SEVENTH EDITION

DIFFERENTIAL EQUATIONS

with Boundary-Value Problems

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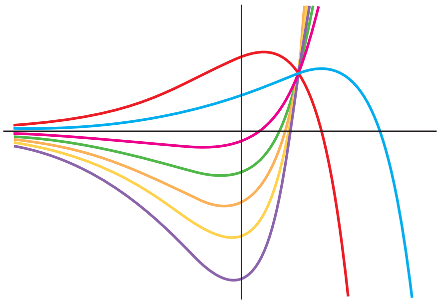

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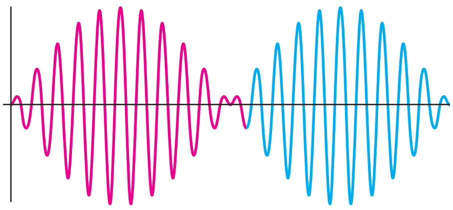

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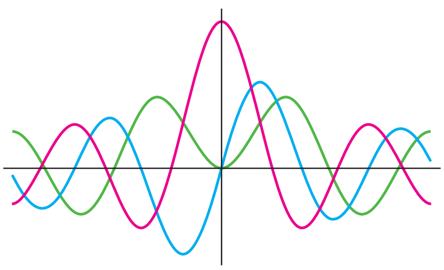

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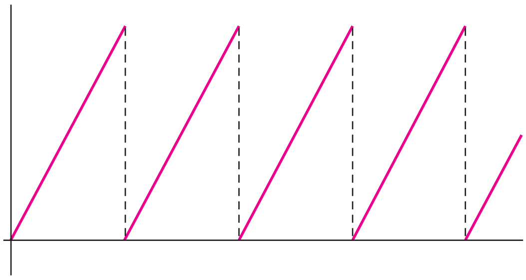

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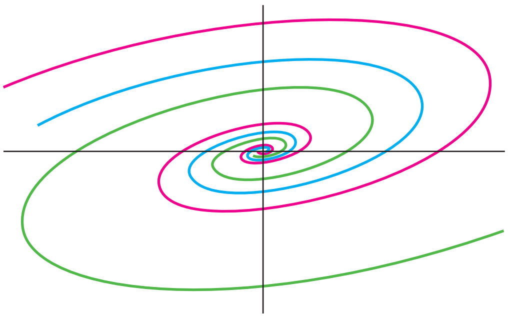

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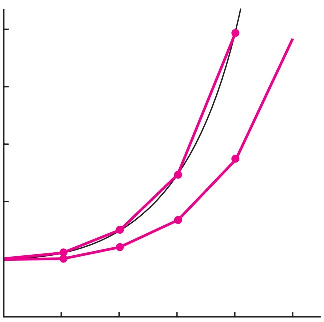

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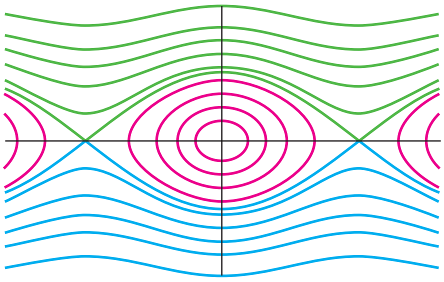

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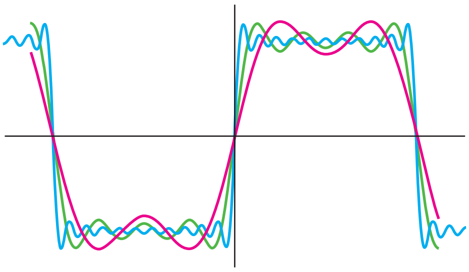

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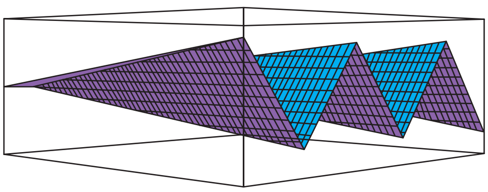

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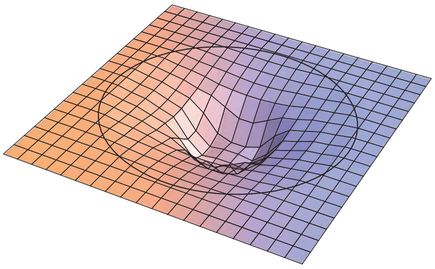

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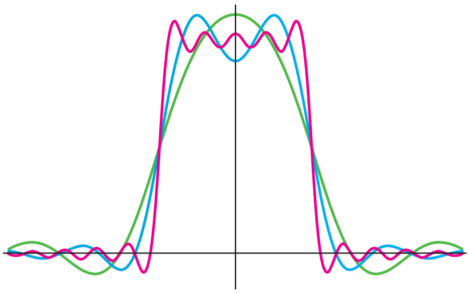

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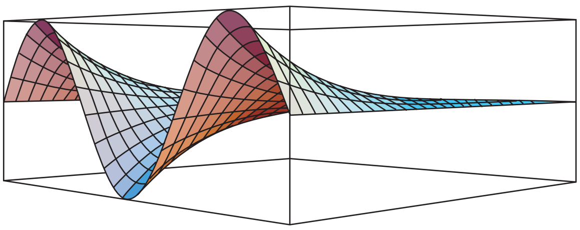

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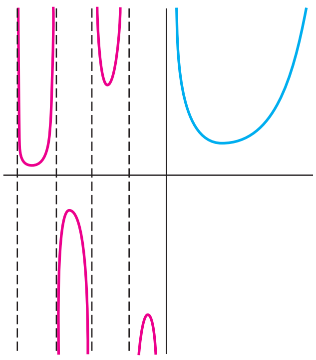
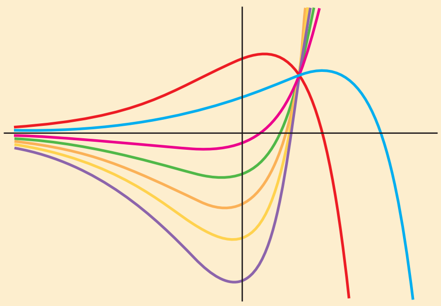


4 HIGHER-ORDER DIFFERENTIAL EQUATIONS



We turn now to the solution of ordinary differential equations of order two or higher. In the first seven sections of this chapter we examine the underlying theory and solution methods for certain kinds of *linear* equations. The elimination method for solving systems of linear equations is introduced in Section 4.8 because this method simply uncouples a system into individual linear equations in each dependent variable. The chapter concludes with a brief examinations of *nonlinear* higher-order equations.

4.1 PRELIMINARY THEORY—LINEAR EQUATIONS

REVIEW MATERIAL

- Reread the *Remarks* at the end of Section 1.1
- Section 2.3 (especially pages 54–58)

INTRODUCTION In Chapter 2 we saw that we could solve a few first-order differential equations by recognizing them as separable, linear, exact, homogeneous, or perhaps Bernoulli equations. Even though the solutions of these equations were in the form of a one-parameter family, this family, with one exception, did not represent the general solution of the differential equation. Only in the case of *linear* first-order differential equations were we able to obtain general solutions, by paying attention to certain continuity conditions imposed on the coefficients. Recall that a **general solution** is a family of solutions defined on some interval I that contains *all* solutions of the DE that are defined on I. Because our primary goal in this chapter is to find general solutions of linear higher-order DEs, we first need to examine some of the theory of linear equations.

4.1.1 INITIAL-VALUE AND BOUNDARY-VALUE PROBLEMS

INITIAL-VALUE PROBLEM In Section 1.2 we defined an initial-value problem for a general nth-order differential equation. For a linear differential equation an ***n*th-order initial-value problem** is

Solve: $$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1}$$

Subject to: $$y(x_0) = y_0, \quad y'(x_0) = y_1, \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

Recall that for a problem such as this one we seek a function defined on some interval I, containing x_0, that satisfies the differential equation and the n initial conditions specified at x_0: $y(x_0) = y_0$, $y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$. We have already seen that in the case of a second-order initial-value problem a solution curve must pass through the point (x_0, y_0) and have slope y_1 at this point.

EXISTENCE AND UNIQUENESS In Section 1.2 we stated a theorem that gave conditions under which the existence and uniqueness of a solution of a first-order initial-value problem were guaranteed. The theorem that follows gives sufficient conditions for the existence of a unique solution of the problem in (1).

THEOREM 4.1.1 Existence of a Unique Solution

Let $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial-value problem (1) exists on the interval and is unique.

EXAMPLE 1 Unique Solution of an IVP

The initial-value problem

$$3y''' + 5y'' - y' + 7y = 0, \quad y(1) = 0, \quad y'(1) = 0, \quad y''(1) = 0$$

possesses the trivial solution $y = 0$. Because the third-order equation is linear with constant coefficients, it follows that all the conditions of Theorem 4.1.1 are fulfilled. Hence $y = 0$ is the *only* solution on any interval containing $x = 1$. ■

EXAMPLE 2 Unique Solution of an IVP

You should verify that the function $y = 3e^{2x} + e^{-2x} - 3x$ is a solution of the initial-value problem

$$y'' - 4y = 12x, \quad y(0) = 4, \quad y'(0) = 1.$$

Now the differential equation is linear, the coefficients as well as $g(x) = 12x$ are continuous, and $a_2(x) = 1 \neq 0$ on any interval I containing $x = 0$. We conclude from Theorem 4.1.1 that the given function is the unique solution on I. ■

The requirements in Theorem 4.1.1 that $a_i(x)$, $i = 0, 1, 2, \ldots, n$ be continuous and $a_n(x) \neq 0$ for every x in I are both important. Specifically, if $a_n(x) = 0$ for some x in the interval, then the solution of a linear initial-value problem may not be unique or even exist. For example, you should verify that the function $y = cx^2 + x + 3$ is a solution of the initial-value problem

$$x^2y'' - 2xy' + 2y = 6, \quad y(0) = 3, \quad y'(0) = 1$$

on the interval $(-\infty, \infty)$ for any choice of the parameter c. In other words, there is no unique solution of the problem. Although most of the conditions of Theorem 4.1.1 are satisfied, the obvious difficulties are that $a_2(x) = x^2$ is zero at $x = 0$ and that the initial conditions are also imposed at $x = 0$.

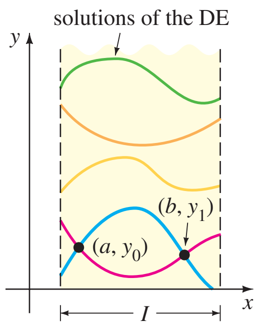


FIGURE 4.1.1 Solution curves of a BVP that pass through two points

BOUNDARY-VALUE PROBLEM Another type of problem consists of solving a linear differential equation of order two or greater in which the dependent variable y or its derivatives are specified at *different points.* A problem such as

$$\text{Solve:} \qquad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$\text{Subject to:} \qquad y(a) = y_0, \quad y(b) = y_1$$

is called a **boundary-value problem (BVP).** The prescribed values $y(a) = y_0$ and $y(b) = y_1$ are called **boundary conditions.** A solution of the foregoing problem is a function satisfying the differential equation on some interval I, containing a and b, whose graph passes through the two points (a, y_0) and (b, y_1). See Figure 4.1.1.

For a second-order differential equation other pairs of boundary conditions could be

$$y'(a) = y_0, \qquad y(b) = y_1$$
$$y(a) = y_0, \qquad y'(b) = y_1$$
$$y'(a) = y_0, \qquad y'(b) = y_1,$$

where y_0 and y_1 denote arbitrary constants. These three pairs of conditions are just special cases of the general boundary conditions

$$\alpha_1 y(a) + \beta_1 y'(a) = \gamma_1$$
$$\alpha_2 y(b) + \beta_2 y'(b) = \gamma_2.$$

The next example shows that even when the conditions of Theorem 4.1.1 are fulfilled, a boundary-value problem may have several solutions (as suggested in Figure 4.1.1), a unique solution, or no solution at all.

EXAMPLE 3 A BVP Can Have Many, One, or No Solutions

In Example 4 of Section 1.1 we saw that the two-parameter family of solutions of the differential equation $x'' + 16x = 0$ is

$$x = c_1 \cos 4t + c_2 \sin 4t. \tag{2}$$

(a) Suppose we now wish to determine the solution of the equation that further satisfies the boundary conditions $x(0) = 0$, $x(\pi/2) = 0$. Observe that the first condition $0 = c_1 \cos 0 + c_2 \sin 0$ implies that $c_1 = 0$, so $x = c_2 \sin 4t$. But when $t = \pi/2$, $0 = c_2 \sin 2\pi$ is satisfied for any choice of c_2, since $\sin 2\pi = 0$. Hence the boundary-value problem

$$x'' + 16x = 0, \quad x(0) = 0, \quad x\left(\frac{\pi}{2}\right) = 0 \tag{3}$$

has infinitely many solutions. Figure 4.1.2 shows the graphs of some of the members of the one-parameter family $x = c_2 \sin 4t$ that pass through the two points (0, 0) and $(\pi/2, 0)$.

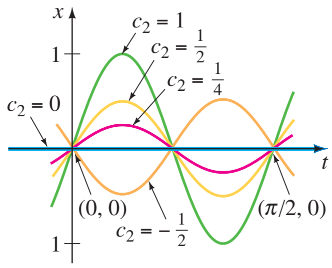


FIGURE 4.1.2 Some solution curves of (3)

(b) If the boundary-value problem in (3) is changed to

$$x'' + 16x = 0, \quad x(0) = 0, \quad x\left(\frac{\pi}{8}\right) = 0, \tag{4}$$

then $x(0) = 0$ still requires $c_1 = 0$ in the solution (2). But applying $x(\pi/8) = 0$ to $x = c_2 \sin 4t$ demands that $0 = c_2 \sin(\pi/2) = c_2 \cdot 1$. Hence $x = 0$ is a solution of this new boundary-value problem. Indeed, it can be proved that $x = 0$ is the *only* solution of (4).

(c) Finally, if we change the problem to

$$x'' + 16x = 0, \quad x(0) = 0, \quad x\left(\frac{\pi}{2}\right) = 1, \tag{5}$$

we find again from $x(0) = 0$ that $c_1 = 0$, but applying $x(\pi/2) = 1$ to $x = c_2 \sin 4t$ leads to the contradiction $1 = c_2 \sin 2\pi = c_2 \cdot 0 = 0$. Hence the boundary-value problem (5) has no solution. ■

4.1.2 HOMOGENEOUS EQUATIONS

A linear nth-order differential equation of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \tag{6}$$

is said to be **homogeneous,** whereas an equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \tag{7}$$

with $g(x)$ not identically zero, is said to be **nonhomogeneous.** For example, $2y'' + 3y' - 5y = 0$ is a homogeneous linear second-order differential equation, whereas $x^3y''' + 6y' + 10y = e^x$ is a nonhomogeneous linear third-order differential equation. The word *homogeneous* in this context does not refer to coefficients that are homogeneous functions, as in Section 2.5.

We shall see that to solve a nonhomogeneous linear equation (7), we must first be able to solve the **associated homogeneous equation** (6).

To avoid needless repetition throughout the remainder of this text, we shall, as a matter of course, make the following important assumptions when stating definitions and theorems about linear equations (1). On some common interval I,

Please remember these two assumptions.

- the coefficient functions $a_i(x)$, $i = 0, 1, 2, \ldots, n$ and $g(x)$ are continuous;
- $a_n(x) \neq 0$ for every x in the interval.

DIFFERENTIAL OPERATORS In calculus differentiation is often denoted by the capital letter D—that is, $dy/dx = Dy$. The symbol D is called a **differential operator** because it transforms a differentiable function into another function. For example, $D(\cos 4x) = -4 \sin 4x$ and $D(5x^3 - 6x^2) = 15x^2 - 12x$. Higher-order derivatives can be expressed in terms of D in a natural manner:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D(Dy) = D^2y \qquad \text{and, in general,} \qquad \frac{d^ny}{dx^n} = D^ny,$$

where y represents a sufficiently differentiable function. Polynomial expressions involving D, such as $D + 3$, $D^2 + 3D - 4$, and $5x^3D^3 - 6x^2D^2 + 4xD + 9$, are also differential operators. In general, we define an ***n*th-order differential operator** or **polynomial operator** to be

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x). \tag{8}$$

As a consequence of two basic properties of differentiation, $D(cf(x)) = cDf(x)$, c is a constant, and $D\{f(x) + g(x)\} = Df(x) + Dg(x)$, the differential operator L possesses a linearity property; that is, L operating on a linear combination of two differentiable functions is the same as the linear combination of L operating on the individual functions. In symbols this means that

$$L\{\alpha f(x) + \beta g(x)\} = \alpha L(f(x)) + \beta L(g(x)), \tag{9}$$

where α and β are constants. Because of (9) we say that the nth-order differential operator L is a **linear operator.**

DIFFERENTIAL EQUATIONS Any linear differential equation can be expressed in terms of the D notation. For example, the differential equation $y'' + 5y' + 6y = 5x - 3$ can be written as $D^2y + 5Dy + 6y = 5x - 3$ or $(D^2 + 5D + 6)y = 5x - 3$. Using (8), we can write the linear nth-order differential equations (6) and (7) compactly as

$$L(y) = 0 \qquad \text{and} \qquad L(y) = g(x),$$

respectively.

SUPERPOSITION PRINCIPLE In the next theorem we see that the sum, or **superposition,** of two or more solutions of a homogeneous linear differential equation is also a solution.

THEOREM 4.1.2 Superposition Principle—Homogeneous Equations

Let $y_1, y_2, \ldots, y_k$ be solutions of the homogeneous nth-order differential equation (6) on an interval I. Then the linear combination

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x),$$

where the c_i, $i = 1, 2, \ldots, k$ are arbitrary constants, is also a solution on the interval.

PROOF We prove the case $k = 2$. Let L be the differential operator defined in (8), and let $y_1(x)$ and $y_2(x)$ be solutions of the homogeneous equation $L(y) = 0$. If we define $y = c_1y_1(x) + c_2y_2(x)$, then by linearity of L we have

$$L(y) = L\{c_1y_1(x) + c_2y_2(x)\} = c_1\,L(y_1) + c_2\,L(y_2) = c_1 \cdot 0 + c_2 \cdot 0 = 0. \qquad \blacksquare$$

COROLLARIES TO THEOREM 4.1.2

(A) A constant multiple $y = c_1y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear differential equation is also a solution.
(B) A homogeneous linear differential equation always possesses the trivial solution $y = 0$.

