos^2 t - 1)}{\cos t} = 4\sin t\left(2\cos t - \frac{1}{\cos t}\right). \tag{12}$$

Having obtained $u_1'(t)$ and $u_2'(t)$, we next integrate so as to find $u_1(t)$ and $u_2(t)$. The result is

$$u_1(t) = 4\sin t\cos t - 4t + c_1 \tag{13}$$

and

$$u_2(t) = 4\ln(\cos t) - 4\cos^2 t + c_2. \tag{14}$$

On substituting these expressions in equation (4), we have

$$y = (4\sin t\cos t)\cos(2t) + (4\ln(\cos t) - 4\cos^2 t)\sin(2t) + c_1\cos(2t) + c_2\sin(2t).$$

Finally, by using the double-angle formulas once more, we obtain

$$y = -2\sin(2t) - 4t\cos(2t) + 4\ln(\cos t)\sin(2t) + c_1\cos(2t) + c_2\sin(2t). \tag{15}$$

The terms in equation (15) involving the arbitrary constants c_1 and c_2 are the general solution of the corresponding homogeneous equation, while the other three terms are a particular solution of the nonhomogeneous equation (1). Thus equation (15) is the general solution of equation (1).

The particular solution identified at the end of Example 1 corresponds to choosing both c_1, and c_2 to be zero in equation (15). Any other choice of c_1 and c_2 is also a particular solution of the same nonhomogeneous differential equation. Notice, in particular, that choosing $c_1 = 0$ and $c_2 = 2$ in equation (15) yields a particular solution with only two terms:

$$-4t\cos(2t) + 4\ln(\cos t)\sin(2t).$$

We conclude this first look at the method of variation of parameters with the observation that the particular solution involves terms that might be difficult to anticipate. This explains why the method of undetermined coefficients is not a good candidate for this problem, and why the method of variation of parameters is needed.

In the preceding example the method of variation of parameters worked well in determining a particular solution, and hence the general solution, of equation (1). The next question is whether this method can be applied effectively to an arbitrary equation. Therefore, we consider

$$y'' + p(t)y' + q(t)y = g(t), \tag{16}$$

where p, q, and g are given continuous functions. As a starting point, we assume that we know the general solution

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t) \tag{17}$$

of the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{18}$$

This is a major assumption. So far we have shown how to solve equation (18) only if it has constant coefficients. If equation (18) has coefficients that depend on t, then usually the methods described in Chapter 5 must be used to obtain $y_c(t)$.

The crucial idea, as illustrated in Example 1, is to replace the constants c_1 and c_2 in equation (17) by functions $u_1(t)$ and $u_2(t)$, respectively; thus we have

$$y = u_1(t)y_1(t) + u_2(t)y_2(t). \tag{19}$$

Then we try to determine $u_1(t)$ and $u_2(t)$ so that the expression in equation (19) is a solution of the nonhomogeneous equation (16) rather than the homogeneous equation (18). Thus we differentiate equation (19), obtaining

$$y' = u_1'(t)y_1(t) + u_1(t)y_1'(t) + u_2'(t)y_2(t) + u_2(t)y_2'(t). \tag{20}$$

As in Example 1, we now set the terms involving $u_1'(t)$ and $u_2'(t)$ in equation (20) equal to zero; that is, we require that

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0. \tag{21}$$

Then, from equation (20), we have

$$y' = u_1(t)y_1'(t) + u_2(t)y_2'(t). \tag{22}$$

Further, by differentiating again, we obtain

$$y'' = u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t). \tag{23}$$

Now we substitute for y, y', and y'' in equation (16) from equations (19), (22), and (23), respectively. After rearranging the terms in the resulting equation, we find that

$$\begin{aligned} &u_1(t)\Big(y_1''(t) \;+\; p(t)y_1'(t) + q(t)y_1(t)\Big) \\ &+ u_2(t)\Big(y_2''(t) \;+\; p(t)y_2'(t) + q(t)y_2(t)\Big) \\ &\qquad + u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t). \end{aligned} \tag{24}$$

Each of the expressions in parentheses in the first two lines of equation (24) is zero because both y_1 and y_2 are solutions of the homogeneous equation (18). Therefore, equation (24) reduces to

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t). \tag{25}$$

Equations (21) and (25) form a system of two linear algebraic equations for the derivatives $u_1'(t)$ and $u_2'(t)$ of the unknown functions. They correspond exactly to equations (6) and (9) in Example 1.

Solving the system of equations (21), (25), we obtain

$$u_1'(t) = -\frac{y_2(t)g(t)}{W[y_1, y_2](t)}, \qquad u_2'(t) = \frac{y_1(t)g(t)}{W[y_1, y_2](t)}, \tag{26}$$

where $W[y_1, y_2]$ is the Wronskian of y_1 and y_2. Note that division by $W[y_1, y_2]$ is permissible since y_1 and y_2 are a fundamental set of solutions, and therefore their Wronskian is nonzero. By integrating equations (26), we find the desired functions $u_1(t)$ and $u_2(t)$, namely,

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W[y_1, y_2](t)}\,dt + c_1, \qquad u_2(t) = \int \frac{y_1(t)g(t)}{W[y_1, y_2](t)}\,dt + c_2. \tag{27}$$

If the integrals in equations (27) can be evaluated in terms of elementary functions, then we substitute the results in equation (19), thereby obtaining the general solution of equation (16). More generally, the solution can always be expressed in terms of integrals, as stated in the following theorem.

Theorem 3.6.1

Consider the nonhomogeneous second-order linear differential equation

$$y'' + p(t)y' + q(t)y = g(t). \tag{28}$$

If the functions p, q, and g are continuous on an open interval I, and if the functions y_1 and y_2 form a fundamental set of solutions of the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0, \tag{29}$$

then a particular solution of equation (28) is

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W[y_1, y_2](s)}\,ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W[y_1, y_2](s)}\,ds, \tag{30}$$

where t_0 is any conveniently chosen point in I. The general solution is

$$y = c_1y_1(t) + c_2y_2(t) + Y(t), \tag{31}$$

as prescribed by Theorem 3.5.2.

By examining the expression (30) and reviewing the process by which we derived it, we can see that there may be two major difficulties in carrying out the method of variation of parameters. As we have mentioned earlier, one is the determination of functions $y_1(t)$ and $y_2(t)$ that form a fundamental set of solutions of the homogeneous equation (29) when the coefficients in that equation are not constants. The other possible difficulty lies in the evaluation of the integrals appearing in equation (30). This depends entirely on the nature of the functions y_1, y_2, and g. In using equation (30), be sure that the differential equation is exactly in the form (28); otherwise, the nonhomogeneous term $g(t)$ will not be correctly identified.

A major advantage of the method of variation of parameters is that equation (30) provides an expression for the particular solution $Y(t)$ in terms of an arbitrary forcing function $g(t)$. This expression is a good starting point if you wish to investigate the effect of variations in the forcing function, or if you wish to analyze the response of a system to a number of different forcing functions. (See Problems 18 to 22.)

Problems

In each of Problems 1 through 3, use the method of variation of parameters to find a particular solution of the given differential equation. Then check your answer by using the method of undetermined coefficients.

1. $y'' - 5y' + 6y = 2e^t$

2. $y'' - y' - 2y = 2e^{-t}$

3. $4y'' - 4y' + y = 16e^{t/2}$

In each of Problems 4 through 9, find the general solution of the given differential equation. In Problems 9, g is an arbitrary continuous function.

4. $y'' + y = \tan t, \quad 0 < t < \pi/2$

5. $y'' + 9y = 9\sec^2(3t), \quad 0 < t < \pi/6$

6. $y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0$

7. $4y'' + y = 2\sec(t/2), \quad -\pi < t < \pi$

8. $y'' - 2y' + y = e^t/(1 + t^2)$

9. $y'' - 5y' + 6y = g(t)$

In each of Problems 10 through 15, verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 14 and 15, g is an arbitrary continuous function.

10. $t^2y'' - 2y = 3t^2 - 1, \ t > 0; \quad y_1(t) = t^2, \ y_2(t) = t^{-1}$

11. $t^2y'' - t(t+2)y' + (t+2)y = 2t^3, \ t > 0;$ $y_1(t) = t, \ y_2(t) = te^t$

12. $ty'' - (1+t)y' + y = t^2e^{2t}, \ t > 0; \quad y_1(t) = 1+t, \ y_2(t) = e^t$

13. $x^2y'' - 3xy' + 4y = x^2\ln x, \ x > 0; \quad y_1(x) = x^2,$ $y_2(x) = x^2\ln x$

14. $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 3x^{3/2}\sin x, \ x > 0;$ $y_1(x) = x^{-1/2}\sin x, \ y_2(x) = x^{-1/2}\cos x$

15. $x^2y'' + xy' + (x^2 - 0.25)y = g(x), \ x > 0;$ $y_1(x) = x^{-1/2}\sin x, \ y_2(x) = x^{-1/2}\cos x$

16. By choosing the lower limit of integration in equation (30) in the text as the initial point t_0, show that $Y(t)$ becomes

$$Y(t) = \int_{t_0}^{t} \frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{y_1(s)y_2'(s) - y_1'(s)y_2(s)}\, g(s)\, ds\,.$$

Show that $Y(t)$ is a solution of the initial value problem

$$L[y] = g(t), \quad y(t_0) = 0, \ \ y'(t_0) = 0.$$

17. Show that the solution of the initial value problem

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \ \ y'(t_0) = y_0' \tag{32}$$

can be written as $y = u(t) + v(t)$, where u and v are solutions of the two initial value problems

$$L[u] = 0, \qquad u(t_0) = y_0, \quad u'(t_0) = y_0', \tag{33}$$

$$L[v] = g(t), \qquad v(t_0) = 0, \quad v'(t_0) = 0, \tag{34}$$

respectively. In other words, the nonhomogeneities in the differential equation and in the initial conditions can be dealt with separately. Observe that u is easy to find if a fundamental set of solutions of $L[u] = 0$ is known. And, as shown in Problem 16, the function v is given by equation (30).

18. a. Use the result of Problem 16 to show that the solution of the initial value problem

$$y'' + y = g(t), \quad y(t_0) = 0, \ \ y'(t_0) = 0 \tag{35}$$

is

$$y = \int_{t_0}^{t} \sin(t - s)g(s)\, ds\,. \tag{36}$$

b. Use the result of Problem 17 to find the solution of the initial value problem

$$y'' + y = g(t), \quad y(0) = y_0, \ \ y'(0) = y_0'.$$

19. Use the result of Problem 16 to find the solution of the initial value problem

$$L[y] = g(t), \quad y(t_0) = 0, \ \ y'(t_0) = 0,$$

where $L[y] = (D-a)(D-b)y$ for real numbers a and b with $a \neq b$. Note that $L[y] = y'' - (a+b)y' + aby$.

20. Use the result of Problem 16 to find the solution of the initial value problem

$$L[y] = g(t), \quad y(t_0) = 0, \ \ y'(t_0) = 0,$$

where $L[y] = (D - (\lambda + i\mu))(D - (\lambda - i\mu))y$; that is, $L[y] = y'' - 2\lambda y' + (\lambda^2 + \mu^2)y$. Note that the roots of the characteristic equation are $\lambda \pm i\mu$.

21. Use the result of Problem 16 to find the solution of the initial value problem

$$L[y] = g(t), \quad y(t_0) = 0, \quad y'(t_0) = 0,$$

where $L[y] = (D - a)^2 y$, that is, $L[y] = y'' - 2ay' + a^2 y$, and a is any real number.