EXAMPLE 4 **Superposition—Homogeneous DE**

The functions $y_1 = x^2$ and $y_2 = x^2 \ln x$ are both solutions of the homogeneous linear equation $x^3y''' - 2xy' + 4y = 0$ on the interval $(0, \infty)$. By the superposition principle the linear combination

$$y = c_1x^2 + c_2x^2 \ln x$$

is also a solution of the equation on the interval. ■

The function $y = e^{7x}$ is a solution of $y'' - 9y' + 14y = 0$. Because the differential equation is linear and homogeneous, the constant multiple $y = ce^{7x}$ is also a solution. For various values of c we see that $y = 9e^{7x}$, $y = 0$, $y = -\sqrt{5}e^{7x}, \ldots$ are all solutions of the equation.

LINEAR DEPENDENCE AND LINEAR INDEPENDENCE The next two concepts are basic to the study of linear differential equations.

DEFINITION 4.1.1 Linear Dependence/Independence

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x) = 0$$

for every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent.**

In other words, a set of functions is linearly independent on an interval I if the only constants for which

$$c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x) = 0$$

for every x in the interval are $c_1 = c_2 = \cdots = c_n = 0$.

It is easy to understand these definitions for a set consisting of two functions $f_1(x)$ and $f_2(x)$. If the set of functions is linearly dependent on an interval, then there exist constants c_1 and c_2 that are not both zero such that for every x in the interval, $c_1f_1(x) + c_2f_2(x) = 0$. Therefore if we assume that $c_1 \neq 0$, it follows that $f_1(x) = (-c_2/c_1)f_2(x)$; that is, *if a set of two functions is linearly dependent, then one function is simply a constant multiple of the other.* Conversely, if $f_1(x) = c_2f_2(x)$ for some constant c_2, then $(-1) \cdot f_1(x) + c_2f_2(x) = 0$ for every x in the interval. Hence the set of functions is linearly dependent because at least one of the constants (namely, $c_1 = -1$) is not zero. We conclude that *a set of two functions $f_1(x)$ and $f_2(x)$ is linearly independent when neither function is a constant multiple of the other* on the interval. For example, the set of functions $f_1(x) = \sin 2x$, $f_2(x) = \sin x \cos x$ is linearly dependent on $(-\infty, \infty)$ because $f_1(x)$ is a constant multiple of $f_2(x)$. Recall from the double-angle formula for the sine that $\sin 2x = 2 \sin x \cos x$. On the other hand, the set of functions $f_1(x) = x$, $f_2(x) = |x|$ is linearly independent on $(-\infty, \infty)$. Inspection of Figure 4.1.3 should convince you that neither function is a constant multiple of the other on the interval.

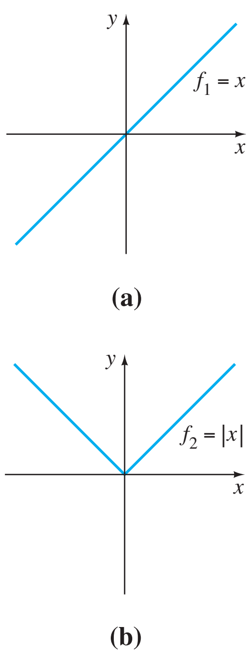


FIGURE 4.1.3 Set consisting of f_1 and f_2 is linearly independent on $(-\infty, \infty)$

It follows from the preceding discussion that the quotient $f_2(x)/f_1(x)$ is not a constant on an interval on which the set $f_1(x), f_2(x)$ is linearly independent. This little fact will be used in the next section.

EXAMPLE 5 Linearly Dependent Set of Functions

The set of functions $f_1(x) = \cos^2 x$, $f_2(x) = \sin^2 x$, $f_3(x) = \sec^2 x$, $f_4(x) = \tan^2 x$ is linearly dependent on the interval $(-\pi/2, \pi/2)$ because

$$c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x = 0$$

when $c_1 = c_2 = 1$, $c_3 = -1$, $c_4 = 1$. We used here $\cos^2 x + \sin^2 x = 1$ and $1 + \tan^2 x = \sec^2 x$. ■

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is linearly dependent on an interval if at least one function can be expressed as a linear combination of the remaining functions.

EXAMPLE 6 Linearly Dependent Set of Functions

The set of functions $f_1(x) = \sqrt{x} + 5$, $f_2(x) = \sqrt{x} + 5x$, $f_3(x) = x - 1$, $f_4(x) = x^2$ is linearly dependent on the interval $(0, \infty)$ because f_2 can be written as a linear combination of f_1, f_3, and f_4. Observe that

$$f_2(x) = 1 \cdot f_1(x) + 5 \cdot f_3(x) + 0 \cdot f_4(x)$$

for every x in the interval $(0, \infty)$. ■

SOLUTIONS OF DIFFERENTIAL EQUATIONS We are primarily interested in linearly independent functions or, more to the point, linearly independent solutions of a linear differential equation. Although we could always appeal directly to Definition 4.1.1, it turns out that the question of whether the set of n solutions $y_1, y_2, \ldots, y_n$ of a homogeneous linear nth-order differential equation (6) is linearly independent can be settled somewhat mechanically by using a determinant.

DEFINITION 4.1.2 Wronskian

Suppose each of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ possesses at least $n - 1$ derivatives. The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

where the primes denote derivatives, is called the **Wronskian** of the functions.

THEOREM 4.1.3 Criterion for Linearly Independent Solutions

Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous linear nth-order differential equation (6) on an interval I. Then the set of solutions is **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in the interval.

It follows from Theorem 4.1.3 that when $y_1, y_2, \ldots, y_n$ are n solutions of (6) on an interval I, the Wronskian $W(y_1, y_2, \ldots, y_n)$ is either identically zero or never zero on the interval.

A set of n linearly independent solutions of a homogeneous linear nth-order differential equation is given a special name.

DEFINITION 4.1.3 Fundamental Set of Solutions

Any set $y_1, y_2, \ldots, y_n$ of n linearly independent solutions of the homogeneous linear nth-order differential equation (6) on an interval I is said to be a **fundamental set of solutions** on the interval.

The basic question of whether a fundamental set of solutions exists for a linear equation is answered in the next theorem.

THEOREM 4.1.4 Existence of a Fundamental Set

There exists a fundamental set of solutions for the homogeneous linear nth-order differential equation (6) on an interval I.

Analogous to the fact that any vector in three dimensions can be expressed as a linear combination of the *linearly independent* vectors **i, j, k,** any solution of an nth-order homogeneous linear differential equation on an interval I can be expressed as a linear combination of n linearly independent solutions on I. In other words, n linearly independent solutions $y_1, y_2, \ldots, y_n$ are the basic building blocks for the general solution of the equation.

THEOREM 4.1.5 General Solution—Homogeneous Equations

Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the homogeneous linear nth-order differential equation (6) on an interval I. Then the **general solution** of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

Theorem 4.1.5 states that if $Y(x)$ is any solution of (6) on the interval, then constants $C_1, C_2, \ldots, C_n$ can always be found so that

$$Y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x).$$

We will prove the case when $n = 2$.

PROOF Let Y be a solution and let y_1 and y_2 be linearly independent solutions of $a_2 y'' + a_1 y' + a_0 y = 0$ on an interval I. Suppose that $x = t$ is a point in I for which $W(y_1(t), y_2(t)) \neq 0$. Suppose also that $Y(t) = k_1$ and $Y'(t) = k_2$. If we now examine the equations

$$C_1 y_1(t) + C_2 y_2(t) = k_1$$
$$C_1 y_1'(t) + C_2 y_2'(t) = k_2,$$

it follows that we can determine C_1 and C_2 uniquely, provided that the determinant of the coefficients satisfies

$$\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \neq 0.$$

But this determinant is simply the Wronskian evaluated at $x = t$, and by assumption, $W \neq 0$. If we define $G(x) = C_1y_1(x) + C_2y_2(x)$, we observe that $G(x)$ satisfies the differential equation since it is a superposition of two known solutions; $G(x)$ satisfies the initial conditions

$$G(t) = C_1y_1(t) + C_2y_2(t) = k_1 \quad \text{and} \quad G'(t) = C_1y_1'(t) + C_2y_2'(t) = k_2;$$

and $Y(x)$ satisfies the *same* linear equation and the *same* initial conditions. Because the solution of this linear initial-value problem is unique (Theorem 4.1.1), we have $Y(x) = G(x)$ or $Y(x) = C_1y_1(x) + C_2y_2(x)$. ■

EXAMPLE 7 General Solution of a Homogeneous DE

The functions $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are both solutions of the homogeneous linear equation $y'' - 9y = 0$ on the interval $(-\infty, \infty)$. By inspection the solutions are linearly independent on the x-axis. This fact can be corroborated by observing that the Wronskian

$$W(e^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0$$

for every x. We conclude that y_1 and y_2 form a fundamental set of solutions, and consequently, $y = c_1e^{3x} + c_2e^{-3x}$ is the general solution of the equation on the interval. ■

EXAMPLE 8 A Solution Obtained from a General Solution

The function $y = 4\sinh 3x - 5e^{3x}$ is a solution of the differential equation in Example 7. (Verify this.) In view of Theorem 4.1.5 we must be able to obtain this solution from the general solution $y = c_1e^{3x} + c_2e^{-3x}$. Observe that if we choose $c_1 = 2$ and $c_2 = -7$, then $y = 2e^{3x} - 7e^{-3x}$ can be rewritten as

$$y = 2e^{3x} - 2e^{-3x} - 5e^{-3x} = 4\left(\frac{e^{3x} - e^{-3x}}{2}\right) - 5e^{-3x}.$$

The last expression is recognized as $y = 4\sinh 3x - 5e^{-3x}$. ■

EXAMPLE 9 General Solution of a Homogeneous DE

The functions $y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ satisfy the third-order equation $y''' - 6y'' + 11y' - 6y = 0$. Since

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0$$

for every real value of x, the functions y_1, y_2, and y_3 form a fundamental set of solutions on $(-\infty, \infty)$. We conclude that $y = c_1e^x + c_2e^{2x} + c_3e^{3x}$ is the general solution of the differential equation on the interval. ■

4.1.3 NONHOMOGENEOUS EQUATIONS

Any function y_p, free of arbitrary parameters, that satisfies (7) is said to be a **particular solution** or **particular integral** of the equation. For example, it is a straightforward task to show that the constant function $y_p = 3$ is a particular solution of the nonhomogeneous equation $y'' + 9y = 27$.

Now if $y_1, y_2, \ldots, y_k$ are solutions of (6) on an interval I and y_p is any particular solution of (7) on I, then the linear combination

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x) + y_p \tag{10}$$

is also a solution of the nonhomogeneous equation (7). If you think about it, this makes sense, because the linear combination $c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$ is transformed into 0 by the operator $L = a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0$, whereas y_p is transformed into $g(x)$. If we use $k = n$ linearly independent solutions of the nth-order equation (6), then the expression in (10) becomes the general solution of (7).

THEOREM 4.1.6 General Solution—Nonhomogeneous Equations

Let y_p be any particular solution of the nonhomogeneous linear nth-order differential equation (7) on an interval I, and let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the associated homogeneous differential equation (6) on I. Then the **general solution** of the equation on the interval is

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p,$$

where the c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

PROOF Let L be the differential operator defined in (8) and let $Y(x)$ and $y_p(x)$ be particular solutions of the nonhomogeneous equation $L(y) = g(x)$. If we define $u(x) = Y(x) - y_p(x)$, then by linearity of L we have

$$L(u) = L\{Y(x) - y_p(x)\} = L(Y(x)) - L(y_p(x)) = g(x) - g(x) = 0.$$

This shows that $u(x)$ is a solution of the homogeneous equation $L(y) = 0$. Hence by Theorem 4.1.5, $u(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$, and so

$$Y(x) - y_p(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

or

$$Y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x). \qquad \blacksquare$$

COMPLEMENTARY FUNCTION We see in Theorem 4.1.6 that the general solution of a nonhomogeneous linear equation consists of the sum of two functions:

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x) = y_c(x) + y_p(x).$$

The linear combination $y_c(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$, which is the general solution of (6), is called the **complementary function** for equation (7). In other words, to solve a nonhomogeneous linear differential equation, we first solve the associated homogeneous equation and then find any particular solution of the nonhomogeneous equation. The general solution of the nonhomogeneous equation is then

$$\begin{aligned} y &= \textit{complementary function} + \textit{any particular solution} \\ &= y_c + y_p. \end{aligned}$$

EXAMPLE 10 General Solution of a Nonhomogeneous DE

By substitution the function $y_p = -\frac{11}{12} - \frac{1}{2}x$ is readily shown to be a particular solution of the nonhomogeneous equation

$$y''' - 6y'' + 11y' - 6y = 3x. \tag{11}$$

To write the general solution of (11), we must also be able to solve the associated homogeneous equation

$$y''' - 6y'' + 11y' - 6y = 0.$$

But in Example 9 we saw that the general solution of this latter equation on the interval $(-\infty, \infty)$ was $y_c = c_1e^x + c_2e^{2x} + c_3e^{3x}$. Hence the general solution of (11) on the interval is

$$y = y_c + y_p = c_1e^x + c_2e^{2x} + c_3e^{3x} - \frac{11}{12} - \frac{1}{2}x. \quad \blacksquare$$

ANOTHER SUPERPOSITION PRINCIPLE The last theorem of this discussion will be useful in Section 4.4 when we consider a method for finding particular solutions of nonhomogeneous equations.

THEOREM 4.1.7 Superposition Principle—Nonhomogeneous Equations

Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be k particular solutions of the nonhomogeneous linear nth-order differential equation (7) on an interval I corresponding, in turn, to k distinct functions $g_1, g_2, \ldots, g_k$. That is, suppose y_{p_i} denotes a particular solution of the corresponding differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_i(x), \quad (12)$$

where $i = 1, 2, \ldots, k$. Then

$$y_p = y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x) \quad (13)$$

is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y$$
$$= g_1(x) + g_2(x) + \cdots + g_k(x). \quad (14)$$

PROOF We prove the case $k = 2$. Let L be the differential operator defined in (8) and let $y_{p_1}(x)$ and $y_{p_2}(x)$ be particular solutions of the nonhomogeneous equations $L(y) = g_1(x)$ and $L(y) = g_2(x)$, respectively. If we define $y_p = y_{p_1}(x) + y_{p_2}(x)$, we want to show that y_p is a particular solution of $L(y) = g_1(x) + g_2(x)$. The result follows again by the linearity of the operator L:

$$L(y_p) = L\{y_{p_1}(x) + y_{p_2}(x)\} = L(y_{p_1}(x)) + L(y_{p_2}(x)) = g_1(x) + g_2(x). \quad \blacksquare$$

EXAMPLE 11 Superposition—Nonhomogeneous DE

You should verify that

$y_{p_1} = -4x^2$ is a particular solution of $y'' - 3y' + 4y = -16x^2 + 24x - 8$,

$y_{p_2} = e^{2x}$ is a particular solution of $y'' - 3y' + 4y = 2e^{2x}$,

$y_{p_3} = xe^x$ is a particular solution of $y'' - 3y' + 4y = 2xe^x - e^x$.

It follows from (13) of Theorem 4.1.7 that the superposition of y_{p_1}, y_{p_2}, and y_{p_3},

$$y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^x,$$

is a solution of

$$y'' - 3y' + 4y = \underbrace{-16x^2 + 24x - 8}_{g_1(x)} + \underbrace{2e^{2x}}_{g_2(x)} + \underbrace{2xe^x - e^x}_{g_3(x)}. \quad \blacksquare$$

NOTE If the y_{p_i} are particular solutions of (12) for $i = 1, 2, \ldots, k$, then the linear combination

$$y_p = c_1 y_{p_1} + c_2 y_{p_2} + \cdots + c_k y_{p_k},$$

where the c_i are constants, is also a particular solution of (14) when the right-hand member of the equation is the linear combination

$$c_1 g_1(x) + c_2 g_2(x) + \cdots + c_k g_k(x).$$

Before we actually start solving homogeneous and nonhomogeneous linear differential equations, we need one additional bit of theory, which is presented in the next section.

REMARKS

This remark is a continuation of the brief discussion of dynamical systems given at the end of Section 1.3.

A dynamical system whose rule or mathematical model is a linear nth-order differential equation

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = g(t)$$

is said to be an nth-order **linear system.** The n time-dependent functions $y(t)$, $y'(t), \ldots, y^{(n-1)}(t)$ are the **state variables** of the system. Recall that their values at some time t give the **state of the system.** The function g is variously called the **input function, forcing function,** or **excitation function.** A solution $y(t)$ of the differential equation is said to be the **output** or **response of the system.** Under the conditions stated in Theorem 4.1.1, the output or response $y(t)$ is uniquely determined by the input and the state of the system prescribed at a time t_0—that is, by the initial conditions $y(t_0), y'(t_0), \ldots, y^{(n-1)}(t_0)$.

For a dynamical system to be a linear system, it is necessary that the superposition principle (Theorem 4.1.7) holds in the system; that is, the response of the system to a superposition of inputs is a superposition of outputs. We have already examined some simple linear systems in Section 3.1 (linear first-order equations); in Section 5.1 we examine linear systems in which the mathematical models are second-order differential equations.

EXERCISES 4.1

Answers to selected odd-numbered problems begin on page ANS-4.

4.1.1 INITIAL-VALUE AND BOUNDARY-VALUE PROBLEMS

In Problems 1–4 the given family of functions is the general solution of the differential equation on the indicated interval. Find a member of the family that is a solution of the initial-value problem.

1. $y = c_1e^x + c_2e^{-x}$, $(-\infty, \infty)$;
$y'' - y = 0, \quad y(0) = 0, \quad y'(0) = 1$

2. $y = c_1e^{4x} + c_2e^{-x}$, $(-\infty, \infty)$;
$y'' - 3y' - 4y = 0, \quad y(0) = 1, \quad y'(0) = 2$

3. $y = c_1x + c_2x \ln x$, $(0, \infty)$;
$x^2y'' - xy' + y = 0, \quad y(1) = 3, \quad y'(1) = -1$

4. $y = c_1 + c_2 \cos x + c_3 \sin x$, $(-\infty, \infty)$;
$y''' + y' = 0, \quad y(\pi) = 0, \quad y'(\pi) = 2, \quad y''(\pi) = -1$

5. Given that $y = c_1 + c_2x^2$ is a two-parameter family of solutions of $xy'' - y' = 0$ on the interval $(-\infty, \infty)$, show that constants c_1 and c_2 cannot be found so that a member of the family satisfies the initial conditions $y(0) = 0$, $y'(0) = 1$. Explain why this does not violate Theorem 4.1.1.

6. Find two members of the family of solutions in Problem 5 that satisfy the initial conditions $y(0) = 0$, $y'(0) = 0$.

7. Given that $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$ is the general solution of $x'' + \omega^2x = 0$ on the interval $(-\infty, \infty)$, show that a solution satisfying the initial conditions $x(0) = x_0$, $x'(0) = x_1$ is given by

$$x(t) = x_0 \cos \omega t + \frac{x_1}{\omega} \sin \omega t.$$

8. Use the general solution of $x'' + \omega^2 x = 0$ given in Problem 7 to show that a solution satisfying the initial conditions $x(t_0) = x_0$, $x'(t_0) = x_1$ is the solution given in Problem 7 shifted by an amount t_0:

$$x(t) = x_0 \cos \omega(t - t_0) + \frac{x_1}{\omega} \sin \omega(t - t_0).$$

In Problems 9 and 10 find an interval centered about $x = 0$ for which the given initial-value problem has a unique solution.

9. $(x - 2)y'' + 3y = x, \quad y(0) = 0, \quad y'(0) = 1$

10. $y'' + (\tan x)y = e^x, \quad y(0) = 1, \quad y'(0) = 0$

11. **(a)** Use the family in Problem 1 to find a solution of $y'' - y = 0$ that satisfies the boundary conditions $y(0) = 0$, $y(1) = 1$.

(b) The DE in part (a) has the alternative general solution $y = c_3 \cosh x + c_4 \sinh x$ on $(-\infty, \infty)$. Use this family to find a solution that satisfies the boundary conditions in part (a).

(c) Show that the solutions in parts (a) and (b) are equivalent

12. Use the family in Problem 5 to find a solution of $xy'' - y' = 0$ that satisfies the boundary conditions $y(0) = 1$, $y'(1) = 6$.

In Problems 13 and 14 the given two-parameter family is a solution of the indicated differential equation on the interval $(-\infty, \infty)$. Determine whether a member of the family can be found that satisfies the boundary conditions.

13. $y = c_1 e^x \cos x + c_2 e^x \sin x; \quad y'' - 2y' + 2y = 0$

(a) $y(0) = 1, \quad y'(\pi) = 0$ **(b)** $y(0) = 1, \quad y(\pi) = -1$

(c) $y(0) = 1, \quad y\left(\frac{\pi}{2}\right) = 1$ **(d)** $y(0) = 0, \quad y(\pi) = 0.$

14. $y = c_1 x^2 + c_2 x^4 + 3; \quad x^2 y'' - 5xy' + 8y = 24$

(a) $y(-1) = 0, \quad y(1) = 4$ **(b)** $y(0) = 1, \quad y(1) = 2$

(c) $y(0) = 3, \quad y(1) = 0$ **(d)** $y(1) = 3, \quad y(2) = 15$

4.1.2 HOMOGENEOUS EQUATIONS

In Problems 15–22 determine whether the given set of functions is linearly independent on the interval $(-\infty, \infty)$.

15. $f_1(x) = x, \quad f_2(x) = x^2, \quad f_3(x) = 4x - 3x^2$

16. $f_1(x) = 0, \quad f_2(x) = x, \quad f_3(x) = e^x$

17. $f_1(x) = 5, \quad f_2(x) = \cos^2 x, \quad f_3(x) = \sin^2 x$

18. $f_1(x) = \cos 2x, \quad f_2(x) = 1, \quad f_3(x) = \cos^2 x$

19. $f_1(x) = x, \quad f_2(x) = x - 1, \quad f_3(x) = x + 3$

20. $f_1(x) = 2 + x, \quad f_2(x) = 2 + |x|$

21. $f_1(x) = 1 + x, \quad f_2(x) = x, \quad f_3(x) = x^2$

22. $f_1(x) = e^x, \quad f_2(x) = e^{-x}, \quad f_3(x) = \sinh x$

In Problems 23–30 verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Form the general solution.

23. $y'' - y' - 12y = 0; \quad e^{-3x}, e^{4x}, (-\infty, \infty)$

24. $y'' - 4y = 0; \quad \cosh 2x, \sinh 2x, (-\infty, \infty)$

25. $y'' - 2y' + 5y = 0; \quad e^x \cos 2x, e^x \sin 2x, (-\infty, \infty)$

26. $4y'' - 4y' + y = 0; \quad e^{x/2}, xe^{x/2}, (-\infty, \infty)$

27. $x^2 y'' - 6xy' + 12y = 0; \quad x^3, x^4, (0, \infty)$

28. $x^2 y'' + xy' + y = 0; \quad \cos(\ln x), \sin(\ln x), (0, \infty)$

29. $x^3 y''' + 6x^2 y'' + 4xy' - 4y = 0; \quad x, x^{-2}, x^{-2} \ln x, (0, \infty)$

30. $y^{(4)} + y'' = 0; \quad 1, x, \cos x, \sin x, (-\infty, \infty)$

4.1.3 NONHOMOGENEOUS EQUATIONS

In Problems 31–34 verify that the given two-parameter family of functions is the general solution of the nonhomogeneous differential equation on the indicated interval.

31. $y'' - 7y' + 10y = 24e^x;$
$y = c_1 e^{2x} + c_2 e^{5x} + 6e^x, (-\infty, \infty)$

32. $y'' + y = \sec x;$
$y = c_1 \cos x + c_2 \sin x + x \sin x + (\cos x) \ln(\cos x),$
$(-\pi/2, \pi/2)$

33. $y'' - 4y' + 4y = 2e^{2x} + 4x - 12;$
$y = c_1 e^{2x} + c_2 x e^{2x} + x^2 e^{2x} + x - 2, (-\infty, \infty)$

34. $2x^2 y'' + 5xy' + y = x^2 - x;$
$y = c_1 x^{-1/2} + c_2 x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x, (0, \infty)$

35. **(a)** Verify that $y_{p_1} = 3e^{2x}$ and $y_{p_2} = x^2 + 3x$ are, respectively, particular solutions of

$$y'' - 6y' + 5y = -9e^{2x}$$

and $$y'' - 6y' + 5y = 5x^2 + 3x - 16.$$

(b) Use part (a) to find particular solutions of

$$y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x}$$

and $$y'' - 6y' + 5y = -10x^2 - 6x + 32 + e^{2x}.$$

36. **(a)** By inspection find a particular solution of

$$y'' + 2y = 10.$$

(b) By inspection find a particular solution of

$$y'' + 2y = -4x.$$

(c) Find a particular solution of $y'' + 2y = -4x + 10$.

(d) Find a particular solution of $y'' + 2y = 8x + 5$.

Discussion Problems

37. Let $n = 1, 2, 3, \ldots$. Discuss how the observations $D^n x^{n-1} = 0$ and $D^n x^n = n!$ can be used to find the general solutions of the given differential equations.

(a) $y'' = 0$ **(b)** $y''' = 0$ **(c)** $y^{(4)} = 0$
(d) $y'' = 2$ **(e)** $y''' = 6$ **(f)** $y^{(4)} = 24$

38. Suppose that $y_1 = e^x$ and $y_2 = e^{-x}$ are two solutions of a homogeneous linear differential equation. Explain why $y_3 = \cosh x$ and $y_4 = \sinh x$ are also solutions of the equation.

39. **(a)** Verify that $y_1 = x^3$ and $y_2 = |x|^3$ are linearly independent solutions of the differential equation $x^2y'' - 4xy' + 6y = 0$ on the interval $(-\infty, \infty)$.

(b) Show that $W(y_1, y_2) = 0$ for every real number x. Does this result violate Theorem 4.1.3? Explain.

(c) Verify that $Y_1 = x^3$ and $Y_2 = x^2$ are also linearly independent solutions of the differential equation in part (a) on the interval $(-\infty, \infty)$.

(d) Find a solution of the differential equation satisfying $y(0) = 0$, $y'(0) = 0$.

(e) By the superposition principle, Theorem 4.1.2, both linear combinations $y = c_1y_1 + c_2y_2$ and $Y = c_1Y_1 + c_2Y_2$ are solutions of the differential equation. Discuss whether one, both, or neither of the linear combinations is a general solution of the differential equation on the interval $(-\infty, \infty)$.

40. Is the set of functions $f_1(x) = e^{x+2}$, $f_2(x) = e^{x-3}$ linearly dependent or linearly independent on $(-\infty, \infty)$? Discuss.

41. Suppose $y_1, y_2, \ldots, y_k$ are k linearly independent solutions on $(-\infty, \infty)$ of a homogeneous linear nth-order differential equation with constant coefficients. By Theorem 4.1.2 it follows that $y_{k+1} = 0$ is also a solution of the differential equation. Is the set of solutions $y_1, y_2, \ldots, y_k, y_{k+1}$ linearly dependent or linearly independent on $(-\infty, \infty)$? Discuss.

42. Suppose that $y_1, y_2, \ldots, y_k$ are k nontrivial solutions of a homogeneous linear nth-order differential equation with constant coefficients and that $k = n + 1$. Is the set of solutions $y_1, y_2, \ldots, y_k$ linearly dependent or linearly independent on $(-\infty, \infty)$? Discuss.

4.2 REDUCTION OF ORDER

REVIEW MATERIAL

- Section 2.5 (using a substitution)
- Section 4.1

INTRODUCTION In the preceding section we saw that the general solution of a homogeneous linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1}$$

is a linear combination $y = c_1y_1 + c_2y_2$, where y_1 and y_2 are solutions that constitute a linearly independent set on some interval I. Beginning in the next section, we examine a method for determining these solutions when the coefficients of the differential equation in (1) are constants. This method, which is a straightforward exercise in algebra, breaks down in a few cases and yields only a single solution y_1 of the DE. It turns out that we can construct a second solution y_2 of a homogeneous equation (1) (even when the coefficients in (1) are variable) provided that we know a nontrivial solution y_1 of the DE. The basic idea described in this section is that *equation* (1) *can be reduced to a linear first-order DE by means of a substitution* involving the known solution y_1. A second solution y_2 of (1) is apparent after this first-order differential equation is solved.

REDUCTION OF ORDER Suppose that y_1 denotes a nontrivial solution of (1) and that y_1 is defined on an interval I. We seek a second solution y_2 so that the set consisting of y_1 and y_2 is linearly independent on I. Recall from Section 4.1 that if y_1 and y_2 are linearly independent, then their quotient y_2/y_1 is nonconstant on I—that is, $y_2(x)/y_1(x) = u(x)$ or $y_2(x) = u(x)y_1(x)$. The function $u(x)$ can be found by substituting $y_2(x) = u(x)y_1(x)$ into the given differential equation. This method is called **reduction of order** because we must solve a linear first-order differential equation to find u.

EXAMPLE 1 A Second Solution by Reduction of Order

Given that $y_1 = e^x$ is a solution of $y'' - y = 0$ on the interval $(-\infty, \infty)$, use reduction of order to find a second solution y_2.

SOLUTION If $y = u(x)y_1(x) = u(x)e^x$, then the Product Rule gives

$$y' = ue^x + e^x u', \quad y'' = ue^x + 2e^x u' + e^x u'',$$

and so
$$y'' - y = e^x(u'' + 2u') = 0.$$

Since $e^x \neq 0$, the last equation requires $u'' + 2u' = 0$. If we make the substitution $w = u'$, this linear second-order equation in u becomes $w' + 2w = 0$, which is a linear first-order equation in w. Using the integrating factor e^{2x}, we can write $\frac{d}{dx}[e^{2x}w] = 0$. After integrating, we get $w = c_1e^{-2x}$ or $u' = c_1e^{-2x}$. Integrating again then yields $u = -\frac{1}{2}c_1e^{-2x} + c_2$. Thus

$$y = u(x)e^x = -\frac{c_1}{2}e^{-x} + c_2e^x. \tag{2}$$

By picking $c_2 = 0$ and $c_1 = -2$, we obtain the desired second solution, $y_2 = e^{-x}$. Because $W(e^x, e^{-x}) \neq 0$ for every x, the solutions are linearly independent on $(-\infty, \infty)$. ■

Since we have shown that $y_1 = e^x$ and $y_2 = e^{-x}$ are linearly independent solutions of a linear second-order equation, the expression in (2) is actually the general solution of $y'' - y = 0$ on $(-\infty, \infty)$.

GENERAL CASE Suppose we divide by $a_2(x)$ to put equation (1) in the **standard form**

$$y'' + P(x)y' + Q(x)y = 0, \tag{3}$$

where $P(x)$ and $Q(x)$ are continuous on some interval I. Let us suppose further that $y_1(x)$ is a known solution of (3) on I and that $y_1(x) \neq 0$ for every x in the interval. If we define $y = u(x)y_1(x)$, it follows that

$$y' = uy_1' + y_1u', \quad y'' = uy_1'' + 2y_1'u' + y_1u''$$

$$y'' + Py' + Qy = u[\underbrace{y_1'' + Py_1' + Qy_1}_{\text{zero}}] + y_1u'' + (2y_1' + Py_1)u' = 0.$$

This implies that we must have

$$y_1u'' + (2y_1' + Py_1)u' = 0 \quad \text{or} \quad y_1w' + (2y_1' + Py_1)w = 0, \tag{4}$$

where we have let $w = u'$. Observe that the last equation in (4) is both linear and separable. Separating variables and integrating, we obtain

$$\frac{dw}{w} + 2\frac{y_1'}{y_1}dx + P\,dx = 0$$

$$\ln|wy_1^2| = -\int P\,dx + c \quad \text{or} \quad wy_1^2 = c_1e^{-\int P\,dx}.$$

We solve the last equation for w, use $w = u'$, and integrate again:

$$u = c_1\int \frac{e^{-\int P\,dx}}{y_1^2}\,dx + c_2.$$

By choosing $c_1 = 1$ and $c_2 = 0$, we find from $y = u(x)y_1(x)$ that a second solution of equation (3) is

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx. \tag{5}$$

It makes a good review of differentiation to verify that the function $y_2(x)$ defined in (5) satisfies equation (3) and that y_1 and y_2 are linearly independent on any interval on which $y_1(x)$ is not zero.

EXAMPLE 2 A Second Solution by Formula (5)

The function $y_1 = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$. Find the general solution of the differential equation on the interval $(0, \infty)$.

SOLUTION From the standard form of the equation,

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0,$$

we find from (5)

$$y_2 = x^2 \int \frac{e^{3\int dx/x}}{x^4}\,dx \quad \leftarrow e^{3\int dx/x} = e^{\ln x^3} = x^3$$

$$= x^2 \int \frac{dx}{x} = x^2 \ln x.$$

The general solution on the interval $(0, \infty)$ is given by $y = c_1y_1 + c_2y_2$; that is, $y = c_1x^2 + c_2x^2 \ln x$. ■

REMARKS

(*i*) The derivation and use of formula (5) have been illustrated here because this formula appears again in the next section and in Sections 4.7 and 6.2. We use (5) simply to save time in obtaining a desired result. Your instructor will tell you whether you should memorize (5) or whether you should know the first principles of reduction of order.

(*ii*) Reduction of order can be used to find the general solution of a nonhomogeneous equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$ whenever a solution y_1 of the associated homogeneous equation is known. See Problems 17–20 in Exercises 4.2.

EXERCISES 4.2

Answers to selected odd-numbered problems begin on page ANS-4.

In Problems 1–16 the indicated function $y_1(x)$ is a solution of the given differential equation. Use reduction of order or formula (5), as instructed, to find a second solution $y_2(x)$.

1. $y'' - 4y' + 4y = 0; \quad y_1 = e^{2x}$
2. $y'' + 2y' + y = 0; \quad y_1 = xe^{-x}$
3. $y'' + 16y = 0; \quad y_1 = \cos 4x$
4. $y'' + 9y = 0; \quad y_1 = \sin 3x$
5. $y'' - y = 0; \quad y_1 = \cosh x$
6. $y'' - 25y = 0; \quad y_1 = e^{5x}$
7. $9y'' - 12y' + 4y = 0; \quad y_1 = e^{2x/3}$
8. $6y'' + y' - y = 0; \quad y_1 = e^{x/3}$
9. $x^2y'' - 7xy' + 16y = 0; \quad y_1 = x^4$
10. $x^2y'' + 2xy' - 6y = 0; \quad y_1 = x^2$
11. $xy'' + y' = 0; \quad y_1 = \ln x$
12. $4x^2y'' + y = 0; \quad y_1 = x^{1/2} \ln x$
13. $x^2y'' - xy' + 2y = 0; \quad y_1 = x \sin(\ln x)$
14. $x^2y'' - 3xy' + 5y = 0; \quad y_1 = x^2 \cos(\ln x)$

15. $(1 - 2x - x^2)y'' + 2(1 + x)y' - 2y = 0; \quad y_1 = x + 1$

16. $(1 - x^2)y'' + 2xy' = 0; \quad y_1 = 1$

In Problems 17–20 the indicated function $y_1(x)$ is a solution of the associated homogeneous equation. Use the method of reduction of order to find a second solution $y_2(x)$ of the homogeneous equation and a particular solution of the given nonhomogeneous equation.

17. $y'' - 4y = 2; \quad y_1 = e^{-2x}$

18. $y'' + y' = 1; \quad y_1 = 1$

19. $y'' - 3y' + 2y = 5e^{3x}; \quad y_1 = e^x$

20. $y'' - 4y' + 3y = x; \quad y_1 = e^x$

Discussion Problems

21. **(a)** Give a convincing demonstration that the second-order equation $ay'' + by' + cy = 0$, a, b, and c constants, always possesses at least one solution of the form $y_1 = e^{m_1 x}$, m_1 a constant.

(b) Explain why the differential equation in part (a) must then have a second solution either of the form $y_2 = e^{m_2 x}$ or of the form $y_2 = xe^{m_1 x}$, m_1 and m_2 constants.