22. By combining the results of Problems 19 through 21, show that the solution of the initial value problem

$$L[y] = (D^2 + bD + c)y = g(t), \quad y(t_0) = 0, \quad y'(t_0) = 0,$$

where b and c are constants, can be written in the form

$$y = \phi(t) = \int_{t_0}^{t} K(t - s)g(s)\,ds, \tag{37}$$

where the function K depends only on the solutions y_1 and y_2 of the corresponding homogeneous equation and is independent of the nonhomogeneous term. Once K is determined, all nonhomogeneous problems involving the same differential operator L are reduced to the evaluation of an integral. Note also that although K depends on both t and s, only the combination $t - s$ appears, so K is actually a function of a single variable. When we think of $g(t)$ as the input to the problem and of $\phi(t)$ as the output, it follows from equation (37) that the output depends on the input over the entire interval from the initial point t_0 to the current value t. The integral in equation (37) is called the **convolution** of K and g, and K is referred to as the **kernel**.

23. The method of reduction of order (Section 3.4) can also be used for the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t), \tag{38}$$

provided one solution y_1 of the corresponding homogeneous equation is known. Let $y = v(t)y_1(t)$ and show that y satisfies equation (38) if v is a solution of

$$y_1(t)v'' + \big(2y_1'(t) + p(t)y_1(t)\big)v' = g(t). \tag{39}$$

Equation (39) is a first-order linear differential equation for v'. By solving equation (39) for v', integrating the result to find v, and then multiplying by $y_1(t)$, you can find the general solution of equation (38). This method simultaneously finds both the second homogeneous solution and a particular solution.

In each of Problems 24 through 26, use the method outlined in Problem 23 to solve the given differential equation.

24. $t^2 y'' - 2ty' + 2y = 4t^2, \quad t > 0; \quad y_1(t) = t$

25. $t^2 y'' + 7ty' + 5y = t, \quad t > 0; \quad y_1(t) = t^{-1}$

26. $ty'' - (1 + t)y' + y = t^2 e^{2t}, \quad t > 0; \quad y_1(t) = 1 + t$ (see Problem 12)

3.7 Mechanical and Electrical Vibrations

One of the reasons why second-order linear differential equations with constant coefficients are worth studying is that they serve as mathematical models of many important physical processes. Two important areas of application are the fields of mechanical and electrical oscillations. For example, the motion of a mass on a vibrating spring, the torsional oscillations of a shaft with a flywheel, the flow of electric current in a simple series circuit, and many other physical problems are all described by the solution of an initial value problem of the form

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_0'. \tag{1}$$

This illustrates a fundamental relationship between mathematics and physics: *many physical problems may have mathematically equivalent models*. Thus, once we know how to solve the initial value problem (1), it is only necessary to make appropriate interpretations of the constants a, b, and c, and of the functions y and g, to obtain solutions of different physical problems.

We will study the motion of a mass on a spring in detail because understanding the behavior of this simple system is the first step in the investigation of more complex vibrating systems. Further, the principles involved are common to many problems.

Consider a mass m hanging at rest on the end of a vertical spring of original length l, as shown in Figure 3.7.1. The mass causes an elongation L of the spring in the downward (positive) direction. In this static situation there are two forces acting at the point where the mass is attached to the spring; see Figure 3.7.2. The gravitational force, or weight of the mass, acts downward and has magnitude $w = mg$, where g is the acceleration due to gravity. There is also a force F_s, due to the spring, that acts upward. If we assume that the elongation L of the spring is small, the spring force is very nearly proportional to L; this is known as **Hooke's**[10] **law**. Thus we write $F_s = -kL$, where the constant of proportionality k is called the

[10]Robert Hooke (1635–1703) was an English scientist with wide-ranging interests. His most important book, *Micrographia*, was published in 1665 and described a variety of microscopical observations. Hooke first published his law of elastic behavior in 1676 as *ceiiinosssttuv*; in 1678 he gave the interpretation *ut tensio sic vis*, which means, roughly, "as the force so is the displacement."

spring constant, and the minus sign is due to the fact that the spring force acts in the upward (negative) direction. Since the mass is in equilibrium, the two forces balance each other, which means that

$$w + F_s = mg - kL = 0. \tag{2}$$

For a given weight $w = mg$, you can measure L and then use equation (2) to determine k. Note that k has the units of force per unit length.

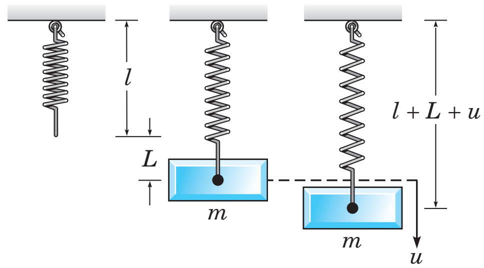

FIGURE 3.7.1 A spring-mass system.

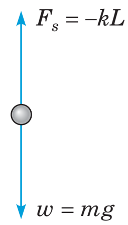

FIGURE 3.7.2 Force diagram for a spring-mass system.

In the corresponding dynamic problem, we are interested in studying the motion of the mass when it is acted on by an external force or is initially displaced. Let $u(t)$, measured positive in the downward direction, denote the displacement of the mass from its equilibrium position at time t; see Figure 3.7.1. Then $u(t)$ is related to the forces acting on the mass through Newton's law of motion

$$mu''(t) = f(t), \tag{3}$$

where u'' is the acceleration of the mass and f is the net force acting on the mass. Observe that both u and f are functions of time. In this dynamic problem there are now four separate forces that must be considered.

1. The weight $w = mg$ of the mass always acts downward.
2. The spring force F_s is assumed to be proportional to the total elongation $L + u$ of the spring and always acts to restore the spring to its natural position. If $L + u > 0$, then the spring is extended, and the spring force is directed upward. In this case

$$F_s = -k(L + u). \tag{4}$$

   On the other hand, if $L + u < 0$, then the spring is compressed a distance $|L + u|$, and the spring force, which is now directed downward, is given by $F_s = k|L+u|$. However, when $L+u < 0$, it follows that $|L+u| = -(L+u)$, so F_s is again given by equation (4). Thus, regardless of the position of the mass, the force exerted by the spring is always expressed by equation (4).
3. The damping or resistive force F_d always acts in the direction opposite to the direction of motion of the mass. This force may arise from several sources: resistance from the air or other medium in which the mass moves, internal energy dissipation due to the extension or compression of the spring, friction between the mass and the guides (if any) that constrain its motion to one dimension, or a mechanical device (dashpot) that imparts a resistive force to the mass. In any case, we assume that the resistive force is proportional to the speed $|du/dt|$ of the mass; this is usually referred to as **viscous damping**. If $du/dt > 0$, then u is increasing, so the mass is moving downward. Then F_d is directed

upward and is given by

$$F_d(t) = -\gamma u'(t), \tag{5}$$

where γ is a positive constant of proportionality known as the damping constant. On the other hand, if $du/dt < 0$, then u is decreasing, the mass is moving upward, and F_d is directed downward. In this case, $F_d = \gamma|u'(t)|$; since $|u'(t)| = -u'(t)$, it follows that $F_d(t)$ is again given by equation (5). Thus, regardless of the direction of motion of the mass, the damping force is always expressed by equation (5).

The damping force may be rather complicated, and the assumption that it is modeled adequately by equation (5) may be open to question. Some dashpots do behave as equation (5) states, and if the other sources of dissipation are small, it may be possible to neglect them altogether or to adjust the damping constant γ to approximate them. An important benefit of the assumption (5) is that it leads to a linear (rather than a nonlinear) differential equation. In turn, this means that a thorough analysis of the system is straightforward, as we will show in this section and in Section 3.8.

4. An applied external force $F(t)$ is directed downward or upward as $F(t)$ is positive or negative. This could be a force due to the motion of the mount to which the spring is attached, or it could be a force applied directly to the mass. Often the external force is periodic.

Taking account of these forces, we can now rewrite Newton's law (3) as

$$\begin{aligned} mu''(t) &= w + F_s(t) + F_d(t) + F(t) \\ &= mg - k(L + u(t)) - \gamma u'(t) + F(t). \end{aligned} \tag{6}$$

Since $mg - kL = 0$ by equation (2), it follows that the equation of motion of the mass is

$$mu''(t) + \gamma u'(t) + ku(t) = F(t), \tag{7}$$

where the constants m, γ, and k are positive. Note that equation (7) has the same form as equation (1), that is, it is a nonhomogeneous second-order linear differential equation with constant coefficients.

It is important to understand that equation (7) is only an approximate equation for the displacement $u(t)$. In particular, both equations (4) and (5) should be viewed as approximations for the spring force and the damping force, respectively. In our derivation we have also neglected the mass of the spring in comparison with the mass of the attached body.

The complete formulation of the vibration problem requires that we specify two initial conditions, namely, the initial position u_0 and the initial velocity v_0 of the mass:

$$u(0) = u_0, \quad u'(0) = v_0. \tag{8}$$

It follows from Theorem 3.2.1 that these conditions give a mathematical problem that has a unique solution for any values of the constants u_0 and v_0. This is consistent with our physical intuition that if the mass is set in motion with a given initial displacement and velocity, then its position will be determined uniquely at all future times. The position of the mass is given (approximately) by the solution of the second-order linear differential equation (7) subject to the prescribed initial conditions (8).

EXAMPLE 1

A mass weighing 4 lb stretches a spring 2 in. Suppose that the mass is given an additional 6-in displacement in the positive direction and then released. The mass is in a medium that exerts a viscous resistance of 6 lb when the mass has a velocity of 3 ft/s. Under the assumptions discussed in this section, formulate the initial value problem that governs the motion of the mass.

Solution:

The required initial value problem consists of the differential equation (7) and initial conditions (8), so our task is to determine the various constants that appear in these equations. The first step is to choose the units of measurement. Based on the statement of the problem, it is natural to use the English rather than the metric system of units. The only time unit mentioned is the second, so we will measure t in seconds. On the other hand, both the foot and the inch appear in the statement as units of length. It is immaterial which one we use, but having made a choice, we must be consistent. To be definite, let us measure the displacement u in feet.

Since nothing is said in the statement of the problem about an external force, we assume that $F(t) = 0$. To determine m, note that

$$m = \frac{w}{g} = \frac{4 \text{ lb}}{32 \text{ ft}/\text{s}^2} = \frac{1}{8} \frac{\text{lb} \cdot \text{s}^2}{\text{ft}}.$$

The damping coefficient γ is determined from the statement that $\gamma u'$ is equal to 6 lb when u' is 3 ft/s. Therefore,

$$\gamma = \frac{6 \text{ lb}}{3 \text{ ft}/\text{s}} = 2 \frac{\text{lb} \cdot \text{s}}{\text{ft}}.$$

The spring constant k is found from the statement that the mass stretches the spring by 2 in or $\frac{1}{6}$ ft. Thus

$$k = \frac{4 \text{ lb}}{1/6 \text{ ft}} = 24 \frac{\text{lb}}{\text{ft}}.$$

Consequently, differential equation (7) becomes

$$\frac{1}{8} u'' + 2u' + 24u = 0,$$

or

$$u'' + 16u' + 192u = 0. \tag{9}$$

The initial conditions are

$$u(0) = \frac{1}{2}, \quad u'(0) = 0. \tag{10}$$

The second initial condition is implied by the word "released" in the statement of the problem, which we interpret to mean that the mass is set in motion with no initial velocity.