(c) Reexamine Problems 1–8. Can you explain why the statements in parts (a) and (b) above are not contradicted by the answers to Problems 3–5?

22. Verify that $y_1(x) = x$ is a solution of $xy'' - xy' + y = 0$. Use reduction of order to find a second solution $y_2(x)$ in the form of an infinite series. Conjecture an interval of definition for $y_2(x)$.

Computer Lab Assignments

23. **(a)** Verify that $y_1(x) = e^x$ is a solution of

$$xy'' - (x + 10)y' + 10y = 0.$$

(b) Use (5) to find a second solution $y_2(x)$. Use a CAS to carry out the required integration.

(c) Explain, using Corollary (A) of Theorem 4.1.2, why the second solution can be written compactly as

$$y_2(x) = \sum_{n=0}^{10} \frac{1}{n!} x^n.$$

4.3 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

REVIEW MATERIAL

- Review Problem 27 in Exercises 1.1 and Theorem 4.1.5
- Review the algebra of solving polynomial equations (see the *Student Resource and Solutions Manual*)

INTRODUCTION As a means of motivating the discussion in this section, let us return to first-order differential equations—more specifically, to *homogeneous* linear equations $ay' + by = 0$, where the coefficients $a \neq 0$ and b are constants. This type of equation can be solved either by separation of variables or with the aid of an integrating factor, but there is another solution method, one that uses only algebra. Before illustrating this alternative method, we make one observation: Solving $ay' + by = 0$ for y' yields $y' = ky$, where k is a constant. This observation reveals the nature of the unknown solution y; the only nontrivial elementary function whose derivative is a constant multiple of itself is an exponential function e^{mx}. Now the new solution method: If we substitute $y = e^{mx}$ and $y' = me^{mx}$ into $ay' + by = 0$, we get

$$ame^{mx} + be^{mx} = 0 \qquad \text{or} \qquad e^{mx}(am + b) = 0.$$

Since e^{mx} is never zero for real values of x, the last equation is satisfied only when m is a solution or root of the first-degree polynomial equation $am + b = 0$. For this single value of m, $y = e^{mx}$ is a solution of the DE. To illustrate, consider the constant-coefficient equation $2y' + 5y = 0$. It is not necessary to go through the differentiation and substitution of $y = e^{mx}$ into the DE; we merely have to form the equation $2m + 5 = 0$ and solve it for m. From $m = -\frac{5}{2}$ we conclude that $y = e^{-5x/2}$ is a solution of $2y' + 5y = 0$, and its general solution on the interval $(-\infty, \infty)$ is $y = c_1 e^{-5x/2}$.

In this section we will see that the foregoing procedure can produce exponential solutions for homogeneous linear higher-order DEs,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \tag{1}$$

where the coefficients a_i, $i = 0, 1, \ldots, n$ are real constants and $a_n \neq 0$.

AUXILIARY EQUATION We begin by considering the special case of the second-order equation

$$ay'' + by' + cy = 0, \tag{2}$$

where a, b, and c are constants. If we try to find a solution of the form $y = e^{mx}$, then after substitution of $y' = me^{mx}$ and $y'' = m^2e^{mx}$, equation (2) becomes

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0 \quad \text{or} \quad e^{mx}(am^2 + bm + c) = 0.$$

As in the introduction we argue that because $e^{mx} \neq 0$ for all x, it is apparent that the only way $y = e^{mx}$ can satisfy the differential equation (2) is when m is chosen as a root of the quadratic equation

$$am^2 + bm + c = 0. \tag{3}$$

This last equation is called the **auxiliary equation** of the differential equation (2). Since the two roots of (3) are $m_1 = (-b + \sqrt{b^2 - 4ac})/2a$ and $m_2 = (-b - \sqrt{b^2 - 4ac})/2a$, there will be three forms of the general solution of (2) corresponding to the three cases:

- m_1 and m_2 real and distinct ($b^2 - 4ac > 0$),
- m_1 and m_2 real and equal ($b^2 - 4ac = 0$), and
- m_1 and m_2 conjugate complex numbers ($b^2 - 4ac < 0$).

We discuss each of these cases in turn.

CASE I: DISTINCT REAL ROOTS Under the assumption that the auxiliary equation (3) has two unequal real roots m_1 and m_2, we find two solutions, $y_1 = e^{m_1x}$ and $y_2 = e^{m_2x}$. We see that these functions are linearly independent on $(-\infty, \infty)$ and hence form a fundamental set. It follows that the general solution of (2) on this interval is

$$y = c_1e^{m_1x} + c_2e^{m_2x}. \tag{4}$$

CASE II: REPEATED REAL ROOTS When $m_1 = m_2$, we necessarily obtain only one exponential solution, $y_1 = e^{m_1x}$. From the quadratic formula we find that $m_1 = -b/2a$ since the only way to have $m_1 = m_2$ is to have $b^2 - 4ac = 0$. It follows from (5) in Section 4.2 that a second solution of the equation is

$$y_2 = e^{m_1x}\int \frac{e^{2m_1x}}{e^{2m_1x}}\,dx = e^{m_1x}\int dx = xe^{m_1x}. \tag{5}$$

In (5) we have used the fact that $-b/a = 2m_1$. The general solution is then

$$y = c_1e^{m_1x} + c_2xe^{m_1x}. \tag{6}$$

CASE III: CONJUGATE COMPLEX ROOTS If m_1 and m_2 are complex, then we can write $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, where α and $\beta > 0$ are real and $i^2 = -1$. Formally, there is no difference between this case and Case I, and hence

$$y = C_1e^{(\alpha+i\beta)x} + C_2e^{(\alpha-i\beta)x}.$$

However, in practice we prefer to work with real functions instead of complex exponentials. To this end we use **Euler's formula:**

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

where θ is any real number.* It follows from this formula that

$$e^{i\beta x} = \cos\beta x + i\sin\beta x \quad \text{and} \quad e^{-i\beta x} = \cos\beta x - i\sin\beta x, \tag{7}$$

*A formal derivation of Euler's formula can be obtained from the Maclaurin series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ by substituting $x = i\theta$, using $i^2 = -1$, $i^3 = -i$, . . . , and then separating the series into real and imaginary parts. The plausibility thus established, we can adopt $\cos\theta + i\sin\theta$ as the *definition* of $e^{i\theta}$.

where we have used $\cos(-\beta x) = \cos\beta x$ and $\sin(-\beta x) = -\sin\beta x$. Note that by first adding and then subtracting the two equations in (7), we obtain, respectively,

$$e^{i\beta x} + e^{-i\beta x} = 2\cos\beta x \quad \text{and} \quad e^{i\beta x} - e^{-i\beta x} = 2i\sin\beta x.$$

Since $y = C_1e^{(\alpha+i\beta)x} + C_2e^{(\alpha-i\beta)x}$ is a solution of (2) for any choice of the constants C_1 and C_2, the choices $C_1 = C_2 = 1$ and $C_1 = 1$, $C_2 = -1$ give, in turn, two solutions:

$$y_1 = e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x} \quad \text{and} \quad y_2 = e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}.$$

But
$$y_1 = e^{\alpha x}(e^{i\beta x} + e^{-i\beta x}) = 2e^{\alpha x}\cos\beta x$$
and
$$y_2 = e^{\alpha x}(e^{i\beta x} - e^{-i\beta x}) = 2ie^{\alpha x}\sin\beta x.$$

Hence from Corollary (A) of Theorem 4.1.2 the last two results show that $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are *real* solutions of (2). Moreover, these solutions form a fundamental set on $(-\infty, \infty)$. Consequently, the general solution is

$$y = c_1e^{\alpha x}\cos\beta x + c_2e^{\alpha x}\sin\beta x = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x). \tag{8}$$

EXAMPLE 1 Second-Order DEs

Solve the following differential equations.

(a) $2y'' - 5y' - 3y = 0$ **(b)** $y'' - 10y' + 25y = 0$ **(c)** $y'' + 4y' + 7y = 0$

SOLUTION We give the auxiliary equations, the roots, and the corresponding general solutions.

(a) $2m^2 - 5m - 3 = (2m + 1)(m - 3) = 0$, $m_1 = -\frac{1}{2}$, $m_2 = 3$

From (4), $y = c_1e^{-x/2} + c_2e^{3x}$.

(b) $m^2 - 10m + 25 = (m - 5)^2 = 0$, $m_1 = m_2 = 5$

From (6), $y = c_1e^{5x} + c_2xe^{5x}$.

(c) $m^2 + 4m + 7 = 0$, $m_1 = -2 + \sqrt{3}i$, $m_2 = -2 - \sqrt{3}i$

From (8) with $\alpha = -2$, $\beta = \sqrt{3}$, $y = e^{-2x}\left(c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x\right)$. ■

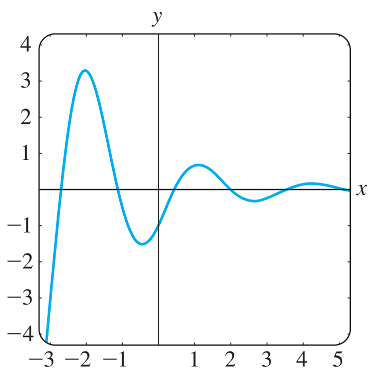


FIGURE 4.3.1 Solution curve of IVP in Example 2

EXAMPLE 2 An Initial-Value Problem

Solve $4y'' + 4y' + 17y = 0$, $y(0) = -1$, $y'(0) = 2$.

SOLUTION By the quadratic formula we find that the roots of the auxiliary equation $4m^2 + 4m + 17 = 0$ are $m_1 = -\frac{1}{2} + 2i$ and $m_2 = -\frac{1}{2} - 2i$. Thus from (8) we have $y = e^{-x/2}(c_1\cos 2x + c_2\sin 2x)$. Applying the condition $y(0) = -1$, we see from $e^0(c_1\cos 0 + c_2\sin 0) = -1$ that $c_1 = -1$. Differentiating $y = e^{-x/2}(-\cos 2x + c_2\sin 2x)$ and then using $y'(0) = 2$ gives $2c_2 + \frac{1}{2} = 2$ or $c_2 = \frac{3}{4}$. Hence the solution of the IVP is $y = e^{-x/2}(-\cos 2x + \frac{3}{4}\sin 2x)$. In Figure 4.3.1 we see that the solution is oscillatory, but $y \to 0$ as $x \to \infty$ and $|y| \to \infty$ as $x \to -\infty$. ■

TWO EQUATIONS WORTH KNOWING The two differential equations

$$y'' + k^2y = 0 \quad \text{and} \quad y'' - k^2y = 0,$$

where k is real, are important in applied mathematics. For $y'' + k^2y = 0$ the auxiliary equation $m^2 + k^2 = 0$ has imaginary roots $m_1 = ki$ and $m_2 = -ki$. With $\alpha = 0$ and $\beta = k$ in (8) the general solution of the DE is seen to be

$$y = c_1 \cos kx + c_2 \sin kx. \tag{9}$$

On the other hand, the auxiliary equation $m^2 - k^2 = 0$ for $y'' - k^2y = 0$ has distinct real roots $m_1 = k$ and $m_2 = -k$, and so by (4) the general solution of the DE is

$$y = c_1 e^{kx} + c_2 e^{-kx}. \tag{10}$$

Notice that if we choose $c_1 = c_2 = \frac{1}{2}$ and $c_1 = \frac{1}{2}$, $c_2 = -\frac{1}{2}$ in (10), we get the particular solutions $y = \frac{1}{2}(e^{kx} + e^{-kx}) = \cosh kx$ and $y = \frac{1}{2}(e^{kx} - e^{-kx}) = \sinh kx$. Since $\cosh kx$ and $\sinh kx$ are linearly independent on any interval of the x-axis, an alternative form for the general solution of $y'' - k^2y = 0$ is

$$y = c_1 \cosh kx + c_2 \sinh kx. \tag{11}$$

See Problems 41 and 42 in Exercises 4.3.

HIGHER-ORDER EQUATIONS In general, to solve an nth-order differential equation (1), where the a_i, $i = 0, 1, \ldots, n$ are real constants, we must solve an nth-degree polynomial equation

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_2 m^2 + a_1 m + a_0 = 0. \tag{12}$$

If all the roots of (12) are real and distinct, then the general solution of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

It is somewhat harder to summarize the analogues of Cases II and III because the roots of an auxiliary equation of degree greater than two can occur in many combinations. For example, a fifth-degree equation could have five distinct real roots, or three distinct real and two complex roots, or one real and four complex roots, or five real but equal roots, or five real roots but two of them equal, and so on. When m_1 is a root of multiplicity k of an nth-degree auxiliary equation (that is, k roots are equal to m_1), it can be shown that the linearly independent solutions are

$$e^{m_1 x}, \quad xe^{m_1 x}, \quad x^2 e^{m_1 x}, \ldots, \quad x^{k-1} e^{m_1 x}$$

and the general solution must contain the linear combination

$$c_1 e^{m_1 x} + c_2 x e^{m_1 x} + c_3 x^2 e^{m_1 x} + \cdots + c_k x^{k-1} e^{m_1 x}.$$

Finally, it should be remembered that when the coefficients are real, complex roots of an auxiliary equation always appear in conjugate pairs. Thus, for example, a cubic polynomial equation can have at most two complex roots.

EXAMPLE 3 Third-Order DE

Solve $y''' + 3y'' - 4y = 0$.

SOLUTION It should be apparent from inspection of $m^3 + 3m^2 - 4 = 0$ that one root is $m_1 = 1$, so $m - 1$ is a factor of $m^3 + 3m^2 - 4$. By division we find

$$m^3 + 3m^2 - 4 = (m - 1)(m^2 + 4m + 4) = (m - 1)(m + 2)^2,$$

so the other roots are $m_2 = m_3 = -2$. Thus the general solution of the DE is $y = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}$. ■

EXAMPLE 4 Fourth-Order DE

Solve $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = 0.$

SOLUTION The auxiliary equation $m^4 + 2m^2 + 1 = (m^2 + 1)^2 = 0$ has roots $m_1 = m_3 = i$ and $m_2 = m_4 = -i$. Thus from Case II the solution is

$$y = C_1e^{ix} + C_2e^{-ix} + C_3xe^{ix} + C_4xe^{-ix}.$$

By Euler's formula the grouping $C_1e^{ix} + C_2e^{-ix}$ can be rewritten as

$$c_1 \cos x + c_2 \sin x$$

after a relabeling of constants. Similarly, $x(C_3e^{ix} + C_4e^{-ix})$ can be expressed as $x(c_3 \cos x + c_4 \sin x)$. Hence the general solution is

$$y = c_1 \cos x + c_2 \sin x + c_3x \cos x + c_4x \sin x.$$ ■

Example 4 illustrates a special case when the auxiliary equation has repeated complex roots. In general, if $m_1 = \alpha + i\beta, \beta > 0$ is a complex root of multiplicity k of an auxiliary equation with real coefficients, then its conjugate $m_2 = \alpha - i\beta$ is also a root of multiplicity k. From the $2k$ complex-valued solutions

$$e^{(\alpha+i\beta)x}, \quad xe^{(\alpha+i\beta)x}, \quad x^2e^{(\alpha+i\beta)x}, \quad \ldots, \quad x^{k-1}e^{(\alpha+i\beta)x},$$
$$e^{(\alpha-i\beta)x}, \quad xe^{(\alpha-i\beta)x}, \quad x^2e^{(\alpha-i\beta)x}, \quad \ldots, \quad x^{k-1}e^{(\alpha-i\beta)x},$$

we conclude, with the aid of Euler's formula, that the general solution of the corresponding differential equation must then contain a linear combination of the $2k$ real linearly independent solutions

$$e^{\alpha x}\cos \beta x, \quad xe^{\alpha x}\cos \beta x, \quad x^2e^{\alpha x}\cos \beta x, \quad \ldots, \quad x^{k-1}e^{\alpha x}\cos \beta x,$$
$$e^{\alpha x}\sin \beta x, \quad xe^{\alpha x}\sin \beta x, \quad x^2e^{\alpha x}\sin \beta x, \quad \ldots, \quad x^{k-1}e^{\alpha x}\sin \beta x.$$

In Example 4 we identify $k = 2$, $\alpha = 0$, and $\beta = 1$.

Of course the most difficult aspect of solving constant-coefficient differential equations is finding roots of auxiliary equations of degree greater than two. For example, to solve $3y''' + 5y'' + 10y' - 4y = 0$, we must solve $3m^3 + 5m^2 + 10m - 4 = 0$. Something we can try is to test the auxiliary equation for rational roots. Recall that if $m_1 = p/q$ is a rational root (expressed in lowest terms) of an auxiliary equation $a_nm^n + \cdots + a_1m + a_0 = 0$ with integer coefficients, then p is a factor of a_0 and q is a factor of a_n. For our specific cubic auxiliary equation, all the factors of $a_0 = -4$ and $a_n = 3$ are p: $\pm1, \pm2, \pm4$ and q: $\pm1, \pm3$, so the possible rational roots are p/q: $\pm1, \pm2, \pm4, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$ Each of these numbers can then be tested—say, by synthetic division. In this way we discover both the root $m_1 = \frac{1}{3}$ and the factorization

$$3m^3 + 5m^2 + 10m - 4 = \left(m - \tfrac{1}{3}\right)(3m^2 + 6m + 12).$$

The quadratic formula then yields the remaining roots $m_2 = -1 + \sqrt{3}i$ and $m_3 = -1 - \sqrt{3}i$. Therefore the general solution of $3y''' + 5y'' + 10y' - 4y = 0$ is $y = c_1e^{x/3} + e^{-x}(c_2 \cos \sqrt{3}x + c_3 \sin \sqrt{3}x)$.

USE OF COMPUTERS Finding roots or approximation of roots of auxiliary equations is a routine problem with an appropriate calculator or computer software. Polynomial equations (in one variable) of degree less than five can be solved by means of algebraic formulas using the *solve* commands in *Mathematica* and *Maple*. For auxiliary equations of degree five or greater it might be necessary to resort to numerical commands such as **NSolve** and **FindRoot** in *Mathematica*. Because of their capability of solving polynomial equations, it is not surprising that these computer algebra

■ There is more on this in the ***SRSM***.

systems are also able, by means of their *dsolve* commands, to provide explicit solutions of homogeneous linear constant-coefficient differential equations.