Undamped Free Vibrations. If there is no external force, then $F(t) = 0$ in equation (7). Let us also suppose that there is no damping so that $\gamma = 0$; this is an idealized configuration of the system, seldom (if ever) completely attainable in practice. However, if the actual damping is very small, then the assumption of no damping may yield satisfactory results over short to moderate time intervals. In this case the equation of motion (7) reduces to

$$mu'' + ku = 0. \tag{11}$$

The characteristic equation for equation (11) is

$$mr^2 + k = 0$$

and its roots are $r = \pm i\sqrt{k/m}$. Thus the general solution of equation (11) is

$$u = A\cos(\omega_0 t) + B\sin(\omega_0 t), \tag{12}$$

where

$$\omega_0^2 = \frac{k}{m}. \tag{13}$$

The arbitrary constants A and B can be determined if initial conditions of the form (8) are given.

In discussing the solution of equation (11), it is convenient to rewrite equation (12) in the form

$$u = R\cos(\omega_0 t - \delta), \tag{14}$$

or

$$u = R\cos\delta\cos\left(\omega_0 t\right) + R\sin\delta\sin\left(\omega_0 t\right). \tag{15}$$

By comparing equation (15) with equation (12), we find that A, B, R, and δ are related by the equations

$$A = R\cos\delta, \quad B = R\sin\delta. \tag{16}$$

Thus

$$R = \sqrt{A^2 + B^2}, \quad \tan\delta = \frac{B}{A}. \tag{17}$$

In calculating δ, we must take care to choose the correct quadrant; this can be done by checking the signs of $\cos\delta$ and $\sin\delta$ in equations (16).

The graph of equation (14), or the equivalent equation (12), for a typical set of initial conditions is shown in Figure 3.7.3. The graph is a displaced cosine wave that describes a periodic, or **simple harmonic**, motion of the mass. The **period** of the motion is

$$T = \frac{2\pi}{\omega_0} = 2\pi\left(\frac{m}{k}\right)^{1/2}. \tag{18}$$

The circular frequency $\omega_0 = \sqrt{k/m}$, measured in radians per unit time, is called the **natural frequency** of the vibration. The maximum displacement R of the mass from equilibrium is the **amplitude** of the motion. The dimensionless parameter δ is called the **phase**, or phase angle, and measures the displacement of the wave from its normal position corresponding to $\delta = 0$.

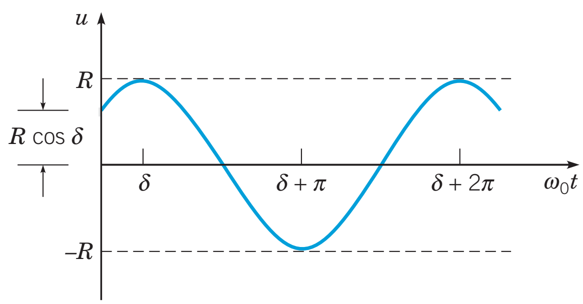

FIGURE 3.7.3 Simple harmonic motion; $u = R\cos(\omega_0 t - \delta)$. Note that the horizontal axis is labeled as $\omega_0 t$.

Note that the motion described by equation (14) has a constant amplitude that does not diminish with time. This reflects the fact that, in the absence of damping, there is no way for the system to dissipate the energy imparted to it by the initial displacement and velocity. Further, for a given mass m and spring constant k, the system always vibrates at the same frequency ω_0, regardless of the initial conditions. However, the initial conditions do help to determine the amplitude of the motion. Finally, observe from equation (18) that the period T increases as the mass m increases, so larger masses vibrate more slowly. On the other hand, the period T decreases as the spring constant k increases, which means that stiffer springs cause the system to vibrate more rapidly.

EXAMPLE 2

Suppose that a mass weighing 10 lb stretches a spring 2 in. If the mass is displaced an additional 2 in and is then set in motion with an initial upward velocity of 1 ft/s, determine the position of the mass at any later time. Also determine the period, amplitude, and phase of the motion.

Solution:

The spring constant is $k = 10$ lb$/2$ in $= 60$ lb/ft, and the mass is $m = w/g = 10/32$ lb·s^2/ft. Hence the equation of motion reduces to

$$u'' + 192u = 0, \tag{19}$$

and the general solution is

$$u = A\cos(8\sqrt{3}t) + B\sin(8\sqrt{3}t).$$

The solution satisfying the initial conditions $u(0) = 1/6$ ft and $u'(0) = -1$ ft/s is

$$u = \frac{1}{6}\cos(8\sqrt{3}t) - \frac{1}{8\sqrt{3}}\sin(8\sqrt{3}t). \tag{20}$$

The natural frequency is $\omega_0 = 8\sqrt{3} \cong 13.856$ rad/s, so the period is $T = 2\pi/\omega_0 \cong 0.453$ s. The amplitude R and phase δ are found from equations (17). We have

$$R^2 = \frac{1}{36} + \frac{1}{192} = \frac{19}{576}, \quad \text{so} \quad R \cong 0.182\text{ft}.$$

The second of equations (17) yields $\tan\delta = -\sqrt{3}/4$. There are two solutions of this equation, one in the second quadrant and one in the fourth. In the present problem, $\cos\delta > 0$ and $\sin\delta < 0$, so δ is in the fourth quadrant. In fact,

$$\delta = -\arctan\left(\frac{\sqrt{3}}{4}\right) \cong -0.40864 \text{ rad}.$$

The graph of the solution (20) is shown in Figure 3.7.4.

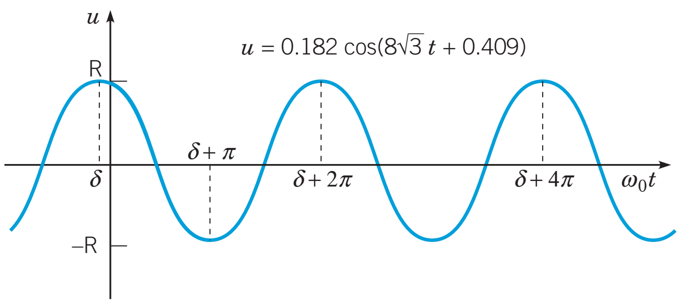

FIGURE 3.7.4 An undamped free vibration: $u'' + 192u = 0,\ u(0) = 1/6,\ u'(0) = -1$. Note that the scale for the horizontal axis is $\omega_0 t$.

Damped Free Vibrations. When the effects of damping are included, the differential equation governing the motion of the mass is

$$mu'' + \gamma u' + ku = 0. \tag{21}$$

We are especially interested in examining the effect of variations in the damping coefficient γ for given values of the mass m and spring constant k. The corresponding characteristic equation is

$$mr^2 + \gamma r + k = 0,$$

and its roots are

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right). \tag{22}$$

Depending on the sign of $\gamma^2 - 4km$, the solution u has one of the following forms:

$$\gamma^2 - 4km > 0, \quad u = Ae^{r_1 t} + Be^{r_2 t}; \tag{23}$$

$$\gamma^2 - 4km = 0, \quad u = (A + Bt)e^{-\gamma t/(2m)}; \tag{24}$$

$$\gamma^2 - 4km < 0, \quad u = e^{-\gamma t/(2m)}(A\cos(\mu t) + B\sin(\mu t)), \quad \mu = \frac{1}{2m}\left(4km - \gamma^2\right)^{1/2} > 0. \tag{25}$$

Since m, γ, and k are positive, $\gamma^2 - 4km$ is always less than γ^2. Hence, if $\gamma^2 - 4km \geq 0$, then the values of r_1 and r_2 given by equation (22) are *negative*. If $\gamma^2 - 4km < 0$, then the values of r_1 and r_2 are complex, but with *negative* real part. Thus, in all cases, the solution u tends to zero as $t \to \infty$; this occurs regardless of the values of the arbitrary constants A and B—that is, regardless of the initial conditions. This confirms our intuitive expectation, namely, that damping gradually dissipates the energy initially imparted to the system, and consequently the motion dies out with increasing time.

The most interesting case is the third one, which occurs when the damping is small. If we let $A = R\cos\delta$ and $B = R\sin\delta$ in equation (25), then we obtain

$$u = Re^{-\gamma t/(2m)}\cos(\mu t - \delta). \tag{26}$$

The displacement u lies between the curves $u = \pm Re^{-\gamma t/(2m)}$; hence it resembles a cosine wave whose amplitude decreases as t increases. A typical example is sketched in Figure 3.7.5. The motion is called a damped oscillation or a damped vibration. The amplitude factor R depends on m, γ, k, and the initial conditions.

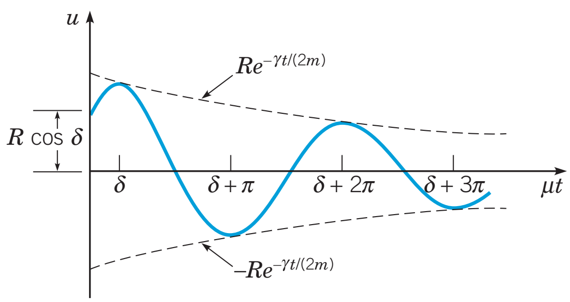

FIGURE 3.7.5 Damped vibration; $u = Re^{-\gamma t/2m}\cos(\mu t - \delta)$. Note that the scale for the horizontal axis is μt.

Although the motion is not periodic, the parameter μ determines the frequency with which the mass oscillates back and forth; consequently, μ is called the **quasi-frequency**. By comparing μ with the frequency ω_0 of undamped motion, we find that

$$\frac{\mu}{\omega_0} = \frac{(4km - \gamma^2)^{1/2}/(2m)}{\sqrt{k/m}} = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2} \cong 1 - \frac{\gamma^2}{8km}. \tag{27}$$

The last approximation is valid when $\gamma^2/4km$ is small; we refer to this situation as "small damping." Thus the effect of small damping is to reduce slightly the frequency of the oscillation. By analogy with equation (18), the quantity $T_d = 2\pi/\mu$ is called the **quasi-period** of the motion. It is the time between successive maxima or successive minima of the position of the mass, or between successive passages of the mass through its equilibrium position while going in the same direction. The relation between T_d and T is given by

$$\frac{T_d}{T} = \frac{\omega_0}{\mu} = \left(1 - \frac{\gamma^2}{4km}\right)^{-1/2} \cong 1 + \frac{\gamma^2}{8km}, \tag{28}$$

where again the last approximation is valid when $\gamma^2/4km$ is small. Thus small damping increases the quasi-period.

Equations (27) and (28) reinforce the significance of the dimensionless ratio $\gamma^2/(4km)$. It is not the magnitude of γ alone that determines whether damping is large or small, but the magnitude of γ^2 compared to $4km$. When $\gamma^2/(4km)$ is small, then damping has a small effect on the quasi-frequency and quasi-period of the motion. On the other hand, if we want to study the detailed motion of the mass for all time, then we can *never* neglect the damping force, no matter how small.

As $\gamma^2/(4km)$ increases, the quasi-frequency μ decreases and the quasi-period T_d increases. In fact, $\mu \to 0$ and $T_d \to \infty$ as $\gamma \to 2\sqrt{km}$. As indicated by equations (23), (24), and (25), the nature of the solution changes as γ passes through the value $2\sqrt{km}$. The motion with $\gamma = 2\sqrt{km}$ is said to be **critically damped**. For larger values of γ, $\gamma > 2\sqrt{km}$, the motion is said to be **overdamped**. In these cases, given by equations (24) and (23), respectively, the mass may pass through its equilibrium position at most once (see Figure 3.7.6) and then creep back to it. The mass does not oscillate about the equilibrium, as it does for small γ. Two typical examples of critically damped motion are shown in Figure 3.7.6, and the situation is discussed further in Problems 15 and 16.

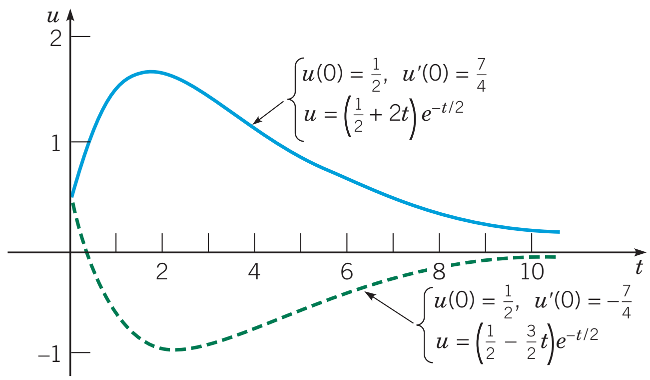

FIGURE 3.7.6 Critically damped motions: $u'' + u' + 0.25u = 0$; $u = (A + Bt)e^{-t/2}$. The solid blue curve is the solution satisfying $u(0) = 1/2$, $u'(0) = 7/4$; the dashed green curve satisfies $u(0) = 1/2$, $u'(0) = -7/4$.

EXAMPLE 3

The motion of a certain spring-mass system is governed by the differential equation

$$u'' + \frac{1}{8}u' + u = 0, \tag{29}$$

where u is measured in feet and t in seconds. If $u(0) = 2$ and $u'(0) = 0$, determine the position of the mass at any time. Find the quasi-frequency and the quasi-period, as well as the time at which the mass first passes through its equilibrium position. Also find the time τ such that $|u(t)| < 0.1$ for all $t > \tau$.

Solution:

The solution of equation (29) is

$$u(t) = e^{-t/16}\left(A\cos\left(\frac{\sqrt{255}}{16}t\right) + B\sin\left(\frac{\sqrt{255}}{16}t\right)\right).$$

To satisfy the initial conditions, we must choose $A = 2$ and $B = 2/\sqrt{255}$; hence the solution of the initial value problem is

$$\begin{aligned} u &= e^{-t/16}\left(2\cos\left(\frac{\sqrt{255}}{16}t\right) + \frac{2}{\sqrt{255}}\sin\left(\frac{\sqrt{255}}{16}t\right)\right) \\ &= \frac{32}{\sqrt{255}}e^{-t/16}\cos\left(\frac{\sqrt{255}}{16}t - \delta\right), \end{aligned} \tag{30}$$

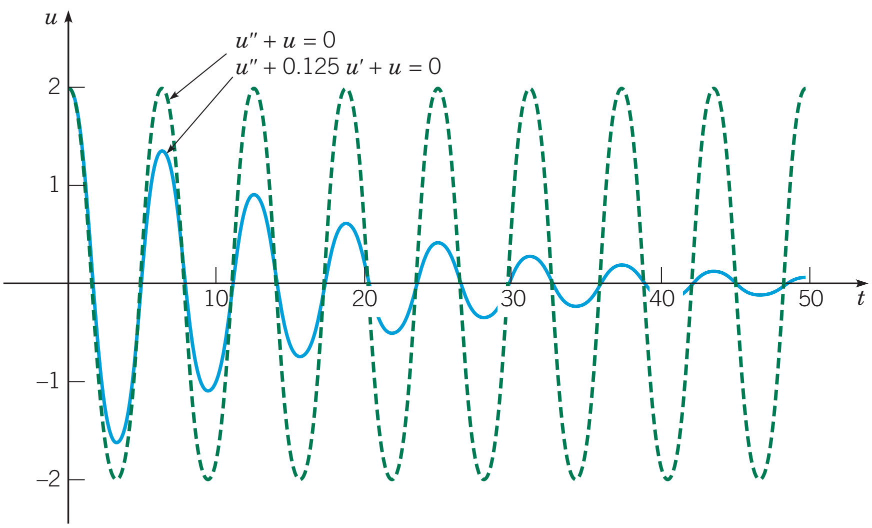

FIGURE 3.7.7 Vibration with small damping (solid blue curve) and with no damping (dashed green curve). Both motions have the same initial conditions: $u(0) = 2$, $u'(0) = 0$.

where δ is in the first quadrant with $\tan\delta = 1/\sqrt{255}$, so $\delta \cong 0.06254$. The displacement of the mass as a function of time is shown in Figure 3.7.7. For purposes of comparison, we also show the motion if the damping term is neglected.

The quasi-frequency is $\mu = \sqrt{255}/16 \cong 0.998$, and the quasi-period is $T_d = 2\pi/\mu \cong 6.295$ s. These values differ only slightly from the corresponding values (1 and 2π, respectively) for the undamped oscillation. This is evident also from the graphs in Figure 3.7.7, which rise and fall almost together. The damping coefficient is small in this example: only one-sixteenth of the critical value, in fact. Nevertheless, the amplitude of the oscillation is reduced rather rapidly.

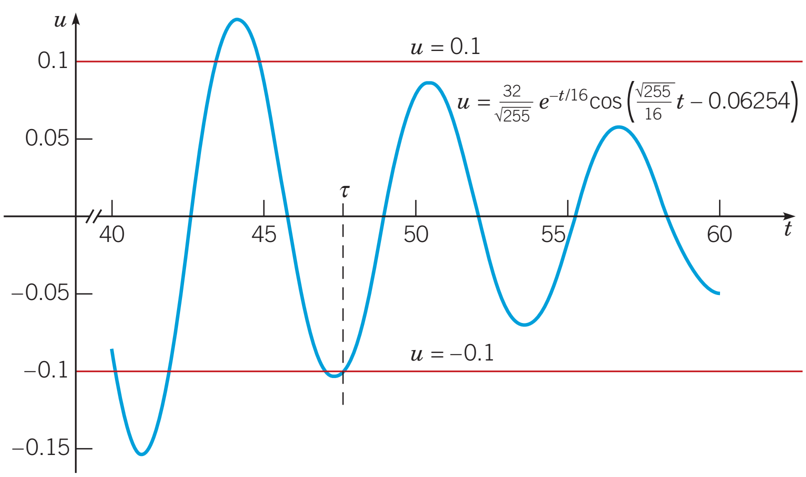

FIGURE 3.7.8 Solution of Example 3 for $40 \le t \le 60$; determination of the time τ after which $|u(t)| < 0.1$.

Figure 3.7.8 shows the graph of the solution for $40 \le t \le 60$, together with the graphs of $u = \pm 0.1$. From the graph it appears that τ is about 47.5, and by a more precise calculation we find that $\tau \cong 47.5149$ s.

To find the time at which the mass first passes through its equilibrium position, we refer to equation (30) and set $\sqrt{255}t/16 - \delta$ equal to $\pi/2$, the smallest positive zero of the cosine function. Then, by solving for t, we obtain

$$t = \frac{16}{\sqrt{255}}\left(\frac{\pi}{2} + \delta\right) \cong 1.637 \text{ s.}$$

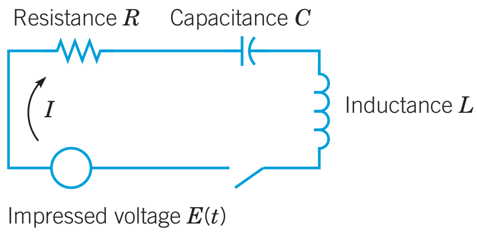

FIGURE 3.7.9 A simple electric circuit.

Electric Circuits. A second example of the occurrence of second-order linear differential equations with constant coefficients is their use as a model of the flow of electric current in the simple series circuit shown in Figure 3.7.9. The current I, measured in amperes (A), is a function of time t. The resistance R in ohms (Ω), the capacitance C in farads (F), and the inductance L in henrys (H) are all positive and are assumed to be known constants. The impressed voltage E in volts (V) is a given function of time. Another physical quantity that enters the discussion is the total charge Q in coulombs (C) on the capacitor at time t. The relation between charge Q and current I is

$$I = \frac{dQ}{dt}. \tag{31}$$

The flow of current in the circuit is governed by Kirchhoff's[11] second law: *In a closed circuit the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit.*

According to the elementary laws of electricity, we know that

The voltage drop across the resistor is RI.

The voltage drop across the capacitor is $\dfrac{Q}{C}$.

The voltage drop across the inductor is $L\dfrac{dI}{dt}$.

Hence, by Kirchhoff's law,

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t). \tag{32}$$

The units for voltage, resistance, current, charge, capacitance, inductance, and time are all related:

$$1 \text{ volt} = 1 \text{ ohm} \cdot 1 \text{ ampere} = 1 \text{ coulomb}/1 \text{ farad} = 1 \text{ henry} \cdot 1 \text{ ampere}/1 \text{ second}.$$

Substituting for I from equation (31), we obtain the differential equation

$$LQ'' + RQ' + \frac{1}{C}Q = E(t) \tag{33}$$

for the charge Q. The initial conditions are

$$Q(t_0) = Q_0, \quad Q'(t_0) = I(t_0) = I_0. \tag{34}$$

Thus to know the charge at any time it is sufficient to know the charge on the capacitor and the current in the circuit at some initial time t_0.

Alternatively, we can obtain a differential equation for the current I by differentiating equation (33) with respect to t, and then substituting for dQ/dt from equation (31). The result is

$$LI'' + RI' + \frac{1}{C}I = E'(t), \tag{35}$$

with the initial conditions

$$I(t_0) = I_0, \quad I'(t_0) = I_0'. \tag{36}$$

[11]Gustav Kirchhoff (1824–1887) was a German physicist and professor at Breslau, Heidelberg, and Berlin. He formulated the basic laws of electric circuits about 1845 while still a student at Albertus University in his native Königsberg. In 1857 he discovered that an electric current in a resistanceless wire travels at the speed of light. He is also famous for fundamental work in electromagnetic absorption and emission and was one of the founders of spectroscopy.

From equation (32) it follows that

$$I_0' = \frac{E(t_0) - RI_0 - \dfrac{Q_0}{C}}{L}. \quad (37)$$

Hence I_0' is also determined by the initial charge and current, which are physically measurable quantities.

The most important conclusion from this discussion is that the flow of current in the circuit is described by an initial value problem of precisely the same form as the one that describes the motion of a spring-mass system. This is a good example of the unifying role of mathematics: once you know how to solve second-order linear equations with constant coefficients, you can interpret the results in terms of mechanical vibrations, electric circuits, or any other physical situation that leads to the same problem.

Problems

In each of Problems 1 and 2, determine ω_0, R, and δ so as to write the given expression in the form $u = R\cos(\omega_0 t - \delta)$.

1. $u = 3\cos(2t) + 4\sin(2t)$

2. $u = -2\cos(\pi t) - 3\sin(\pi t)$

3. A mass of 100 g stretches a spring 5 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/s, and if there is no damping, determine the position u of the mass at any time t. When does the mass first return to its equilibrium position?

4. A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in and then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position u of the mass at any time t. Determine the frequency, period, amplitude, and phase of the motion.