In the classic text *Differential Equations* by Ralph Palmer Agnew* (used by the author as a student) the following statement is made:

> *It is not reasonable to expect students in this course to have computing skill and equipment necessary for efficient solving of equations such as*

$$4.317\frac{d^4y}{dx^4} + 2.179\frac{d^3y}{dx^3} + 1.416\frac{d^2y}{dx^2} + 1.295\frac{dy}{dx} + 3.169y = 0. \tag{13}$$

Although it is debatable whether computing skills have improved in the intervening years, it is a certainty that technology has. If one has access to a computer algebra system, equation (13) could now be considered reasonable. After simplification and some relabeling of output, *Mathematica* yields the (approximate) general solution

$$\begin{aligned} y = {} & c_1e^{-0.728852x}\cos(0.618605x) + c_2e^{-0.728852x}\sin(0.618605x) \\ & + c_3e^{0.476478x}\cos(0.759081x) + c_4e^{0.476478x}\sin(0.759081x). \end{aligned}$$

Finally, if we are faced with an initial-value problem consisting of, say, a fourth-order equation, then to fit the general solution of the DE to the four initial conditions, we must solve four linear equations in four unknowns (the c_1, c_2, c_3, c_4 in the general solution). Using a CAS to solve the system can save lots of time. See Problems 59 and 60 in Exercises 4.3 and Problem 35 in Chapter 4 in Review.

*McGraw-Hill, New York, 1960.

EXERCISES 4.3

Answers to selected odd-numbered problems begin on page ANS-4.

In Problems 1–14 find the general solution of the given second-order differential equation.

1. $4y'' + y' = 0$

2. $y'' - 36y = 0$

3. $y'' - y' - 6y = 0$

4. $y'' - 3y' + 2y = 0$

5. $y'' + 8y' + 16y = 0$

6. $y'' - 10y' + 25y = 0$

7. $12y'' - 5y' - 2y = 0$

8. $y'' + 4y' - y = 0$

9. $y'' + 9y = 0$

10. $3y'' + y = 0$

11. $y'' - 4y' + 5y = 0$

12. $2y'' + 2y' + y = 0$

13. $3y'' + 2y' + y = 0$

14. $2y'' - 3y' + 4y = 0$

In Problems 15–28 find the general solution of the given higher-order differential equation.

15. $y''' - 4y'' - 5y' = 0$

16. $y''' - y = 0$

17. $y''' - 5y'' + 3y' + 9y = 0$

18. $y''' + 3y'' - 4y' - 12y = 0$

19. $\dfrac{d^3u}{dt^3} + \dfrac{d^2u}{dt^2} - 2u = 0$

20. $\dfrac{d^3x}{dt^3} - \dfrac{d^2x}{dt^2} - 4x = 0$

21. $y''' + 3y'' + 3y' + y = 0$

22. $y''' - 6y'' + 12y' - 8y = 0$

23. $y^{(4)} + y''' + y'' = 0$

24. $y^{(4)} - 2y'' + y = 0$

25. $16\dfrac{d^4y}{dx^4} + 24\dfrac{d^2y}{dx^2} + 9y = 0$

26. $\dfrac{d^4y}{dx^4} - 7\dfrac{d^2y}{dx^2} - 18y = 0$

27. $\dfrac{d^5u}{dr^5} + 5\dfrac{d^4u}{dr^4} - 2\dfrac{d^3u}{dr^3} - 10\dfrac{d^2u}{dr^2} + \dfrac{du}{dr} + 5u = 0$

28. $2\dfrac{d^5x}{ds^5} - 7\dfrac{d^4x}{ds^4} + 12\dfrac{d^3x}{ds^3} + 8\dfrac{d^2x}{ds^2} = 0$

In Problems 29–36 solve the given initial-value problem.

29. $y'' + 16y = 0, \quad y(0) = 2, y'(0) = -2$

30. $\dfrac{d^2y}{d\theta^2} + y = 0, \quad y\left(\dfrac{\pi}{3}\right) = 0, y'\left(\dfrac{\pi}{3}\right) = 2$

31. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} - 5y = 0, \quad y(1) = 0, y'(1) = 2$

32. $4y'' - 4y' - 3y = 0, \quad y(0) = 1, y'(0) = 5$

33. $y'' + y' + 2y = 0, \quad y(0) = y'(0) = 0$

34. $y'' - 2y' + y = 0, \quad y(0) = 5, y'(0) = 10$

35. $y''' + 12y'' + 36y' = 0, \quad y(0) = 0, y'(0) = 1, y''(0) = -7$

36. $y''' + 2y'' - 5y' - 6y = 0, \quad y(0) = y'(0) = 0, \; y''(0) = 1$

In Problems 37–40 solve the given boundary-value problem.

37. $y'' - 10y' + 25y = 0, \quad y(0) = 1, y(1) = 0$

38. $y'' + 4y = 0, \quad y(0) = 0, y(\pi) = 0$

39. $y'' + y = 0, \quad y'(0) = 0, y'\left(\dfrac{\pi}{2}\right) = 0$

40. $y'' - 2y' + 2y = 0, \quad y(0) = 1, y(\pi) = 1$

In Problems 41 and 42 solve the given problem first using the form of the general solution given in (10). Solve again, this time using the form given in (11).

41. $y'' - 3y = 0, \quad y(0) = 1, y'(0) = 5$

42. $y'' - y = 0, \quad y(0) = 1, y'(1) = 0$

In Problems 43–48 each figure represents the graph of a particular solution of one of the following differential equations:

(a) $y'' - 3y' - 4y = 0$ **(b)** $y'' + 4y = 0$
(c) $y'' + 2y' + y = 0$ **(d)** $y'' + y = 0$
(e) $y'' + 2y' + 2y = 0$ **(f)** $y'' - 3y' + 2y = 0$

Match a solution curve with one of the differential equations. Explain your reasoning.

43.

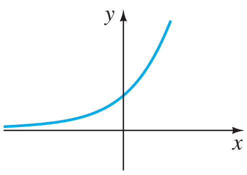


FIGURE 4.3.2 Graph for Problem 43

44.

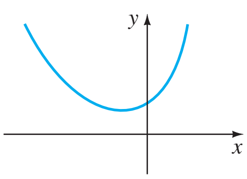


FIGURE 4.3.3 Graph for Problem 44

45.

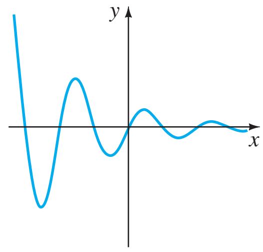


FIGURE 4.3.4 Graph for Problem 45

46.

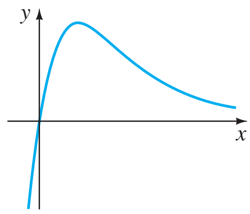


FIGURE 4.3.5 Graph for Problem 46

47.

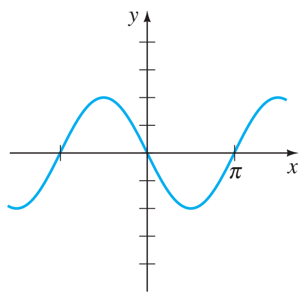


FIGURE 4.3.6 Graph for Problem 47

48.

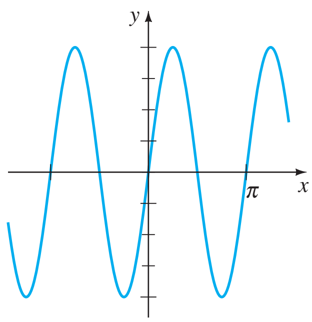


FIGURE 4.3.7 Graph for Problem 48

Discussion Problems

49. The roots of a cubic auxiliary equation are $m_1 = 4$ and $m_2 = m_3 = -5$. What is the corresponding homogeneous linear differential equation? Discuss: Is your answer unique?

50. Two roots of a cubic auxiliary equation with real coefficients are $m_1 = -\frac{1}{2}$ and $m_2 = 3 + i$. What is the corresponding homogeneous linear differential equation?

51. Find the general solution of $y''' + 6y'' + y' - 34y = 0$ if it is known that $y_1 = e^{-4x} \cos x$ is one solution.

52. To solve $y^{(4)} + y = 0$, we must find the roots of $m^4 + 1 = 0$. This is a trivial problem using a CAS but can also be done by hand working with complex numbers. Observe that $m^4 + 1 = (m^2 + 1)^2 - 2m^2$. How does this help? Solve the differential equation.

53. Verify that $y = \sinh x - 2 \cos (x + \pi/6)$ is a particular solution of $y^{(4)} - y = 0$. Reconcile this particular solution with the general solution of the DE.

54. Consider the boundary-value problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(\pi/2) = 0$. Discuss: Is it possible to determine values of λ so that the problem possesses **(a)** trivial solutions? **(b)** nontrivial solutions?

Computer Lab Assignments

In Problems 55–58 use a computer either as an aid in solving the auxiliary equation or as a means of directly obtaining the general solution of the given differential equation. If you use a CAS to obtain the general solution, simplify the output and, if necessary, write the solution in terms of real functions.

55. $y''' - 6y'' + 2y' + y = 0$

56. $6.11y''' + 8.59y'' + 7.93y' + 0.778y = 0$

57. $3.15y^{(4)} - 5.34y'' + 6.33y' - 2.03y = 0$

58. $y^{(4)} + 2y'' - y' + 2y = 0$

In Problems 59 and 60 use a CAS as an aid in solving the auxiliary equation. Form the general solution of the differential equation. Then use a CAS as an aid in solving the system of equations for the coefficients c_i, $i = 1, 2, 3, 4$ that results when the initial conditions are applied to the general solution.

59. $2y^{(4)} + 3y''' - 16y'' + 15y' - 4y = 0$,
$y(0) = -2$, $y'(0) = 6$, $y''(0) = 3$, $y'''(0) = \frac{1}{2}$

60. $y^{(4)} - 3y''' + 3y'' - y' = 0$,
$y(0) = y'(0) = 0$, $y''(0) = y'''(0) = 1$

4.4 UNDETERMINED COEFFICIENTS—SUPERPOSITION APPROACH*

REVIEW MATERIAL

- Review Theorems 4.1.6 and 4.1.7 (Section 4.1)

INTRODUCTION To solve a nonhomogeneous linear differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x), \tag{1}$$

we must do two things:

- find the complementary function y_c and
- find *any* particular solution y_p of the nonhomogeneous equation (1).

Then, as was discussed in Section 4.1, the general solution of (1) is $y = y_c + y_p$. The complementary function y_c is the general solution of the associated homogeneous DE of (1), that is,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

In Section 4.3 we saw how to solve these kinds of equations when the coefficients were constants. Our goal in the present section is to develop a method for obtaining particular solutions.

*Note to the Instructor:** In this section the method of undetermined coefficients is developed from the viewpoint of the superposition principle for nonhomogeneous equations (Theorem 4.7.1). In Section 4.5 an entirely different approach will be presented, one utilizing the concept of differential annihilator operators. Take your pick.

METHOD OF UNDETERMINED COEFFICIENTS The first of two ways we shall consider for obtaining a particular solution y_p for a nonhomogeneous linear DE is called the **method of undetermined coefficients.** The underlying idea behind this method is a conjecture about the form of y_p, an educated guess really, that is motivated by the kinds of functions that make up the input function $g(x)$. The general method is limited to linear DEs such as (1) where

- the coefficients a_i, $i = 0, 1, \ldots, n$ are constants and
- $g(x)$ is a constant k, a polynomial function, an exponential function $e^{\alpha x}$, a sine or cosine function $\sin \beta x$ or $\cos \beta x$, or finite sums and products of these functions.

NOTE Strictly speaking, $g(x) = k$ (constant) is a polynomial function. Since a constant function is probably not the first thing that comes to mind when you think of polynomial functions, for emphasis we shall continue to use the redundancy "constant functions, polynomials, ... "

The following functions are some examples of the types of inputs $g(x)$ that are appropriate for this discussion:

$$g(x) = 10, \quad g(x) = x^2 - 5x, \qquad g(x) = 15x - 6 + 8e^{-x},$$
$$g(x) = \sin 3x - 5x \cos 2x, \qquad g(x) = xe^x \sin x + (3x^2 - 1)e^{-4x}.$$

That is, $g(x)$ is a linear combination of functions of the type

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \qquad P(x)\, e^{\alpha x}, \quad P(x)\, e^{\alpha x} \sin \beta x, \quad \text{and} \quad P(x)\, e^{\alpha x} \cos \beta x,$$

where n is a nonnegative integer and α and β are real numbers. The method of undetermined coefficients is not applicable to equations of form (1) when

$$g(x) = \ln x, \quad g(x) = \frac{1}{x}, \quad g(x) = \tan x, \quad g(x) = \sin^{-1} x,$$

and so on. Differential equations in which the input $g(x)$ is a function of this last kind will be considered in Section 4.6.

The set of functions that consists of constants, polynomials, exponentials $e^{\alpha x}$, sines, and cosines has the remarkable property that derivatives of their sums and products are again sums and products of constants, polynomials, exponentials $e^{\alpha x}$, sines, and cosines. Because the linear combination of derivatives $a_n y_p^{(n)} + a_{n-1} y_p^{(n-1)} + \cdots + a_1 y_p' + a_0 y_p$ must be identical to $g(x)$, it seems reasonable to assume that y_p *has the same form as* $g(x)$.

The next two examples illustrate the basic method.

EXAMPLE 1 General Solution Using Undetermined Coefficients

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6$. (2)

SOLUTION **Step 1.** We first solve the associated homogeneous equation $y'' + 4y' - 2y = 0$. From the quadratic formula we find that the roots of the auxiliary equation $m^2 + 4m - 2 = 0$ are $m_1 = -2 - \sqrt{6}$ and $m_2 = -2 + \sqrt{6}$. Hence the complementary function is

$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}.$$

Step 2. Now, because the function $g(x)$ is a quadratic polynomial, let us assume a particular solution that is also in the form of a quadratic polynomial:

$$y_p = Ax^2 + Bx + C.$$

We seek to determine *specific* coefficients A, B, and C for which y_p is a solution of (2). Substituting y_p and the derivatives

$$y_p' = 2Ax + B \qquad \text{and} \qquad y_p'' = 2A$$

into the given differential equation (2), we get

$$y_p'' + 4y_p' - 2y_p = 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6.$$

Because the last equation is supposed to be an identity, the coefficients of like powers of x must be equal:

equal

$$\boxed{-2A}\, x^2 + \boxed{8A - 2B}\, x + \boxed{2A + 4B - 2C} = 2x^2 - 3x + 6$$

That is, $\quad -2A = 2, \qquad 8A - 2B = -3, \qquad 2A + 4B - 2C = 6.$

Solving this system of equations leads to the values $A = -1$, $B = -\frac{5}{2}$, and $C = -9$. Thus a particular solution is

$$y_p = -x^2 - \frac{5}{2}x - 9.$$

Step 3. The general solution of the given equation is

$$y = y_c + y_p = c_1 e^{-(2+\sqrt{6})x} + c_1 e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9. \qquad ■$$

EXAMPLE 2 Particular Solution Using Undetermined Coefficients

Find a particular solution of $y'' - y' + y = 2 \sin 3x$.

SOLUTION A natural first guess for a particular solution would be $A \sin 3x$. But because successive differentiations of $\sin 3x$ produce $\sin 3x$ *and* $\cos 3x$, we are prompted instead to assume a particular solution that includes both of these terms:

$$y_p = A \cos 3x + B \sin 3x.$$

Differentiating y_p and substituting the results into the differential equation gives, after regrouping,

$$y_p'' - y_p' + y_p = (-8A - 3B) \cos 3x + (3A - 8B) \sin 3x = 2 \sin 3x$$

or

equal

$$\boxed{-8A - 3B} \cos 3x + \boxed{3A - 8B} \sin 3x = 0 \cos 3x + 2 \sin 3x.$$

From the resulting system of equations,

$$-8A - 3B = 0, \qquad 3A - 8B = 2,$$

we get $A = \frac{6}{73}$ and $B = -\frac{16}{73}$. A particular solution of the equation is

$$y_p = \frac{6}{73} \cos 3x - \frac{16}{73} \sin 3x. \qquad ■$$

As we mentioned, the form that we assume for the particular solution y_p is an educated guess; it is not a blind guess. This educated guess must take into consideration not only the types of functions that make up $g(x)$ but also, as we shall see in Example 4, the functions that make up the complementary function y_c.

EXAMPLE 3 Forming y_p by Superposition

Solve $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$. (3)

SOLUTION **Step 1.** First, the solution of the associated homogeneous equation $y'' - 2y' - 3y = 0$ is found to be $y_c = c_1e^{-x} + c_2e^{3x}$.

Step 2. Next, the presence of $4x - 5$ in $g(x)$ suggests that the particular solution includes a linear polynomial. Furthermore, because the derivative of the product xe^{2x} produces $2xe^{2x}$ and e^{2x}, we also assume that the particular solution includes both xe^{2x} and e^{2x}. In other words, g is the sum of two basic kinds of functions:

$$g(x) = g_1(x) + g_2(x) = \textit{polynomial} + \textit{exponentials}.$$

Correspondingly, the superposition principle for nonhomogeneous equations (Theorem 4.1.7) suggests that we seek a particular solution

$$y_p = y_{p_1} + y_{p_2},$$

where $y_{p_1} = Ax + B$ and $y_{p_2} = Cxe^{2x} + Ee^{2x}$. Substituting

$$y_p = Ax + B + Cxe^{2x} + Ee^{2x}$$

into the given equation (3) and grouping like terms gives

$$y_p'' - 2y_p' - 3y_p = -3Ax - 2A - 3B - 3Cxe^{2x} + (2C - 3E)e^{2x} = 4x - 5 + 6xe^{2x}. \quad (4)$$

From this identity we obtain the four equations

$$-3A = 4, \qquad -2A - 3B = -5, \qquad -3C = 6, \qquad 2C - 3E = 0.$$

The last equation in this system results from the interpretation that the coefficient of e^{2x} in the right member of (4) is zero. Solving, we find $A = -\frac{4}{3}$, $B = \frac{23}{9}$, $C = -2$, and $E = -\frac{4}{3}$. Consequently,

$$y_p = -\frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}.$$

Step 3. The general solution of the equation is

$$y = c_1e^{-x} + c_2e^{3x} - \frac{4}{3}x + \frac{23}{9} - \left(2x + \frac{4}{3}\right)e^{2x}.$$ ■

In light of the superposition principle (Theorem 4.1.7) we can also approach Example 3 from the viewpoint of solving two simpler problems. You should verify that substituting

$$y_{p_1} = Ax + B \qquad \text{into} \qquad y'' - 2y' - 3y = 4x - 5$$

and

$$y_{p_2} = Cxe^{2x} + Ee^{2x} \qquad \text{into} \qquad y'' - 2y' - 3y = 6xe^{2x}$$

yields, in turn, $y_{p_1} = -\frac{4}{3}x + \frac{23}{9}$ and $y_{p_2} = -\left(2x + \frac{4}{3}\right)e^{2x}$. A particular solution of (3) is then $y_p = y_{p_1} + y_{p_2}$.