G **5.** A mass of 20 g stretches a spring 5 cm. Suppose that the mass is also attached to a viscous damper with a damping constant of 400 dyn·s/cm. If the mass is pulled down an additional 2 cm and then released, find its position u at any time t. Plot u versus t. Determine the quasi-frequency and the quasi-period. Determine the ratio of the quasi-period to the period of the corresponding undamped motion. Also find the time τ such that $|u(t)| < 0.05$ cm for all $t > \tau$.

6. A spring is stretched 10 cm by a force of 3 N. A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass is 5 m/s. If the mass is pulled down 5 cm below its equilibrium position and given an initial downward velocity of 10 cm/s, determine its position u at any time t. Find the quasi-frequency μ and the ratio of μ to the natural frequency of the corresponding undamped motion.

7. A series circuit has a capacitor of 10^{-5} F, a resistor of $3 \times 10^2\ \Omega$, and an inductor of 0.2 H. The initial charge on the capacitor is 10^{-6} C and there is no initial current. Find the charge Q on the capacitor at any time t.

8. A vibrating system satisfies the equation $u'' + \gamma u' + u = 0$. Find the value of the damping coefficient γ for which the quasi-period of the damped motion is 50% greater than the period of the corresponding undamped motion.

9. Show that the period of motion of an undamped vibration of a mass hanging from a vertical spring is $2\pi\sqrt{L/g}$, where L is the elongation of the spring due to the mass, and g is the acceleration due to gravity.

10. Show that the solution of the initial value problem

$$mu'' + \gamma u' + ku = 0, \quad u(t_0) = u_0, \quad u'(t_0) = u_0'$$

can be expressed as the sum $u = v + w$, where v satisfies the initial conditions $v(t_0) = u_0$, $v'(t_0) = 0$, w satisfies the initial conditions $w(t_0) = 0$, $w'(t_0) = u_0'$, and both v and w satisfy the same differential equation as u. This is another instance of superposing solutions of simpler problems to obtain the solution of a more general problem.

11. **a.** Show that $A\cos(\omega_0 t) + B\sin(\omega_0 t)$ can be written in the form $r\sin(\omega_0 t - \theta)$. Determine r and θ in terms of A and B.

b. If $R\cos(\omega_0 t - \delta) = r\sin(\omega_0 t - \theta)$, determine the relationship among R, r, δ, and θ.

12. If a series circuit has a capacitor of $C = 0.8 \times 10^{-6}$ F and an inductor of $L = 0.2$ H, find the resistance R so that the circuit is critically damped.

13. Assume that the system described by the differential equation $mu'' + \gamma u' + ku = 0$ is either critically damped or overdamped. Show that the mass can pass through the equilibrium position at most once, regardless of the initial conditions.
Hint: Determine all possible values of t for which $u = 0$.

14. Assume that the system described by the differential equation $mu'' + \gamma u' + ku = 0$ is critically damped and that the initial conditions are $u(0) = u_0$, $u'(0) = v_0$. If $v_0 = 0$, show that $u \to 0$ as $t \to \infty$ but that u is never zero. If u_0 is positive, determine a condition on v_0 that will ensure that the mass passes through its equilibrium position after it is released.

15. **Logarithmic Decrement.** **a.** For the damped oscillation described by equation (26), show that the time between successive maxima is $T_d = 2\pi/\mu$.

b. Show that the ratio of the displacements at two successive maxima is given by $\exp(\gamma T_d/(2m))$. Observe that this ratio does not depend on which pair of maxima is chosen. The natural logarithm of this ratio is called the **logarithmic decrement** and is denoted by Δ.

c. Show that $\Delta = \pi\gamma/(m\mu)$. Since m, μ, and Δ are quantities that can be measured easily for a mechanical system, this result provides a convenient and *practical* method for determining the damping constant of the system, which is more difficult to measure directly. In particular, for the motion of a vibrating mass in a viscous fluid, the damping constant depends on the viscosity of the fluid; for simple geometric shapes the form of this dependence is known, and the preceding relation allows the experimental determination of the viscosity. This is one of the most accurate ways of determining the viscosity of a gas at high pressure.

16. Referring to Problem 15, find the logarithmic decrement of the system in Problem 5.

17. The position of a certain spring-mass system satisfies the initial value problem

$$\frac{3}{2}u'' + ku = 0, \quad u(0) = 2, \ u'(0) = v.$$

If the period and amplitude of the resulting motion are observed to be π and 3, respectively, determine the values of k and v.

18. Consider the initial value problem

$$mu'' + \gamma u' + ku = 0, \quad u(0) = u_0, \ u'(0) = v_0.$$

Assume that $\gamma^2 < 4km$.

a. Solve the initial value problem.

b. Write the solution in the form $u(t) = Re^{-\gamma t/(2m)} \cos(\mu t - \delta)$. Determine R in terms of m, γ, k, u_0, and v_0.

c. Investigate the dependence of R on the damping coefficient γ for fixed values of the other parameters.

19. A cubic block of side l and mass density ρ per unit volume is floating in a fluid of mass density ρ_0 per unit volume, where $\rho_0 > \rho$. If the block is slightly depressed and then released, it oscillates in the vertical direction. Assuming that the viscous damping of the fluid and air can be neglected, derive the differential equation of motion and determine the period of the motion.
Hint: Use Archimedes'[12] principle: an object that is completely or partially submerged in a fluid is acted on by an upward (buoyant) force equal to the weight of the displaced fluid.

20. The position of a certain undamped spring-mass system satisfies the initial value problem

$$u'' + 2u = 0, \quad u(0) = 0, \ u'(0) = 2.$$

a. Find the solution of this initial value problem.

b. Plot u versus t and u' versus t on the same axes.

c. Plot u' versus u; that is, plot $u(t)$ and $u'(t)$ parametrically with t as the parameter. This plot is known as a **phase plot**, and the uu'-plane is called the **phase plane**. Observe that a closed curve in the phase plane corresponds to a periodic solution $u(t)$. What is the direction of motion on the phase plot as t increases?

21. The position of a certain spring-mass system satisfies the initial value problem

$$u'' + \frac{1}{4}u' + 2u = 0, \quad u(0) = 0, \ u'(0) = 2.$$

a. Find the solution of this initial value problem.

b. Plot u versus t and u' versus t on the same axes.

c. Plot u' versus u in the phase plane (see Problem 20). Identify several corresponding points on the curves in parts b and c. What is the direction of motion on the phase plot as t increases?

22. In the absence of damping, the motion of a spring-mass system satisfies the initial value problem

$$mu'' + ku = 0, \quad u(0) = a, \ u'(0) = b.$$

a. Show that the kinetic energy initially imparted to the mass is $mb^2/2$ and that the potential energy initially stored in the spring is $ka^2/2$, so initially the total energy in the system is $(ka^2 + mb^2)/2$.

b. Solve the given initial value problem.

c. Using the solution in part b, determine the total energy in the system at any time t. Your result should confirm the principle of conservation of energy for this system.

23. Suppose that a mass m slides without friction on a horizontal surface. The mass is attached to a spring with spring constant k, as shown in Figure 3.7.10, and is also subject to viscous air resistance with coefficient γ. Show that the displacement $u(t)$ of the mass from its equilibrium position satisfies equation (21). How does the derivation of the equation of motion in this case differ from the derivation given in the text?

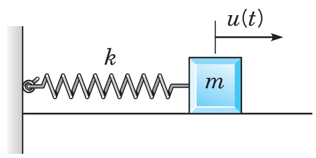

FIGURE 3.7.10 A spring-mass system.

24. In the spring-mass system of Problem 23, suppose that the spring force is not given by Hooke's law but instead satisfies the relation

$$F_s = -(ku + \epsilon u^3),$$

where $k > 0$ and ϵ is small but may be of either sign. The spring is called a hardening spring if $\epsilon > 0$ and a softening spring if $\epsilon < 0$. Why are these terms appropriate?

a. Show that the displacement $u(t)$ of the mass from its equilibrium position satisfies the differential equation

$$mu'' + \gamma u' + ku + \epsilon u^3 = 0.$$

Suppose that the initial conditions are

$$u(0) = 0, \quad u'(0) = 1.$$

In the remainder of this problem, assume that $m = 1$, $k = 1$, and $\gamma = 0$.

b. Find $u(t)$ when $\epsilon = 0$ and also determine the amplitude and period of the motion.

c. Let $\epsilon = 0.1$. Plot a numerical approximation to the solution. Does the motion appear to be periodic? Estimate the amplitude and period.

d. Repeat part c for $\epsilon = 0.2$ and $\epsilon = 0.3$.

e. Plot your estimated values of the amplitude A and the period T versus ϵ. Describe the way in which A and T, respectively, depend on ϵ.

f. Repeat parts c, d, and e for negative values of ϵ.

[12]Archimedes (287–212 BCE) was the foremost of the ancient Greek mathematicians. He lived in Syracuse on the island of Sicily. His most notable discoveries were in geometry, but he also made important contributions to hydrostatics and other branches of mechanics. His method of exhaustion is a precursor of the integral calculus developed by Newton and Leibniz almost two millennia later. He died at the hands of a Roman soldier during the Second Punic War.

3.8 Forced Periodic Vibrations

We will now investigate the situation in which a periodic external force is applied to a spring-mass system. The behavior of this simple system models that of many oscillatory systems with an external force due, for example, to a motor attached to the system. We will first consider the case in which damping is present and will look later at the idealized special case in which there is assumed to be no damping.

Forced Vibrations with Damping. The algebraic calculations can be fairly complicated in this kind of problem, so we will begin with a relatively simple example.

EXAMPLE 1

Suppose that the motion of a certain spring-mass system satisfies the differential equation

$$u'' + u' + \frac{5}{4}u = 3\cos t \tag{1}$$

and the initial conditions

$$u(0) = 2, \quad u'(0) = 3. \tag{2}$$

Find the solution of this initial value problem and describe the behavior of the solution for large t.

Solution:

The homogeneous equation corresponding to equation (1) has the characteristic equation $r^2 + r + \frac{5}{4} = 0$ with roots $r = -\frac{1}{2} \pm i$. Thus a general solution $u_c(t)$ of this homogeneous equation is

$$u_c(t) = c_1 e^{-t/2}\cos t + c_2 e^{-t/2}\sin t. \tag{3}$$

A particular solution of equation (1) has the form $U(t) = A\cos t + B\sin t$, where A and B are found by substituting $U(t)$ for u in equation (1). We have $U'(t) = -A\sin t + B\cos t$ and $U''(t) = -A\cos t - B\sin t$. Thus, from equation (1) we obtain

$$\left(\frac{1}{4}A + B\right)\cos t + \left(-A + \frac{1}{4}B\right)\sin t = 3\cos t.$$

Consequently, A and B must satisfy the equations

$$\frac{1}{4}A + B = 3, \quad -A + \frac{1}{4}B = 0,$$

with the result that $A = \frac{12}{17}$ and $B = \frac{48}{17}$. Therefore, the particular solution is

$$U(t) = \frac{12}{17}\cos t + \frac{48}{17}\sin t, \tag{4}$$

and the general solution of equation (1) is

$$u = u_c(t) + U(t) = c_1 e^{-t/2}\cos t + c_2 e^{-t/2}\sin t + \frac{12}{17}\cos t + \frac{48}{17}\sin t. \tag{5}$$

The remaining constants c_1 and c_2 are determined by the initial conditions (2). From equation (5), and its first derivative, we have

$$u(0) = c_1 + \frac{12}{17} = 2, \quad u'(0) = -\frac{1}{2}c_1 + c_2 + \frac{48}{17} = 3,$$

so $c_1 = \frac{22}{17}$ and $c_2 = \frac{14}{17}$. Thus we finally arrive at the solution of the given initial value problem (1), (2), namely,

$$u = \frac{22}{17}e^{-t/2}\cos t + \frac{14}{17}e^{-t/2}\sin t + \frac{12}{17}\cos t + \frac{48}{17}\sin t. \tag{6}$$

The graph of the solution (6) is shown by the green curve in Figure 3.8.1.

It is important to note that the solution consists of two distinct parts. The first two terms on the right-hand side of equation (6) contain the exponential factor $e^{-t/2}$; as a result, they rapidly approach zero. It is customary to call these terms the **transient solution**. The remaining terms in equation (6) involve only sines and cosines, so they represent an oscillation that continues indefinitely. We refer to them as the **steady-state solution**. The dotted red and dashed blue curves in Figure 3.8.1 show the transient and the steady-state parts of the solution, respectively. The transient part comes from the solution of the homogeneous equation corresponding to equation (1) and is needed to satisfy the initial conditions. The steady-state solution is the particular solution of the full nonhomogeneous equation. After a fairly short time, the transient solution is vanishingly small and the full solution is essentially indistinguishable from the steady state.

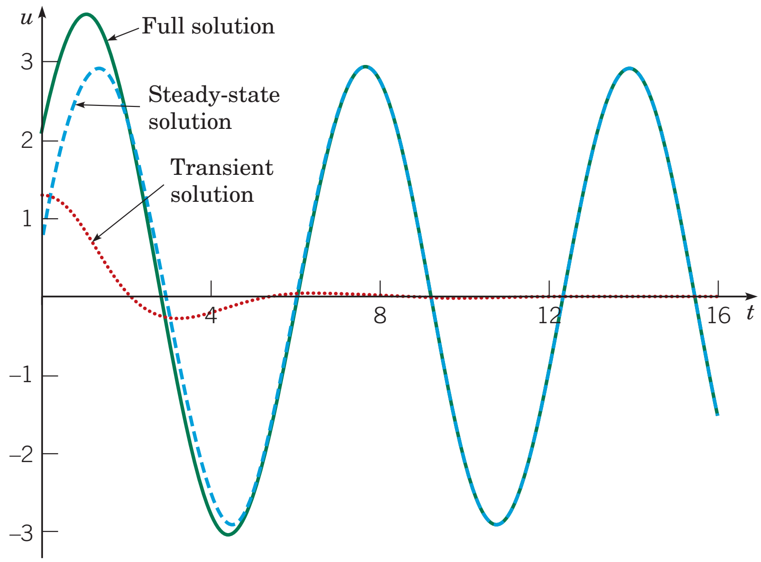

FIGURE 3.8.1 Solution of the initial value problem (1), (2): $u'' + u' + 5u/4 = 3\cos t$, $u(0) = 2$, $u'(0) = 3$. The full solution (solid green) is the sum of the transient solution (dotted red) and steady-state solution (dashed blue).

The equation of motion of a general spring-mass system subject to an external force $F(t)$ is equation (7) in Section 3.7:

$$mu''(t) + \gamma u'(t) + ku(t) = F(t), \tag{7}$$

where m, γ, and k are the mass, damping coefficient, and spring constant of the spring-mass system. Suppose now that the external force is given by $F_0 \cos(\omega t)$, where F_0 and ω are positive constants representing the amplitude and frequency, respectively, of the force. Then equation (7) becomes

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t). \tag{8}$$

Solutions of equation (8) behave very much like the solution in the preceding example. The general solution of equation (8) must have the form

$$u = c_1 u_1(t) + c_2 u_2(t) + A\cos(\omega t) + B\sin(\omega t) = u_c(t) + U(t). \tag{9}$$

The first two terms on the right-hand side of equation (9) are the general solution $u_c(t)$ of the homogeneous equation corresponding to equation (8), and the latter two terms are a particular solution $U(t)$ of the full nonhomogeneous equation. The coefficients A and B can be found, as usual, by substituting these terms into the differential equation (8), while the arbitrary constants c_1 and c_2 are available to satisfy initial conditions, if any are prescribed. The solutions $u_1(t)$

and $u_2(t)$ of the homogeneous equation depend on the roots r_1 and r_2 of the characteristic equation $mr^2 + \gamma r + k = 0$. Since m, γ, and k are all positive, it follows that r_1 and r_2 either are real and negative or are complex conjugates with a negative real part. In either case, both $u_1(t)$ and $u_2(t)$ approach zero as $t \to \infty$. Since $u_c(t)$ dies out as t increases, it is called the **transient solution**. In many applications, it is of little importance and (depending on the value of γ) may well be undetectable after only a few seconds.

The remaining terms in equation (9)—namely, $U(t) = A\cos(\omega t) + B\sin(\omega t)$—do not die out as t increases but persist indefinitely, or as long as the external force is applied. They represent a steady oscillation with the same frequency as the external force and are called the **steady-state solution** or the **forced response** of the system. The transient solution enables us to satisfy whatever initial conditions may be imposed. With increasing time, the energy put into the system by the initial displacement and velocity is dissipated through the damping force, and the motion then becomes the response of the system to the external force. Without damping, the effect of the initial conditions would persist for all time.

It is convenient to express $U(t)$ as a single trigonometric term rather than as a sum of two terms. Recall that we did this for other similar expressions in Section 3.7. Thus we write

$$U(t) = R\cos(\omega t - \delta). \tag{10}$$

The amplitude R and phase δ depend directly on A and B and indirectly on the parameters in the differential equation (8). It is possible to show, by straightforward but somewhat lengthy algebraic computations, that

$$R = \frac{F_0}{\Delta}, \quad \cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \quad \text{and} \quad \sin\delta = \frac{\gamma\omega}{\Delta}, \tag{11}$$

where

$$\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \quad \text{and} \quad \omega_0^2 = \frac{k}{m}. \tag{12}$$

Recall that ω_0 is the natural frequency of the unforced system in the absence of damping.

We now investigate how the amplitude R of the steady-state oscillation depends on the frequency ω of the external force. Substituting from equation (12) into the expression for R in equation (11) and executing some algebraic manipulations, we find that

$$\frac{Rk}{F_0} = \left(\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \Gamma\left(\frac{\omega}{\omega_0}\right)^2\right)^{-1/2} \quad \text{where } \Gamma = \frac{\gamma^2}{mk}. \tag{13}$$

Observe that the quantity Rk/F_0 is the ratio of the amplitude R of the forced response to F_0/k, the static displacement of the spring produced by a force F_0.

For low frequency excitation—that is, as $\omega \to 0$—it follows from equation (13) that $Rk/F_0 \to 1$ or $R \to F_0/k$. At the other extreme, for very high frequency excitation, equation (13) implies that $R \to 0$ as $\omega \to \infty$. At an intermediate value of ω the amplitude may have a maximum. To find this maximum point, we can differentiate R with respect to ω and set the result equal to zero. In this way we find that the maximum amplitude occurs when $\omega = \omega_{\max}$, where

$$\omega_{\max}^2 = \omega_0^2 - \frac{\gamma^2}{2m^2} = \omega_0^2\left(1 - \frac{\gamma^2}{2mk}\right). \tag{14}$$

Note that $\omega_{\max} < \omega_0$ and that $\omega_{\max}$ is close to ω_0 when γ is small. The maximum value of R is

$$R_{\max} = \frac{F_0}{\gamma\omega_0\sqrt{1 - (\gamma^2/4mk)}} \cong \frac{F_0}{\gamma\omega_0}\left(1 + \frac{\gamma^2}{8mk}\right), \tag{15}$$

where the last expression is an approximation that is valid when γ is small (see Problem 5). If $\frac{\gamma^2}{mk} > 2$, then $\omega_{\max}$ as given by equation (14) is imaginary; in this case the maximum value of R occurs for $\omega = 0$, and R is a monotone decreasing function of ω. Recall that critical damping occurs when $\frac{\gamma^2}{mk} = 4$.

For small γ it follows from equation (15) that $R_{\max} \cong \frac{F_0}{\gamma\omega_0}$. Thus, for lightly damped systems, the amplitude R of the forced response when ω is near ω_0 is quite large even for relatively small external forces, and the smaller the value of γ, the more pronounced is this effect. This phenomenon is known as **resonance**, and it is often an important design consideration. Resonance can be either good or bad, depending on the circumstances. It must be taken very seriously in the design of structures, such as buildings and bridges, where it can produce instabilities that might lead to the catastrophic failure of the structure. On the other hand, resonance can be put to good use in the design of instruments, such as seismographs, that are intended to detect weak periodic incoming signals.

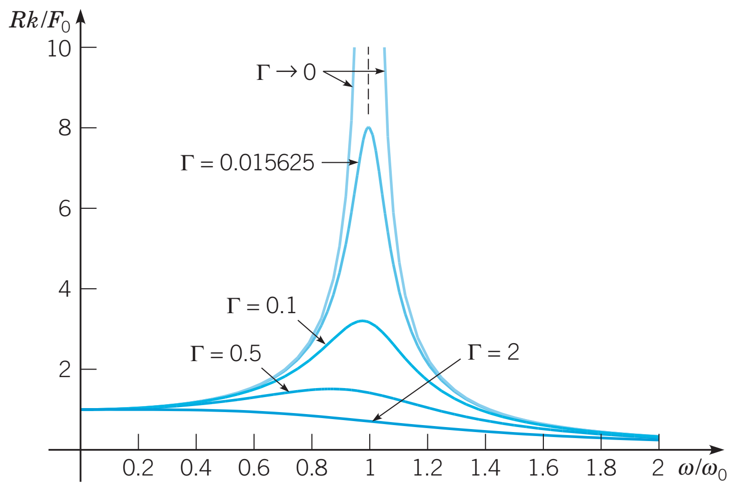

FIGURE 3.8.2 Forced vibration with damping: amplitude of steady-state response versus frequency of driving force for several values of the dimensionless damping parameter $\Gamma = \gamma^2/mk$.

Figure 3.8.2 contains some representative graphs of $\frac{Rk}{F_0}$ versus $\frac{\omega}{\omega_0}$ for several values of $\Gamma = \frac{\gamma^2}{mk}$. We refer to Γ as a damping parameter, as the following examples will explain. The graph corresponding to $\Gamma = 0.015625$ is included because this is the value of Γ that occurs in Example 2 below. Note particularly the sharp peak in the curve corresponding to $\Gamma = 0.015625$ near $\frac{\omega}{\omega_0} = 1$. The limiting case as $\Gamma \to 0$ is also shown. It follows from equation (13), or from equations (11) and (12), that $R \to \frac{F_0}{m\left|\omega_0^2 - \omega^2\right|}$ as $\gamma \to 0$ and hence $\frac{Rk}{F_0}$ is asymptotic to the vertical line $\omega = \omega_0$, as shown in the figure. As the damping in the system increases, the peak response gradually diminishes.

Figure 3.8.2 also illustrates the usefulness of dimensionless variables. You can easily verify that each of the quantities $\frac{Rk}{F_0}$, $\frac{\omega}{\omega_0}$, and Γ is dimensionless (see Problem 9d). The importance of this observation is that the number of significant parameters in the problem has been reduced to three rather than the five that appear in equation (8). Thus only one family of curves, of which a few are shown in Figure 3.8.2, is needed to describe the response-versus-frequency behavior of all systems governed by equation (8).