The next example illustrates that sometimes the "obvious" assumption for the form of y_p is not a correct assumption.

EXAMPLE 4 A Glitch in the Method

Find a particular solution of $y'' - 5y' + 4y = 8e^x$.

SOLUTION Differentiation of e^x produces no new functions. Therefore proceeding as we did in the earlier examples, we can reasonably assume a particular solution of the form $y_p = Ae^x$. But substitution of this expression into the differential equation yields the contradictory statement $0 = 8e^x$, so we have clearly made the wrong guess for y_p.

The difficulty here is apparent on examining the complementary function $y_c = c_1e^x + c_2e^{4x}$. Observe that our assumption Ae^x is already present in y_c. This means that e^x is a solution of the associated homogeneous differential equation, and a constant multiple Ae^x when substituted into the differential equation necessarily produces zero.

What then should be the form of y_p? Inspired by Case II of Section 4.3, let's see whether we can find a particular solution of the form

$$y_p = Axe^x.$$

Substituting $y_p' = Axe^x + Ae^x$ and $y_p'' = Axe^x + 2Ae^x$ into the differential equation and simplifying gives

$$y_p'' - 5y_p' + 4y_p = -3Ae^x = 8e^x.$$

From the last equality we see that the value of A is now determined as $A = -\frac{8}{3}$. Therefore a particular solution of the given equation is $y_p = -\frac{8}{3}xe^x$. ■

The difference in the procedures used in Examples 1–3 and in Example 4 suggests that we consider two cases. The first case reflects the situation in Examples 1–3.

CASE I No function in the assumed particular solution is a solution of the associated homogeneous differential equation.

In Table 4.1 we illustrate some specific examples of $g(x)$ in (1) along with the corresponding form of the particular solution. We are, of course, taking for granted that no function in the assumed particular solution y_p is duplicated by a function in the complementary function y_c.

TABLE 4.1 Trial Particular Solutions

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A\cos 4x + B\sin 4x$
6. $\cos 4x$	$A\cos 4x + B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

EXAMPLE 5 Forms of Particular Solutions—Case I

Determine the form of a particular solution of

(a) $y'' - 8y' + 25y = 5x^3e^{-x} - 7e^{-x}$ **(b)** $y'' + 4y = x\cos x$

SOLUTION **(a)** We can write $g(x) = (5x^3 - 7)e^{-x}$. Using entry 9 in Table 4.1 as a model, we assume a particular solution of the form

$$y_p = (Ax^3 + Bx^2 + Cx + E)e^{-x}.$$

Note that there is no duplication between the terms in y_p and the terms in the complementary function $y_c = e^{4x}(c_1\cos 3x + c_2\sin 3x)$.

(b) The function $g(x) = x \cos x$ is similar to entry 11 in Table 4.1 except, of course, that we use a linear rather than a quadratic polynomial and $\cos x$ and $\sin x$ instead of $\cos 4x$ and $\sin 4x$ in the form of y_p:

$$y_p = (Ax + B)\cos x + (Cx + E)\sin x.$$

Again observe that there is no duplication of terms between y_p and $y_c = c_1 \cos 2x + c_2 \sin 2x$. ■

If $g(x)$ consists of a sum of, say, m terms of the kind listed in the table, then (as in Example 3) the assumption for a particular solution y_p consists of the sum of the trial forms $y_{p_1}, y_{p_2}, \ldots, y_{p_m}$ corresponding to these terms:

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_m}.$$

The foregoing sentence can be put another way.

Form Rule for Case I *The form of y_p is a linear combination of all linearly independent functions that are generated by repeated differentiations of $g(x)$.*

EXAMPLE 6 Forming y_p by Superposition—Case I

Determine the form of a particular solution of

$$y'' - 9y' + 14y = 3x^2 - 5\sin 2x + 7xe^{6x}.$$

SOLUTION

Corresponding to $3x^2$ we assume $\quad y_{p_1} = Ax^2 + Bx + C.$

Corresponding to $-5\sin 2x$ we assume $\quad y_{p_2} = E\cos 2x + F\sin 2x.$

Corresponding to $7xe^{6x}$ we assume $\quad y_{p_3} = (Gx + H)e^{6x}.$

The assumption for the particular solution is then

$$y_p = y_{p_1} + y_{p_2} + y_{p_3} = Ax^2 + Bx + C + E\cos 2x + F\sin 2x + (Gx + H)e^{6x}.$$

No term in this assumption duplicates a term in $y_c = c_1e^{2x} + c_2e^{7x}$. ■

CASE II A function in the assumed particular solution is also a solution of the associated homogeneous differential equation.

The next example is similar to Example 4.

EXAMPLE 7 Particular Solution—Case II

Find a particular solution of $y'' - 2y' + y = e^x$.

SOLUTION The complementary function is $y_c = c_1e^x + c_2xe^x$. As in Example 4, the assumption $y_p = Ae^x$ will fail, since it is apparent from y_c that e^x is a solution of the associated homogeneous equation $y'' - 2y' + y = 0$. Moreover, we will not be able to find a particular solution of the form $y_p = Axe^x$, since the term xe^x is also duplicated in y_c. We next try

$$y_p = Ax^2e^x.$$

Substituting into the given differential equation yields $2Ae^x = e^x$, so $A = \frac{1}{2}$. Thus a particular solution is $y_p = \frac{1}{2}x^2e^x$. ■

Suppose again that $g(x)$ consists of m terms of the kind given in Table 4.1, and suppose further that the usual assumption for a particular solution is

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_m},$$

where the y_{p_i}, $i = 1, 2, \ldots, m$ are the trial particular solution forms corresponding to these terms. Under the circumstances described in Case II, we can make up the following general rule.

Multiplication Rule for Case II *If any y_{p_i} contains terms that duplicate terms in y_c, then that y_{p_i} must be multiplied by x^n, where n is the smallest positive integer that eliminates that duplication.*

EXAMPLE 8 An Initial-Value Problem

Solve $y'' + y = 4x + 10 \sin x$, $y(\pi) = 0$, $y'(\pi) = 2$.

SOLUTION The solution of the associated homogeneous equation $y'' + y = 0$ is $y_c = c_1 \cos x + c_2 \sin x$. Because $g(x) = 4x + 10 \sin x$ is the sum of a linear polynomial and a sine function, our normal assumption for y_p, from entries 2 and 5 of Table 4.1, would be the sum of $y_{p_1} = Ax + B$ and $y_{p_2} = C \cos x + E \sin x$:

$$y_p = Ax + B + C \cos x + E \sin x. \tag{5}$$

But there is an obvious duplication of the terms cos x and sin x in this assumed form and two terms in the complementary function. This duplication can be eliminated by simply multiplying y_{p_2} by x. Instead of (5) we now use

$$y_p = Ax + B + Cx \cos x + Ex \sin x. \tag{6}$$

Differentiating this expression and substituting the results into the differential equation gives

$$y_p'' + y_p = Ax + B - 2C \sin x + 2E \cos x = 4x + 10 \sin x,$$

and so $A = 4$, $B = 0$, $-2C = 10$, and $2E = 0$. The solutions of the system are immediate: $A = 4$, $B = 0$, $C = -5$, and $E = 0$. Therefore from (6) we obtain $y_p = 4x - 5x \cos x$. The general solution of the given equation is

$$y = y_c + y_p = c_1 \cos x + c_2 \sin x + 4x - 5x \cos x.$$

We now apply the prescribed initial conditions to the general solution of the equation. First, $y(\pi) = c_1 \cos \pi + c_2 \sin \pi + 4\pi - 5\pi \cos \pi = 0$ yields $c_1 = 9\pi$, since $\cos \pi = -1$ and $\sin \pi = 0$. Next, from the derivative

$$y' = -9\pi \sin x + c_2 \cos x + 4 + 5x \sin x - 5 \cos x$$

and $$y'(\pi) = -9\pi \sin \pi + c_2 \cos \pi + 4 + 5\pi \sin \pi - 5 \cos \pi = 2$$

we find $c_2 = 7$. The solution of the initial-value is then

$$y = 9\pi \cos x + 7 \sin x + 4x - 5x \cos x.$$ ■

EXAMPLE 9 Using the Multiplication Rule

Solve $y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$.

SOLUTION The complementary function is $y_c = c_1e^{3x} + c_2xe^{3x}$. And so, based on entries 3 and 7 of Table 4.1, the usual assumption for a particular solution would be

$$y_p = \underbrace{Ax^2 + Bx + C}_{y_{p_1}} + \underbrace{Ee^{3x}}_{y_{p_2}}.$$

Inspection of these functions shows that the one term in y_{p_2} is duplicated in y_c. If we multiply y_{p_2} by x, we note that the term xe^{3x} is still part of y_c. But multiplying y_{p_2} by x^2 eliminates all duplications. Thus the operative form of a particular solution is

$$y_p = Ax^2 + Bx + C + Ex^2e^{3x}.$$

Differentiating this last form, substituting into the differential equation, and collecting like terms gives

$$y_p'' - 6y_p' + 9y_p = 9Ax^2 + (-12A + 9B)x + 2A - 6B + 9C + 2Ee^{3x} = 6x^2 + 2 - 12e^{3x}.$$

It follows from this identity that $A = \frac{2}{3}$, $B = \frac{8}{9}$, $C = \frac{2}{3}$, and $E = -6$. Hence the general solution $y = y_c + y_p$ is $y = c_1e^{3x} + c_2xe^{3x} + \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} - 6x^2e^{3x}$. ■

EXAMPLE 10 Third-Order DE—Case I

Solve $y''' + y'' = e^x \cos x$.

SOLUTION From the characteristic equation $m^3 + m^2 = 0$ we find $m_1 = m_2 = 0$ and $m_3 = -1$. Hence the complementary function of the equation is $y_c = c_1 + c_2x + c_3e^{-x}$. With $g(x) = e^x \cos x$, we see from entry 10 of Table 4.1 that we should assume that

$$y_p = Ae^x \cos x + Be^x \sin x.$$

Because there are no functions in y_p that duplicate functions in the complementary solution, we proceed in the usual manner. From

$$y_p''' + y_p'' = (-2A + 4B)e^x \cos x + (-4A - 2B)e^x \sin x = e^x \cos x$$

we get $-2A + 4B = 1$ and $-4A - 2B = 0$. This system gives $A = -\frac{1}{10}$ and $B = \frac{1}{5}$, so a particular solution is $y_p = -\frac{1}{10}e^x \cos x + \frac{1}{5}e^x \sin x$. The general solution of the equation is

$$y = y_c + y_p = c_1 + c_2x + c_3e^{-x} - \frac{1}{10}e^x \cos x + \frac{1}{5}e^x \sin x. \quad ■$$

EXAMPLE 11 Fourth-Order DE—Case II

Determine the form of a particular solution of $y^{(4)} + y''' = 1 - x^2e^{-x}$.

SOLUTION Comparing $y_c = c_1 + c_2x + c_3x^2 + c_4e^{-x}$ with our normal assumption for a particular solution

$$y_p = \underbrace{A}_{y_{p_1}} + \underbrace{Bx^2e^{-x} + Cxe^{-x} + Ee^{-x}}_{y_{p_2}},$$

we see that the duplications between y_c and y_p are eliminated when y_{p_1} is multiplied by x^3 and y_{p_2} is multiplied by x. Thus the correct assumption for a particular solution is $y_p = Ax^3 + Bx^3e^{-x} + Cx^2e^{-x} + Exe^{-x}$. ■

REMARKS

(*i*) In Problems 27–36 in Exercises 4.4 you are asked to solve initial-value problems, and in Problems 37–40 you are asked to solve boundary-value problems. As illustrated in Example 8, be sure to apply the initial conditions or the boundary conditions to the general solution $y = y_c + y_p$. Students often make the mistake of applying these conditions only to the complementary function y_c because it is that part of the solution that contains the constants $c_1, c_2, \ldots, c_n$.

(*ii*) From the "Form Rule for Case I" on page 145 of this section you see why the method of undetermined coefficients is not well suited to nonhomogeneous linear DEs when the input function $g(x)$ is something other than one of the four basic types highlighted in color on page 141. For example, if $P(x)$ is a polynomial, then continued differentiation of $P(x)e^{\alpha x}\sin\beta x$ will generate an independent set containing only a *finite* number of functions—all of the same type, namely, a polynomial times $e^{\alpha x}\sin\beta x$ or a polynomial times $e^{\alpha x}\cos\beta x$. On the other hand, repeated differentiation of input functions such as $g(x) = \ln x$ or $g(x) = \tan^{-1}x$ generates an independent set containing an *infinite* number of functions:

$$\textit{derivatives of } \ln x\text{:} \quad \frac{1}{x}, \frac{-1}{x^2}, \frac{2}{x^3}, \ldots,$$

$$\textit{derivatives of } \tan^{-1}x\text{:} \quad \frac{1}{1+x^2}, \frac{-2x}{(1+x^2)^2}, \frac{-2+6x^2}{(1+x^2)^3}, \ldots.$$

EXERCISES 4.4

Answers to selected odd-numbered problems begin on page ANS-5.

In Problems 1–26 solve the given differential equation by undetermined coefficients.

1. $y'' + 3y' + 2y = 6$
2. $4y'' + 9y = 15$
3. $y'' - 10y' + 25y = 30x + 3$
4. $y'' + y' - 6y = 2x$
5. $\frac{1}{4}y'' + y' + y = x^2 - 2x$
6. $y'' - 8y' + 20y = 100x^2 - 26xe^x$
7. $y'' + 3y = -48x^2e^{3x}$
8. $4y'' - 4y' - 3y = \cos 2x$
9. $y'' - y' = -3$
10. $y'' + 2y' = 2x + 5 - e^{-2x}$
11. $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$
12. $y'' - 16y = 2e^{4x}$
13. $y'' + 4y = 3\sin 2x$
14. $y'' - 4y = (x^2 - 3)\sin 2x$
15. $y'' + y = 2x\sin x$
16. $y'' - 5y' = 2x^3 - 4x^2 - x + 6$
17. $y'' - 2y' + 5y = e^x\cos 2x$
18. $y'' - 2y' + 2y = e^{2x}(\cos x - 3\sin x)$
19. $y'' + 2y' + y = \sin x + 3\cos 2x$
20. $y'' + 2y' - 24y = 16 - (x + 2)e^{4x}$
21. $y''' - 6y'' = 3 - \cos x$
22. $y''' - 2y'' - 4y' + 8y = 6xe^{2x}$
23. $y''' - 3y'' + 3y' - y = x - 4e^x$
24. $y''' - y'' - 4y' + 4y = 5 - e^x + e^{2x}$
25. $y^{(4)} + 2y'' + y = (x - 1)^2$
26. $y^{(4)} - y'' = 4x + 2xe^{-x}$

In Problems 27–36 solve the given initial-value problem.

27. $y'' + 4y = -2, \quad y\left(\frac{\pi}{8}\right) = \frac{1}{2}, y'\left(\frac{\pi}{8}\right) = 2$
28. $2y'' + 3y' - 2y = 14x^2 - 4x - 11, \quad y(0) = 0, y'(0) = 0$
29. $5y'' + y' = -6x, \quad y(0) = 0, y'(0) = -10$
30. $y'' + 4y' + 4y = (3 + x)e^{-2x}, \quad y(0) = 2, y'(0) = 5$
31. $y'' + 4y' + 5y = 35e^{-4x}, \quad y(0) = -3, y'(0) = 1$

32. $y'' - y = \cosh x, \quad y(0) = 2, y'(0) = 12$

33. $\dfrac{d^2x}{dt^2} + \omega^2 x = F_0 \sin \omega t, \quad x(0) = 0, x'(0) = 0$

34. $\dfrac{d^2x}{dt^2} + \omega^2 x = F_0 \cos \gamma t, \quad x(0) = 0, x'(0) = 0$

35. $y''' - 2y'' + y' = 2 - 24e^x + 40e^{5x}, \quad y(0) = \frac{1}{2}, y'(0) = \frac{5}{2}, y''(0) = -\frac{9}{2}$

36. $y''' + 8y = 2x - 5 + 8e^{-2x}, \quad y(0) = -5, y'(0) = 3, y''(0) = -4$

In Problems 37–40 solve the given boundary-value problem.

37. $y'' + y = x^2 + 1, \quad y(0) = 5, y(1) = 0$

38. $y'' - 2y' + 2y = 2x - 2, \quad y(0) = 0, y(\pi) = \pi$

39. $y'' + 3y = 6x, \quad y(0) = 0, y(1) + y'(1) = 0$

40. $y'' + 3y = 6x, \quad y(0) + y'(0) = 0, y(1) = 0$

In Problems 41 and 42 solve the given initial-value problem in which the input function $g(x)$ is discontinuous. [*Hint*: Solve each problem on two intervals, and then find a solution so that y and y' are continuous at $x = \pi/2$ (Problem 41) and at $x = \pi$ (Problem 42).]

41. $y'' + 4y = g(x), \quad y(0) = 1, y'(0) = 2,$ where

$$g(x) = \begin{cases} \sin x, & 0 \le x \le \pi/2 \\ 0, & x > \pi/2 \end{cases}$$

42. $y'' - 2y' + 10y = g(x), \quad y(0) = 0, y'(0) = 0,$ where

$$g(x) = \begin{cases} 20, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$$

Discussion Problems

43. Consider the differential equation $ay'' + by' + cy = e^{kx}$, where a, b, c, and k are constants. The auxiliary equation of the associated homogeneous equation is $am^2 + bm + c = 0$.