The phase angle δ also depends in an interesting way on ω. For ω near zero, it follows from equations (11) and (12) that $\cos\delta \cong 1$ and $\sin\delta \cong 0$. Thus $\delta \cong 0$, and the response is nearly in phase with the excitation, meaning that they rise and fall together and, in particular, assume their respective maxima nearly together and their respective minima nearly together.

For $\omega = \omega_0$ we find that $\cos\delta = 0$ and $\sin\delta = 1$, so $\delta = \pi/2$. In this case the response lags behind the excitation by $\pi/2$; that is, the peaks of the response occur $\pi/2$ later than the peaks of the excitation, and similarly for the valleys. Finally, for ω very large, we have $\cos\delta \cong -1$ and $\sin\delta \cong 0$. Thus $\delta \cong \pi$ so that the response is nearly out of phase with the excitation; this means that the response is minimum when the excitation is maximum, and vice versa. Figure 3.8.3 shows the graphs of δ versus ω/ω_0 for several values of Γ. For small damping, the phase transition from near $\delta = 0$ to near $\delta = \pi$ occurs rather abruptly, whereas for larger values of the damping parameter, the transition takes place more gradually.

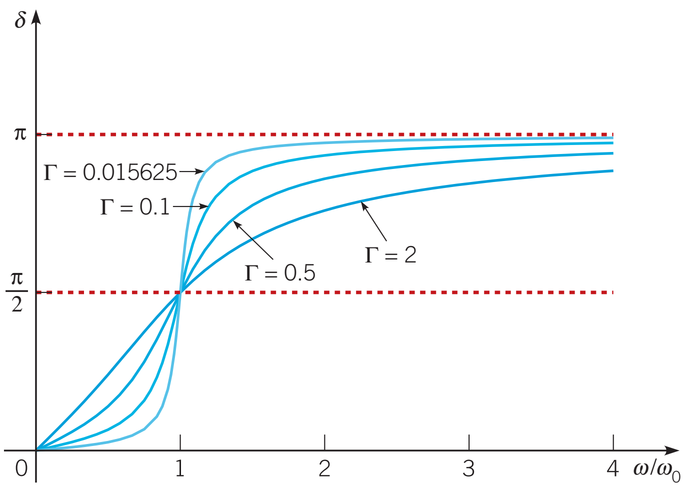

FIGURE 3.8.3 Forced vibration with damping: phase of steady-state response versus frequency of driving force for several values of the dimensionless damping parameter $\Gamma = \gamma^2/mk$.

EXAMPLE 2

Consider the initial value problem

$$u'' + \frac{1}{8}u' + u = 3\cos(\omega t), \quad u(0) = 2, \quad u'(0) = 0. \tag{16}$$

Show plots of the solution for different values of the forcing frequency ω, and compare them with corresponding plots of the forcing function.

Solution:

For this system we have $\omega_0 = 1$ and $\Gamma = 1/64 = 0.015625$. Its unforced motion was discussed in Example 3 of Section 3.7, and Figure 3.7.7 shows the graph of the solution of the unforced problem. Figures 3.8.4, 3.8.5, and 3.8.6 show the solution of the forced problem (16) for $\omega = 0.3$, $\omega = 1$, and $\omega = 2$, respectively. The graph of the corresponding forcing function is also shown in each figure. In this example the static displacement, F_0/k, is equal to 3.

Figure 3.8.4 shows the low frequency case, $\omega/\omega_0 = 0.3$. After the initial transient response is substantially damped out, the remaining steady-state response is essentially in phase with the excitation, and the amplitude of the response is somewhat larger than the static displacement. To be specific, $R \cong 3.2939$ and $\delta \cong 0.041185$.

The resonant case, $\omega/\omega_0 = 1$, is shown in Figure 3.8.5. Here, the amplitude of the steady-state response is eight times the static displacement, and the figure also shows the predicted phase lag of $\pi/2$ relative to the external force.

The case of comparatively high frequency excitation is shown in Figure 3.8.6. Observe that the amplitude of the steady forced response is approximately one-third the static displacement and that the phase difference between the excitation and the response is approximately π. More precisely, we find that $R \cong 0.99655$ and that $\delta \cong 3.0585$.

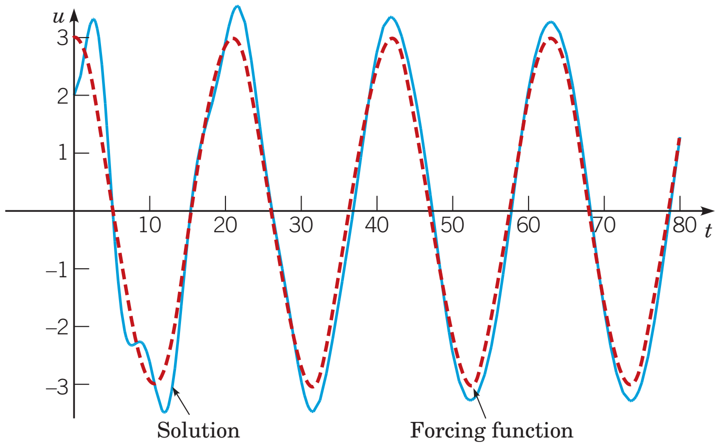

FIGURE 3.8.4 A forced vibration with damping; the solution (solid blue) of equation (16) with $\omega = 0.3$: $u'' + \frac{1}{8}u' + u = 3\cos(0.3t)$, $u(0) = 2$, $u'(0) = 0$. The dashed red curve is the external force: $F(t) = 3\cos(0.3t)$.

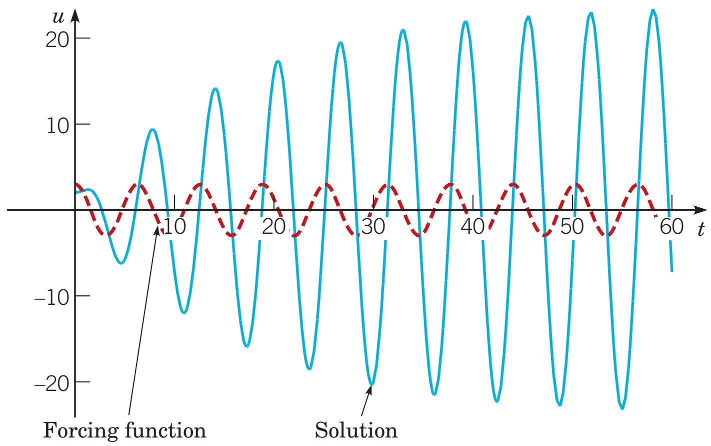

FIGURE 3.8.5 A forced vibration with damping; the solution (solid blue) of equation (16) with $\omega = 1$: $u'' + \frac{1}{8}u' + u = 3\cos t$, $u(0) = 2$, $u'(0) = 0$. The dashed red curve is the external force: $F(t) = 3\cos t$.

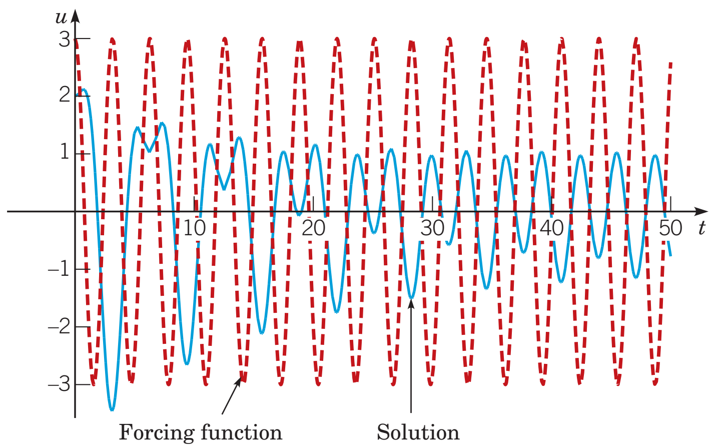

FIGURE 3.8.6 A forced vibration with damping; the solution (solid blue) of equation (16) with $\omega = 2$: $u'' + \frac{1}{8}u' + u = 3\cos(2t)$, $u(0) = 2$, $u'(0) = 0$. The dashed red curve is the external force: $F(t) = 3\cos(2t)$.

Forced Vibrations Without Damping. We now assume that $\gamma = 0$ in equation (8), thereby obtaining the equation of motion of an undamped forced oscillator,

$$mu'' + ku = F_0 \cos(\omega t). \tag{17}$$

The form of the general solution of equation (17) is different, depending on whether the forcing frequency ω is different from or equal to the natural frequency $\omega_0 = \sqrt{k/m}$ of the unforced system. First consider the case $\omega \neq \omega_0$; then the general solution of equation (17) is

$$u = c_1 \cos\big(\omega_0 t\big) + c_2 \sin\big(\omega_0 t\big) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t). \tag{18}$$

The constants c_1 and c_2 are determined by the initial conditions. The resulting motion is, in general, the sum of two periodic motions of different frequencies (ω_0 and ω) and different amplitudes as well.

It is particularly interesting to suppose that the mass is initially at rest so that the initial conditions are $u(0) = 0$ and $u'(0) = 0$. Then the energy driving the system comes entirely from the external force, with no contribution from the initial conditions. In this case it turns out that the constants c_1 and c_2 in equation (18) are given by

$$c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}, \qquad c_2 = 0, \tag{19}$$

and the solution of equation (17) is

$$u = \frac{F_0}{m(\omega_0^2 - \omega^2)}\big(\cos(\omega t) - \cos(\omega_0 t)\big). \tag{20}$$

This is the sum of two periodic functions of different periods but the same amplitude. Making use of the trigonometric identities for $\cos(A \pm B)$ with $A = \frac{1}{2}(\omega_0 + \omega)t$ and $B = \frac{1}{2}(\omega_0 - \omega)t$, we can write equation (20) in the form

$$u = \frac{2F_0}{m}\left(\omega_0^2 - \omega^2\right) \sin\left(\frac{1}{2}(\omega_0 - \omega)t\right) \sin\left(\frac{1}{2}(\omega_0 + \omega)t\right). \tag{21}$$

If $|\omega_0 - \omega|$ is small, then $\omega_0 + \omega$ is much greater than $|\omega_0 - \omega|$. Consequently, $\sin\left(\frac{1}{2}(\omega_0 + \omega)t\right)$ is a rapidly oscillating function compared to $\sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)$. Thus the motion is a rapid oscillation with frequency $\frac{1}{2}(\omega_0 + \omega)$ but with a slowly varying sinusoidal amplitude

$$\frac{2F_0}{m\left|\omega_0^2 - \omega^2\right|}\left|\sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)\right|.$$

This type of motion, possessing a periodic variation of amplitude, exhibits what is called a **beat**. For example, such a phenomenon occurs in acoustics when two tuning forks of nearly equal frequency are excited simultaneously. In this case the periodic variation of amplitude is quite apparent to the unaided ear. In electronics, the variation of the amplitude with time is called **amplitude modulation**.

EXAMPLE 3

Solve the initial value problem

$$u'' + u = \frac{1}{2}\cos(0.8t), \quad u(0) = 0, \ \ u'(0) = 0, \tag{22}$$

and plot the solution.

Solution:

In this case $\omega_0 = 1$, $\omega = 0.8$, and $F_0 = \frac{1}{2}$, so from equation (21) the solution of the given problem is

$$u = 2.778 \sin(0.1t) \sin(0.9t). \tag{23}$$

A graph of this solution is shown in Figure 3.8.7. The amplitude variation has a slow frequency of 0.1 and a corresponding slow period of $2\pi/0.1 = 20\pi$. Note that a half-period of 10π corresponds to a single cycle of increasing and then decreasing amplitude. The displacement of the spring-mass system oscillates with a relatively fast frequency of 0.9, which is only slightly less than the natural frequency ω_0.

Now imagine that the forcing frequency ω is increased, say, to $\omega = 0.9$. Then the slow frequency is halved to 0.05, and the corresponding slow half-period is doubled to 20π. The multiplier 2.7778 also increases substantially, to 5.263. However, the fast frequency is only marginally increased, to 0.95. Can you visualize what happens as ω takes on values closer and closer to the natural frequency $\omega_0 = 1$?

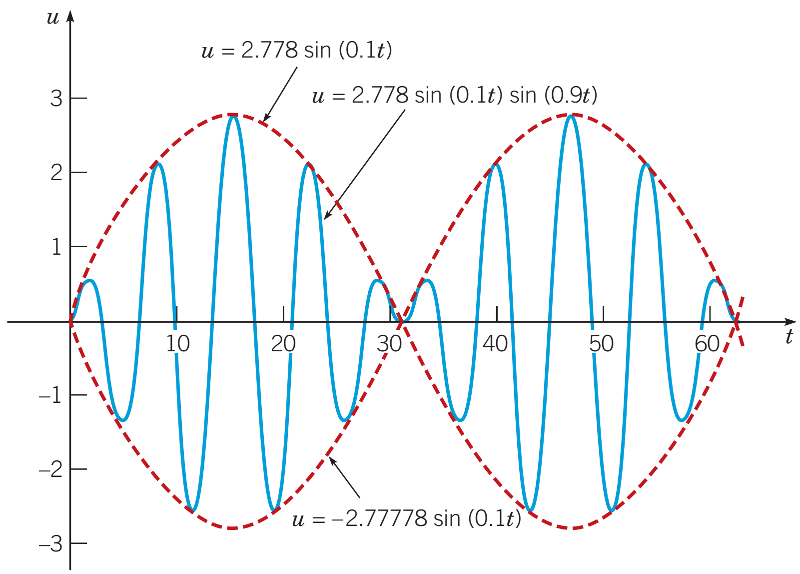

FIGURE 3.8.7 A beat; the solution (solid blue) of equation (22): $u'' + u = \frac{1}{2}\cos(0.8t)$, $u(0) = 0$, $u'(0) = 0$ is $u = 2.778 \sin(0.1t) \sin(0.9t)$. The dashed red curve is the external force $F(t) = \frac{1}{2}\cos(0.8t)$.

Now let us return to equation (17) and consider the case of resonance, where $\omega = \omega_0$; that is, the frequency of the forcing function is the same as the natural frequency of the system. Then the nonhomogeneous term $F_0 \cos(\omega t)$ is a solution of the homogeneous equation. In this case the solution of equation (17) is

$$u = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t). \tag{24}$$

Consider the following example.

EXAMPLE 4

Solve the initial value problem

$$u'' + u = \frac{1}{2}\cos t, \quad u(0) = 0, \quad u'(0) = 0, \tag{25}$$

and plot the graph of the solution.

Solution:

The general solution of the differential equation is

$$u = c_1 \cos t + c_2 \sin t + \frac{t}{4} \sin t,$$

and the initial conditions require that $c_1 = c_2 = 0$. Thus the solution of the given initial value problem is

$$u = \frac{t}{4}\sin t. \tag{26}$$

The graph of the solution is shown in Figure 3.8.8.

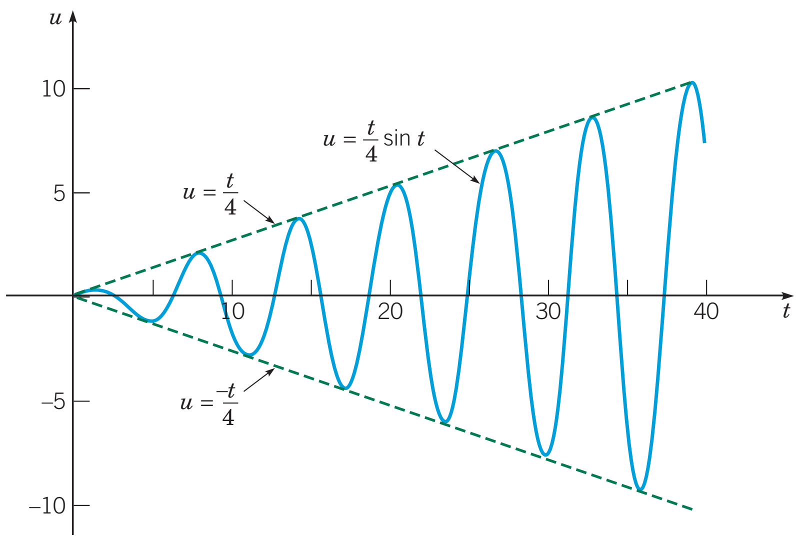

FIGURE 3.8.8 Resonance; the solution (solid blue) of equation (25): $u'' + u = \frac{1}{2}\cos t$, $u(0) = 0$, $u'(0) = 0$ is $u = \frac{t}{4}\sin t$.

Because of the term $t\sin(\omega_0 t)$, the solution (24) predicts that the motion will become unbounded as $t \to \infty$ regardless of the values of c_1 and c_2, and Figure 3.8.8 bears this out. Of course, in reality, unbounded oscillations do not occur, because the spring cannot stretch infinitely far. Moreover, as soon as u becomes large, the mathematical model on which equation (17) is based is no longer valid, since the assumption that the spring force depends linearly on the displacement requires that u be small. As we have seen, if damping is included in the model, the predicted motion remains bounded; however, the response to the input function $F_0\cos(\omega t)$ may be quite large if the damping is small and ω is close to ω_0.

Problems

In each of Problems 1 through 3, write the given expression as a product of two trigonometric functions of different frequencies.

1. $\sin(7t) - \sin(6t)$

2. $\cos(\pi t) + \cos(2\pi t)$

3. $\sin(3t) + \sin(4t)$

4. A mass of 5 kg stretches a spring 10 cm. The mass is acted on by an external force of $10\sin(t/2)$ N (newtons) and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 4 cm/s. If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/s, formulate the initial value problem describing the motion of the mass.

5. a. Find the solution of the initial value problem in Problem 4.

b. Identify the transient and steady-state parts of the solution.

G **c.** Plot the graph of the steady-state solution.

N **d.** If the given external force is replaced by a force of $2\cos(\omega t)$ of frequency ω, find the value of ω for which the amplitude of the forced response is maximum.

N **6.** A mass that weighs 8 lb stretches a spring 6 in. The system is acted on by an external force of $8\sin(8t)$ lb. If the mass is pulled down 3 in and then released, determine the position of the mass at any time. Determine the first four times at which the velocity of the mass is zero.

7. A spring is stretched 6 in by a mass that weighs 8 lb. The mass is attached to a dashpot mechanism that has a damping constant of $\frac{1}{4}$ lb·s/ft and is acted on by an external force of $4\cos(2t)$ lb.

a. Determine the steady-state response of this system.

b. If the given mass is replaced by a mass m, determine the value of m for which the amplitude of the steady-state response is maximum.

8. A spring-mass system has a spring constant of 3 N/m. A mass of 2 kg is attached to the spring, and the motion takes place in a viscous fluid that offers a resistance numerically equal to the magnitude of the instantaneous velocity. If the system is driven by an external force of $(3\cos(3t) - 2\sin(3t))$ N, determine the steady-state response. Express your answer in the form $R\cos(\omega t - \delta)$.

9. In this problem we ask you to supply some of the details in the analysis of a forced damped oscillator.

a. Derive equations (10), (11), and (12) for the steady-state solution of equation (8).
b. Derive the expression in equation (13) for Rk/F_0.
c. Show that ω_{max}^2 and R_{max} are given by equations (14) and (15), respectively.
d. Verify that Rk/F_0, ω/ω_0, and $\Gamma = \gamma^2/(mk)$ are all dimensionless quantities.

10. Find the velocity of the steady-state response given by equation (10). Then show that the velocity is maximum when $\omega = \omega_0$.

11. Find the solution of the initial value problem

$$u'' + u = F(t), \quad u(0) = 0, \quad u'(0) = 0,$$

where

$$F(t) = \begin{cases} F_0 t, & 0 \le t \le \pi, \\ F_0(2\pi - t), & \pi < t \le 2\pi, \\ 0, & 2\pi < t. \end{cases}$$

Hint: Treat each time interval separately, and match the solutions in the different intervals by requiring u and u' to be continuous functions of t.

N **12.** A series circuit has a capacitor of 0.25×10^{-6} F, a resistor of $5 \times 10^3\ \Omega$, and an inductor of 1 H. The initial charge on the capacitor is zero. If a 12 V battery is connected to the circuit and the circuit is closed at $t = 0$, determine the charge on the capacitor at $t = 0.001$ s, at $t = 0.01$ s, and at any time t. Also determine the limiting charge as $t \to \infty$.

N **13.** Consider the forced but undamped system described by the initial value problem

$$u'' + u = 3\cos(\omega t), \quad u(0) = 0, \quad u'(0) = 0.$$

a. Find the solution $u(t)$ for $\omega \neq 1$.
G **b.** Plot the solution $u(t)$ versus t for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Describe how the response $u(t)$ changes as ω varies in this interval. What happens as ω takes on values closer and closer to 1? Note that the natural frequency of the unforced system is $\omega_0 = 1$.

14. Consider the vibrating system described by the initial value problem

$$u'' + u = 3\cos(\omega t), \quad u(0) = 1, \quad u'(0) = 1.$$

a. Find the solution for $\omega \neq 1$.
G **b.** Plot the solution $u(t)$ versus t for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Compare the results with those of Problem 13; that is, describe the effect of the nonzero initial conditions.

G **15.** For the initial value problem in Problem 13, plot u' versus u for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. (Recall that such a plot is called a phase plot.) Use a t interval that is long enough so that the phase plot appears as a closed curve. Mark your curve with arrows to show the direction in which it is traversed as t increases.

Problems 16 through 18 deal with the initial value problem

$$u'' + \frac{1}{8}u' + 4u = F(t), \quad u(0) = 2, \quad u'(0) = 0.$$

In each of these problems:

G **a.** Plot the given forcing function $F(t)$ versus t, and also plot the solution $u(t)$ versus t on the same set of axes. Use a t interval that is long enough so the initial transients are substantially eliminated. Observe the relation between the amplitude and phase of the forcing term and the amplitude and phase of the response. Note that $\omega_0 = \sqrt{k/m} = 2$.
G **b.** Draw the phase plot of the solution; that is, plot u' versus u.

16. $F(t) = 3\cos(t/4)$

17. $F(t) = 3\cos(2t)$

18. $F(t) = 3\cos(6t)$

G **19.** A spring-mass system with a hardening spring (Problem 24 of Section 3.7) is acted on by a periodic external force. In the absence of damping, suppose that the displacement of the mass satisfies the initial value problem

$$u'' + u + \frac{1}{5}u^3 = \cos\omega t, \quad u(0) = 0, \quad u'(0) = 0.$$

a. Let $\omega = 1$ and plot a computer-generated solution of the given problem. Does the system exhibit a beat?
b. Plot the solution for several values of ω between $1/2$ and 2. Describe how the solution changes as ω increases.

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