(a) If k is not a root of the auxiliary equation, show that we can find a particular solution of the form $y_p = Ae^{kx}$, where $A = 1/(ak^2 + bk + c)$.

(b) If k is a root of the auxiliary equation of multiplicity one, show that we can find a particular solution of the form $y_p = Axe^{kx}$, where $A = 1/(2ak + b)$. Explain how we know that $k \neq -b/(2a)$.

(c) If k is a root of the auxiliary equation of multiplicity two, show that we can find a particular solution of the form $y = Ax^2e^{kx}$, where $A = 1/(2a)$.

44. Discuss how the method of this section can be used to find a particular solution of $y'' + y = \sin x \cos 2x$. Carry out your idea.

45. Without solving, match a solution curve of $y'' + y = f(x)$ shown in the figure with one of the following functions:

(*i*) $f(x) = 1$, (*ii*) $f(x) = e^{-x}$,
(*iii*) $f(x) = e^x$, (*iv*) $f(x) = \sin 2x$,
(*v*) $f(x) = e^x \sin x$, (*vi*) $f(x) = \sin x$.

Briefly discuss your reasoning.

(a)

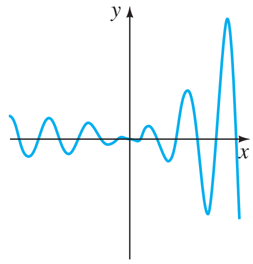


FIGURE 4.4.1 Solution curve

(b)

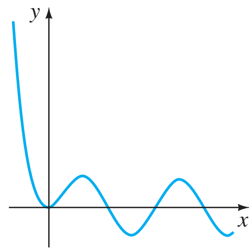


FIGURE 4.4.2 Solution curve

(c)

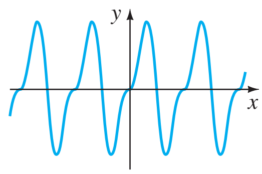


FIGURE 4.4.3 Solution curve

(d)

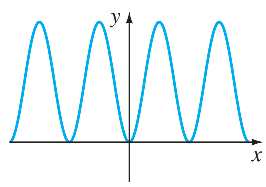


FIGURE 4.4.4 Solution curve

Computer Lab Assignments

In Problems 46 and 47 find a particular solution of the given differential equation. Use a CAS as an aid in carrying out differentiations, simplifications, and algebra.

46. $y'' - 4y' + 8y = (2x^2 - 3x)e^{2x} \cos 2x + (10x^2 - x - 1)e^{2x} \sin 2x$

47. $y^{(4)} + 2y'' + y = 2 \cos x - 3x \sin x$

4.5 UNDETERMINED COEFFICIENTS—ANNIHILATOR APPROACH

REVIEW MATERIAL

- Review Theorems 4.1.6 and 4.1.7 (Section 4.1)

INTRODUCTION We saw in Section 4.1 that an nth-order differential equation can be written

$$a_nD^ny + a_{n-1}D^{n-1}y + \cdots + a_1Dy + a_0y = g(x), \tag{1}$$

where $D^ky = d^ky/dx^k$, $k = 0, 1, \ldots, n$. When it suits our purpose, (1) is also written as $L(y) = g(x)$, where L denotes the linear nth-order differential, or polynomial, operator

$$a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0. \tag{2}$$

Not only is the operator notation a helpful shorthand, but also on a very practical level the application of differential operators enables us to justify the somewhat mind-numbing rules for determining the form of particular solution y_p that were presented in the preceding section. In this section there are no special rules; the form of y_p follows almost automatically once we have found an appropriate linear differential operator that *annihilates* $g(x)$ in (1). Before investigating how this is done, we need to examine two concepts.

FACTORING OPERATORS When the coefficients a_i, $i = 0, 1, \ldots, n$ are real constants, a linear differential operator (1) can be factored whenever the characteristic polynomial $a_nm^n + a_{n-1}m^{n-1} + \cdots + a_1m + a_0$ factors. In other words, if r_1 is a root of the auxiliary equation

$$a_nm^n + a_{n-1}m^{n-1} + \cdots + a_1m + a_0 = 0,$$

then $L = (D - r_1)\,P(D)$, where the polynomial expression $P(D)$ is a linear differential operator of order $n - 1$. For example, if we treat D as an algebraic quantity, then the operator $D^2 + 5D + 6$ can be factored as $(D + 2)(D + 3)$ or as $(D + 3)(D + 2)$. Thus if a function $y = f(x)$ possesses a second derivative, then

$$(D^2 + 5D + 6)y = (D + 2)(D + 3)y = (D + 3)(D + 2)y.$$

This illustrates a general property:

Factors of a linear differential operator with constant coefficients commute.

A differential equation such as $y'' + 4y' + 4y = 0$ can be written as

$$(D^2 + 4D + 4)y = 0 \quad \text{or} \quad (D + 2)(D + 2)y = 0 \quad \text{or} \quad (D + 2)^2y = 0.$$

ANNIHILATOR OPERATOR If L is a linear differential operator with constant coefficients and f is a sufficiently differentiable function such that

$$L(f(x)) = 0,$$

then L is said to be an **annihilator** of the function. For example, a constant function $y = k$ is annihilated by D, since $Dk = 0$. The function $y = x$ is annihilated by the differential operator D^2 since the first and second derivatives of x are 1 and 0, respectively. Similarly, $D^3x^2 = 0$, and so on.

The differential operator D^n annihilates each of the functions

$$1, \quad x, \quad x^2, \quad \ldots, \quad x^{n-1}. \tag{3}$$

As an immediate consequence of (3) and the fact that differentiation can be done term by term, a polynomial

$$c_0 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1} \tag{4}$$

can be annihilated by finding an operator that annihilates the highest power of x.

The functions that are annihilated by a linear nth-order differential operator L are simply those functions that can be obtained from the general solution of the homogeneous differential equation $L(y) = 0$.

> The differential operator $(D - \alpha)^n$ annihilates each of the functions
>
> $$e^{\alpha x}, \quad xe^{\alpha x}, \quad x^2e^{\alpha x}, \quad \ldots, \quad x^{n-1}e^{\alpha x}. \tag{5}$$

To see this, note that the auxiliary equation of the homogeneous equation $(D - \alpha)^n y = 0$ is $(m - \alpha)^n = 0$. Since α is a root of multiplicity n, the general solution is

$$y = c_1e^{\alpha x} + c_2xe^{\alpha x} + \cdots + c_nx^{n-1}e^{\alpha x}. \tag{6}$$

EXAMPLE 1 Annihilator Operators

Find a differential operator that annihilates the given function.

(a) $1 - 5x^2 + 8x^3$ **(b)** e^{-3x} **(c)** $4e^{2x} - 10xe^{2x}$

SOLUTION **(a)** From (3) we know that $D^4x^3 = 0$, so it follows from (4) that

$$D^4(1 - 5x^2 + 8x^3) = 0.$$

(b) From (5), with $\alpha = -3$ and $n = 1$, we see that

$$(D + 3)e^{-3x} = 0.$$

(c) From (5) and (6), with $\alpha = 2$ and $n = 2$, we have

$$(D - 2)^2(4e^{2x} - 10xe^{2x}) = 0.$$ ■

When α and β, $\beta > 0$ are real numbers, the quadratic formula reveals that $[m^2 - 2\alpha m + (\alpha^2 + \beta^2)]^n = 0$ has complex roots $\alpha + i\beta$, $\alpha - i\beta$, both of multiplicity n. From the discussion at the end of Section 4.3 we have the next result.

> The differential operator $[D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^n$ annihilates each of the functions
>
> $$\begin{aligned} &e^{\alpha x}\cos\beta x, \quad xe^{\alpha x}\cos\beta x, \quad x^2e^{\alpha x}\cos\beta x, \quad \ldots, \quad x^{n-1}e^{\alpha x}\cos\beta x, \\ &e^{\alpha x}\sin\beta x, \quad xe^{\alpha x}\sin\beta x, \quad x^2e^{\alpha x}\sin\beta x, \quad \ldots, \quad x^{n-1}e^{\alpha x}\sin\beta x. \end{aligned} \tag{7}$$

EXAMPLE 2 Annihilator Operator

Find a differential operator that annihilates $5e^{-x}\cos 2x - 9e^{-x}\sin 2x$.

SOLUTION Inspection of the functions $e^{-x}\cos 2x$ and $e^{-x}\sin 2x$ shows that $\alpha = -1$ and $\beta = 2$. Hence from (7) we conclude that $D^2 + 2D + 5$ will annihilate each function. Since $D^2 + 2D + 5$ is a linear operator, it will annihilate *any* linear combination of these functions such as $5e^{-x}\cos 2x - 9e^{-x}\sin 2x$. ■

When $\alpha = 0$ and $n = 1$, a special case of (7) is

$$(D^2 + \beta^2)\begin{cases}\cos \beta x \\ \sin \beta x\end{cases} = 0. \tag{8}$$

For example, $D^2 + 16$ will annihilate any linear combination of $\sin 4x$ and $\cos 4x$.

We are often interested in annihilating the sum of two or more functions. As we have just seen in Examples 1 and 2, if L is a linear differential operator such that $L(y_1) = 0$ and $L(y_2) = 0$, then L will annihilate the linear combination $c_1y_1(x) + c_2y_2(x)$. This is a direct consequence of Theorem 4.1.2. Let us now suppose that L_1 and L_2 are linear differential operators with constant coefficients such that L_1 annihilates $y_1(x)$ and L_2 annihilates $y_2(x)$, but $L_1(y_2) \neq 0$ and $L_2(y_1) \neq 0$. Then the *product* of differential operators L_1L_2 annihilates the sum $c_1y_1(x) + c_2y_2(x)$. We can easily demonstrate this, using linearity and the fact that $L_1L_2 = L_2L_1$:

$$\begin{aligned} L_1L_2(y_1 + y_2) &= L_1L_2(y_1) + L_1L_2(y_2) \\ &= L_2L_1(y_1) + L_1L_2(y_2) \\ &= L_2[\underbrace{L_1(y_1)}_{\text{zero}}] + L_1[\underbrace{L_2(y_2)}_{\text{zero}}] = 0. \end{aligned}$$

For example, we know from (3) that D^2 annihilates $7 - x$ and from (8) that $D^2 + 16$ annihilates $\sin 4x$. Therefore the product of operators $D^2(D^2 + 16)$ will annihilate the linear combination $7 - x + 6 \sin 4x$.

NOTE The differential operator that annihilates a function is not unique. We saw in part (b) of Example 1 that $D + 3$ will annihilate e^{-3x}, but so will differential operators of higher order as long as $D + 3$ is one of the factors of the operator. For example, $(D + 3)(D + 1)$, $(D + 3)^2$, and $D^3(D + 3)$ all annihilate e^{-3x}. (Verify this.) As a matter of course, when we seek a differential annihilator for a function $y = f(x)$, we want the operator of *lowest possible order* that does the job.

UNDETERMINED COEFFICIENTS This brings us to the point of the preceding discussion. Suppose that $L(y) = g(x)$ is a linear differential equation with constant coefficients and that the input $g(x)$ consists of finite sums and products of the functions listed in (3), (5), and (7)—that is, $g(x)$ is a linear combination of functions of the form

$$k \text{ (constant)}, \quad x^m, \quad x^me^{\alpha x}, \quad x^me^{\alpha x}\cos \beta x, \quad \text{and} \quad x^me^{\alpha x}\sin \beta x,$$

where m is a nonnegative integer and α and β are real numbers. We now know that such a function $g(x)$ can be annihilated by a differential operator L_1 of lowest order, consisting of a product of the operators D^n, $(D - \alpha)^n$, and $(D^2 - 2\alpha D + \alpha^2 + \beta^2)^n$. Applying L_1 to both sides of the equation $L(y) = g(x)$ yields $L_1L(y) = L_1(g(x)) = 0$. By solving the *homogeneous higher-order* equation $L_1L(y) = 0$, we can discover the *form* of a particular solution y_p for the original *nonhomogeneous* equation $L(y) = g(x)$. We then substitute this assumed form into $L(y) = g(x)$ to find an explicit particular solution. This procedure for determining y_p, called the **method of undetermined coefficients,** is illustrated in the next several examples.

Before proceeding, recall that the general solution of a nonhomogeneous linear differential equation $L(y) = g(x)$ is $y = y_c + y_p$, where y_c is the complementary function—that is, the general solution of the associated homogeneous equation $L(y) = 0$. The general solution of each equation $L(y) = g(x)$ is defined on the interval $(-\infty, \infty)$.

EXAMPLE 3 General Solution Using Undetermined Coefficients

Solve $y'' + 3y' + 2y = 4x^2$. (9)

SOLUTION **Step 1.** First, we solve the homogeneous equation $y'' + 3y' + 2y = 0$. Then, from the auxiliary equation $m^2 + 3m + 2 = (m + 1)(m + 2) = 0$ we find $m_1 = -1$ and $m_2 = -2$, and so the complementary function is

$$y_c = c_1e^{-x} + c_2e^{-2x}.$$

Step 2. Now, since $4x^2$ is annihilated by the differential operator D^3, we see that $D^3(D^2 + 3D + 2)y = 4D^3x^2$ is the same as

$$D^3(D^2 + 3D + 2)y = 0. \tag{10}$$

The auxiliary equation of the fifth-order equation in (10),

$$m^3(m^2 + 3m + 2) = 0 \quad \text{or} \quad m^3(m + 1)(m + 2) = 0,$$

has roots $m_1 = m_2 = m_3 = 0$, $m_4 = -1$, and $m_5 = -2$. Thus its general solution must be

$$y = c_1 + c_2x + c_3x^2 + \boxed{c_4e^{-x} + c_5e^{-2x}}. \tag{11}$$

The terms in the shaded box in (11) constitute the complementary function of the original equation (9). We can then argue that a particular solution y_p of (9) should also satisfy equation (10). This means that the terms remaining in (11) must be the basic form of y_p:

$$y_p = A + Bx + Cx^2, \tag{12}$$

where, for convenience, we have replaced c_1, c_2, and c_3 by A, B, and C, respectively. For (12) to be a particular solution of (9), it is necessary to find *specific* coefficients A, B, and C. Differentiating (12), we have

$$y_p' = B + 2Cx, \qquad y_p'' = 2C,$$

and substitution into (9) then gives

$$y_p'' + 3y_p' + 2y_p = 2C + 3B + 6Cx + 2A + 2Bx + 2Cx^2 = 4x^2.$$

Because the last equation is supposed to be an identity, the coefficients of like powers of x must be equal:

equal

$$\boxed{2C}\,x^2 + \boxed{2B + 6C}\,x + \boxed{2A + 3B + 2C} = 4x^2 + 0x + 0.$$

That is $\qquad 2C = 4, \qquad 2B + 6C = 0, \qquad 2A + 3B + 2C = 0.$ (13)

Solving the equations in (13) gives $A = 7$, $B = -6$, and $C = 2$. Thus $y_p = 7 - 6x + 2x^2$.

Step 3. The general solution of the equation in (9) is $y = y_c + y_p$ or

$$y = c_1e^{-x} + c_2e^{-2x} + 7 - 6x + 2x^2.$$ ■

EXAMPLE 4 General Solution Using Undetermined Coefficients

Solve $y'' - 3y' = 8e^{3x} + 4\sin x$. (14)

SOLUTION **Step 1.** The auxiliary equation for the associated homogeneous equation $y'' - 3y' = 0$ is $m^2 - 3m = m(m - 3) = 0$, so $y_c = c_1 + c_2e^{3x}$.

Step 2. Now, since $(D - 3)e^{3x} = 0$ and $(D^2 + 1)\sin x = 0$, we apply the differential operator $(D - 3)(D^2 + 1)$ to both sides of (14):

$$(D - 3)(D^2 + 1)(D^2 - 3D)y = 0. \tag{15}$$

The auxiliary equation of (15) is

$$(m - 3)(m^2 + 1)(m^2 - 3m) = 0 \qquad \text{or} \qquad m(m - 3)^2(m^2 + 1) = 0.$$

Thus $$y = \boxed{c_1 + c_2e^{3x}} + c_3xe^{3x} + c_4\cos x + c_5\sin x.$$

After excluding the linear combination of terms in the box that corresponds to y_c, we arrive at the form of y_p:

$$y_p = Axe^{3x} + B\cos x + C\sin x.$$

Substituting y_p in (14) and simplifying yield

$$y_p'' - 3y_p' = 3Ae^{3x} + (-B - 3C)\cos x + (3B - C)\sin x = 8e^{3x} + 4\sin x.$$

Equating coefficients gives $3A = 8$, $-B - 3C = 0$, and $3B - C = 4$. We find $A = \frac{8}{3}$, $B = \frac{6}{5}$, and $C = -\frac{2}{5}$, and consequently,

$$y_p = \frac{8}{3}xe^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x.$$

Step 3. The general solution of (14) is then

$$y = c_1 + c_2e^{3x} + \frac{8}{3}xe^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x.$$ ■

EXAMPLE 5 General Solution Using Undetermined Coefficients

Solve $y'' + y = x\cos x - \cos x$. (16)

SOLUTION The complementary function is $y_c = c_1\cos x + c_2\sin x$. Now by comparing $\cos x$ and $x\cos x$ with the functions in the first row of (7), we see that $\alpha = 0$ and $n = 1$, and so $(D^2 + 1)^2$ is an annihilator for the right-hand member of the equation in (16). Applying this operator to the differential equation gives

$$(D^2 + 1)^2(D^2 + 1)y = 0 \qquad \text{or} \qquad (D^2 + 1)^3y = 0.$$

Since i and $-i$ are both complex roots of multiplicity 3 of the auxiliary equation of the last differential equation, we conclude that

$$y = \boxed{c_1\cos x + c_2\sin x} + c_3x\cos x + c_4x\sin x + c_5x^2\cos x + c_6x^2\sin x.$$

We substitute

$$y_p = Ax\cos x + Bx\sin x + Cx^2\cos x + Ex^2\sin x$$

into (16) and simplify:

$$\begin{aligned} y_p'' + y_p &= 4Ex\cos x - 4Cx\sin x + (2B + 2C)\cos x + (-2A + 2E)\sin x \\ &= x\cos x - \cos x. \end{aligned}$$

Equating coefficients gives the equations $4E = 1$, $-4C = 0$, $2B + 2C = -1$, and $-2A + 2E = 0$, from which we find $A = \frac{1}{4}$, $B = -\frac{1}{2}$, $C = 0$, and $E = \frac{1}{4}$. Hence the general solution of (16) is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{4} x \cos x - \frac{1}{2} x \sin x + \frac{1}{4} x^2 \sin x. \qquad \blacksquare$$

EXAMPLE 6 Form of a Particular Solution

Determine the form of a particular solution for

$$y'' - 2y' + y = 10e^{-2x} \cos x. \tag{17}$$

SOLUTION The complementary function for the given equation is $y_c = c_1 e^x + c_2 x e^x$.

Now from (7), with $\alpha = -2$, $\beta = 1$, and $n = 1$, we know that

$$(D^2 + 4D + 5)e^{-2x} \cos x = 0.$$

Applying the operator $D^2 + 4D + 5$ to (17) gives

$$(D^2 + 4D + 5)(D^2 - 2D + 1)y = 0. \tag{18}$$

Since the roots of the auxiliary equation of (18) are $-2 - i$, $-2 + i$, 1, and 1, we see from

$$y = c_1 e^x + c_2 x e^x + c_3 e^{-2x} \cos x + c_4 e^{-2x} \sin x$$

that a particular solution of (17) can be found with the form

$$y_p = Ae^{-2x} \cos x + Be^{-2x} \sin x. \qquad \blacksquare$$

EXAMPLE 7 Form of a Particular Solution

Determine the form of a particular solution for

$$y''' - 4y'' + 4y' = 5x^2 - 6x + 4x^2 e^{2x} + 3e^{5x}. \tag{19}$$

SOLUTION Observe that

$$D^3(5x^2 - 6x) = 0, \qquad (D - 2)^3 x^2 e^{2x} = 0, \qquad \text{and} \qquad (D - 5)e^{5x} = 0.$$

Therefore $D^3(D - 2)^3(D - 5)$ applied to (19) gives

$$D^3(D - 2)^3(D - 5)(D^3 - 4D^2 + 4D)y = 0$$

or

$$D^4(D - 2)^5(D - 5)y = 0.$$

The roots of the auxiliary equation for the last differential equation are easily seen to be 0, 0, 0, 0, 2, 2, 2, 2, 2, and 5. Hence

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{2x} + c_6 x e^{2x} + c_7 x^2 e^{2x} + c_8 x^3 e^{2x} + c_9 x^4 e^{2x} + c_{10} e^{5x}. \tag{20}$$

Because the linear combination $c_1 + c_5 e^{2x} + c_6 x e^{2x}$ corresponds to the complementary function of (19), the remaining terms in (20) give the form of a particular solution of the differential equation:

$$y_p = Ax + Bx^2 + Cx^3 + Ex^2 e^{2x} + Fx^3 e^{2x} + Gx^4 e^{2x} + He^{5x}. \qquad \blacksquare$$

SUMMARY OF THE METHOD For your convenience the method of undetermined coefficients is summarized as follows.

UNDETERMINED COEFFICIENTS—ANNIHILATOR APPROACH

The differential equation $L(y) = g(x)$ has constant coefficients, and the function $g(x)$ consists of finite sums and products of constants, polynomials, exponential functions $e^{\alpha x}$, sines, and cosines.

(*i*) Find the complementary solution y_c for the homogeneous equation $L(y) = 0$.

(*ii*) Operate on both sides of the nonhomogeneous equation $L(y) = g(x)$ with a differential operator L_1 that annihilates the function $g(x)$.

(*iii*) Find the general solution of the higher-order homogeneous differential equation $L_1L(y) = 0$.

(*iv*) Delete from the solution in step (*iii*) all those terms that are duplicated in the complementary solution y_c found in step (*i*). Form a linear combination y_p of the terms that remain. This is the form of a particular solution of $L(y) = g(x)$.

(*v*) Substitute y_p found in step (*iv*) into $L(y) = g(x)$. Match coefficients of the various functions on each side of the equality, and solve the resulting system of equations for the unknown coefficients in y_p.

(*vi*) With the particular solution found in step (*v*), form the general solution $y = y_c + y_p$ of the given differential equation.

REMARKS

The method of undetermined coefficients is not applicable to linear differential equations with variable coefficients nor is it applicable to linear equations with constant coefficients when $g(x)$ is a function such as

$$g(x) = \ln x, \qquad g(x) = \frac{1}{x}, \qquad g(x) = \tan x, \qquad g(x) = \sin^{-1} x,$$

and so on. Differential equations in which the input $g(x)$ is a function of this last kind will be considered in the next section.

EXERCISES 4.5

Answers to selected odd-numbered problems begin on page ANS-5.

In Problems 1–10 write the given differential equation in the form $L(y) = g(x)$, where L is a linear differential operator with constant coefficients. If possible, factor L.

1. $9y'' - 4y = \sin x$

2. $y'' - 5y = x^2 - 2x$

3. $y'' - 4y' - 12y = x - 6$

4. $2y'' - 3y' - 2y = 1$

5. $y''' + 10y'' + 25y' = e^x$

6. $y''' + 4y' = e^x \cos 2x$

7. $y''' + 2y'' - 13y' + 10y = xe^{-x}$

8. $y''' + 4y'' + 3y' = x^2 \cos x - 3x$

9. $y^{(4)} + 8y' = 4$

10. $y^{(4)} - 8y'' + 16y = (x^3 - 2x)e^{4x}$

In Problems 11–14 verify that the given differential operator annihilates the indicated functions.

11. D^4; $y = 10x^3 - 2x$

12. $2D - 1$; $y = 4e^{x/2}$

13. $(D - 2)(D + 5)$; $y = e^{2x} + 3e^{-5x}$

14. $D^2 + 64$; $y = 2 \cos 8x - 5 \sin 8x$

In Problems 15–26 find a linear differential operator that annihilates the given function.

15. $1 + 6x - 2x^3$

16. $x^3(1 - 5x)$

17. $1 + 7e^{2x}$

18. $x + 3xe^{6x}$

19. $\cos 2x$

20. $1 + \sin x$

21. $13x + 9x^2 - \sin 4x$

22. $8x - \sin x + 10 \cos 5x$

23. $e^{-x} + 2xe^x - x^2e^x$

24. $(2 - e^x)^2$

25. $3 + e^x \cos 2x$

26. $e^{-x} \sin x - e^{2x} \cos x$

In Problems 27–34 find linearly independent functions that are annihilated by the given differential operator.

27. D^5

28. $D^2 + 4D$

29. $(D - 6)(2D + 3)$

30. $D^2 - 9D - 36$

31. $D^2 + 5$

32. $D^2 - 6D + 10$

33. $D^3 - 10D^2 + 25D$

34. $D^2(D - 5)(D - 7)$

In Problems 35–64 solve the given differential equation by undetermined coefficients.

35. $y'' - 9y = 54$

36. $2y'' - 7y' + 5y = -29$

37. $y'' + y' = 3$

38. $y''' + 2y'' + y' = 10$

39. $y'' + 4y' + 4y = 2x + 6$

40. $y'' + 3y' = 4x - 5$

41. $y''' + y'' = 8x^2$

42. $y'' - 2y' + y = x^3 + 4x$

43. $y'' - y' - 12y = e^{4x}$

44. $y'' + 2y' + 2y = 5e^{6x}$

45. $y'' - 2y' - 3y = 4e^x - 9$

46. $y'' + 6y' + 8y = 3e^{-2x} + 2x$

47. $y'' + 25y = 6 \sin x$

48. $y'' + 4y = 4 \cos x + 3 \sin x - 8$

49. $y'' + 6y' + 9y = -xe^{4x}$

50. $y'' + 3y' - 10y = x(e^x + 1)$

51. $y'' - y = x^2e^x + 5$

52. $y'' + 2y' + y = x^2e^{-x}$

53. $y'' - 2y' + 5y = e^x \sin x$

54. $y'' + y' + \frac{1}{4}y = e^x(\sin 3x - \cos 3x)$

55. $y'' + 25y = 20 \sin 5x$

56. $y'' + y = 4 \cos x - \sin x$

57. $y'' + y' + y = x \sin x$

58. $y'' + 4y = \cos^2 x$

59. $y''' + 8y'' = -6x^2 + 9x + 2$

60. $y''' - y'' + y' - y = xe^x - e^{-x} + 7$

61. $y''' - 3y'' + 3y' - y = e^x - x + 16$

62. $2y''' - 3y'' - 3y' + 2y = (e^x + e^{-x})^2$

63. $y^{(4)} - 2y''' + y'' = e^x + 1$

64. $y^{(4)} - 4y'' = 5x^2 - e^{2x}$

In Problems 65–72 solve the given initial-value problem.

65. $y'' - 64y = 16, \quad y(0) = 1, y'(0) = 0$

66. $y'' + y' = x, \quad y(0) = 1, y'(0) = 0$

67. $y'' - 5y' = x - 2, \quad y(0) = 0, y'(0) = 2$

68. $y'' + 5y' - 6y = 10e^{2x}, \quad y(0) = 1, y'(0) = 1$

69. $y'' + y = 8 \cos 2x - 4 \sin x, \quad y\left(\frac{\pi}{2}\right) = -1, y'\left(\frac{\pi}{2}\right) = 0$

70. $y''' - 2y'' + y' = xe^x + 5, \quad y(0) = 2, y'(0) = 2, y''(0) = -1$

71. $y'' - 4y' + 8y = x^3, \quad y(0) = 2, y'(0) = 4$

72. $y^{(4)} - y''' = x + e^x, \quad y(0) = 0, y'(0) = 0, y''(0) = 0, y'''(0) = 0$

Discussion Problems

73. Suppose L is a linear differential operator that factors but has variable coefficients. Do the factors of L commute? Defend your answer.

4.6 VARIATION OF PARAMETERS

REVIEW MATERIAL

- Variation of parameters was first introduced in Section 2.3 and used again in Section 4.2. A review of those sections is recommended.

INTRODUCTION The procedure that we used to find a particular solution y_p of a linear first-order differential equation on an interval is applicable to linear higher-order DEs as well. To adapt the method of **variation of parameters** to a linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x), \tag{1}$$

we begin by putting the equation into the standard form

$$y'' + P(x)y' + Q(x)y = f(x) \tag{2}$$

by dividing through by the lead coefficient $a_2(x)$. Equation (2) is the second-order analogue of the standard form of a linear first-order equation: $dy/dx + P(x)y = f(x)$. In (2) we suppose that $P(x)$, $Q(x)$, and $f(x)$ are continuous on some common interval I. As we have already seen in Section 4.3, there is no difficulty in obtaining the complementary function y_c, the general solution of the associated homogeneous equation of (2), when the coefficients are constant.

ASSUMPTIONS Corresponding to the assumption $y_p = u_1(x)y_1(x)$ that we used in Section 2.3 to find a particular solution y_p of $dy/dx + P(x)y = f(x)$, for the linear second-order equation (2) we seek a solution of the form

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x), \tag{3}$$

where y_1 and y_2 form a fundamental set of solutions on I of the associated homogeneous form of (1). Using the Product Rule to differentiate y_p twice, we get

$$y_p' = u_1y_1' + y_1u_1' + u_2y_2' + y_2u_2'$$
$$y_p'' = u_1y_1'' + y_1'u_1' + y_1u_1'' + u_1'y_1' + u_2y_2'' + y_2'u_2' + y_2u_2'' + u_2'y_2'.$$

Substituting (3) and the foregoing derivatives into (2) and grouping terms yields

$$\begin{aligned} y_p'' + P(x)y_p' + Q(x)y_p &= u_1[\overbrace{y_1'' + Py_1' + Qy_1}^{\text{zero}}] + u_2[\overbrace{y_2'' + Py_2' + Qy_2}^{\text{zero}}] + y_1u_1'' + u_1'y_1' \\ &\quad + y_2u_2'' + u_2'y_2' + P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2' \\ &= \frac{d}{dx}[y_1u_1'] + \frac{d}{dx}[y_2u_2'] + P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2' \\ &= \frac{d}{dx}[y_1u_1' + y_2u_2'] + P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2' = f(x). \end{aligned} \tag{4}$$

Because we seek to determine two unknown functions u_1 and u_2, reason dictates that we need two equations. We can obtain these equations by making the further assumption that the functions u_1 and u_2 satisfy $y_1u_1' + y_2u_2' = 0$. This assumption does not come out of the blue but is prompted by the first two terms in (4), since if we demand that $y_1u_1' + y_2u_2' = 0$, then (4) reduces to $y_1'u_1' + y_2'u_2' = f(x)$. We now have our desired two equations, albeit two equations for determining the derivatives u_1' and u_2'. By Cramer's Rule, the solution of the system

$$y_1u_1' + y_2u_2' = 0$$
$$y_1'u_1' + y_2'u_2' = f(x)$$

can be expressed in terms of determinants:

$$u_1' = \frac{W_1}{W} = -\frac{y_2f(x)}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W} = \frac{y_1f(x)}{W}, \tag{5}$$

where $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}. \tag{6}$$

The functions u_1 and u_2 are found by integrating the results in (5). The determinant W is recognized as the Wronskian of y_1 and y_2. By linear independence of y_1 and y_2 on I, we know that $W(y_1(x), y_2(x)) \neq 0$ for every x in the interval.

SUMMARY OF THE METHOD Usually, it is not a good idea to memorize formulas in lieu of understanding a procedure. However, the foregoing procedure is too long and complicated to use each time we wish to solve a differential equation. In this case it is more efficient to simply use the formulas in (5). Thus to solve $a_2y'' + a_1y' + a_0y = g(x)$, first find the complementary function $y_c = c_1y_1 + c_2y_2$ and then compute the Wronskian $W(y_1(x), y_2(x))$. By dividing by a_2, we put the equation into the standard form $y'' + Py' + Qy = f(x)$ to determine $f(x)$. We find u_1 and u_2 by integrating $u_1' = W_1/W$ and $u_2' = W_2/W$, where W_1 and W_2 are defined as in (6). A particular solution is $y_p = u_1y_1 + u_2y_2$. The general solution of the equation is then $y = y_c + y_p$.

EXAMPLE 1 General Solution Using Variation of Parameters

Solve $y'' - 4y' + 4y = (x + 1)e^{2x}$.

SOLUTION From the auxiliary equation $m^2 - 4m + 4 = (m - 2)^2 = 0$ we have $y_c = c_1e^{2x} + c_2xe^{2x}$. With the identifications $y_1 = e^{2x}$ and $y_2 = xe^{2x}$, we next compute the Wronskian:

$$W(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}.$$

Since the given differential equation is already in form (2) (that is, the coefficient of y'' is 1), we identify $f(x) = (x + 1)e^{2x}$. From (6) we obtain

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -(x + 1)xe^{4x}, \qquad W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x + 1)e^{4x},$$

and so from (5)

$$u_1' = -\frac{(x+1)xe^{4x}}{e^{4x}} = -x^2 - x, \qquad u_2' = \frac{(x+1)e^{4x}}{e^{4x}} = x + 1.$$

It follows that $u_1 = -\frac{1}{3}x^3 - \frac{1}{2}x^2$ and $u_2 = \frac{1}{2}x^2 + x$. Hence

$$y_p = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x} = \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

and

$$y = y_c + y_p = c_1e^{2x} + c_2xe^{2x} + \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}.$$ ■

EXAMPLE 2 General Solution Using Variation of Parameters

Solve $4y'' + 36y = \csc 3x$.

SOLUTION We first put the equation in the standard form (2) by dividing by 4:

$$y'' + 9y = \frac{1}{4}\csc 3x.$$

Because the roots of the auxiliary equation $m^2 + 9 = 0$ are $m_1 = 3i$ and $m_2 = -3i$, the complementary function is $y_c = c_1 \cos 3x + c_2 \sin 3x$. Using $y_1 = \cos 3x$, $y_2 = \sin 3x$, and $f(x) = \frac{1}{4}\csc 3x$, we obtain

$$W(\cos 3x, \sin 3x) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3,$$

$$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\csc 3x & 3\cos 3x \end{vmatrix} = -\frac{1}{4}, \qquad W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \frac{1}{4}\csc 3x \end{vmatrix} = \frac{1}{4}\frac{\cos 3x}{\sin 3x}.$$

Integrating $\quad u_1' = \dfrac{W_1}{W} = -\dfrac{1}{12} \quad$ and $\quad u_2' = \dfrac{W_2}{W} = \dfrac{1}{12}\dfrac{\cos 3x}{\sin 3x}$

gives $u_1 = -\frac{1}{12}x$ and $u_2 = \frac{1}{36}\ln|\sin 3x|$. Thus a particular solution is

$$y_p = -\frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|.$$

The general solution of the equation is

$$y = y_c + y_p = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|. \quad (7)$$ ■

Equation (7) represents the general solution of the differential equation on, say, the interval $(0, \pi/6)$.

CONSTANTS OF INTEGRATION When computing the indefinite integrals of u_1' and u_2', we need not introduce any constants. This is because

$$\begin{aligned} y = y_c + y_p &= c_1y_1 + c_2y_2 + (u_1 + a_1)y_1 + (u_2 + b_1)y_2 \\ &= (c_1 + a_1)y_1 + (c_2 + b_1)y_2 + u_1y_1 + u_2y_2 \\ &= C_1y_1 + C_2y_2 + u_1y_1 + u_2y_2. \end{aligned}$$

EXAMPLE 3 General Solution Using Variation of Parameters

Solve $y'' - y = \dfrac{1}{x}$.

SOLUTION The auxiliary equation $m^2 - 1 = 0$ yields $m_1 = -1$ and $m_2 = 1$. Therefore $y_c = c_1e^x + c_2e^{-x}$. Now $W(e^x, e^{-x}) = -2$, and

$$u_1' = -\frac{e^{-x}(1/x)}{-2}, \qquad u_1 = \frac{1}{2}\int_{x_0}^{x} \frac{e^{-t}}{t}\,dt,$$

$$u_2' = \frac{e^{x}(1/x)}{-2}, \qquad u_2 = -\frac{1}{2}\int_{x_0}^{x} \frac{e^{t}}{t}\,dt.$$

Since the foregoing integrals are nonelementary, we are forced to write

$$y_p = \frac{1}{2}e^x\int_{x_0}^{x}\frac{e^{-t}}{t}\,dt - \frac{1}{2}e^{-x}\int_{x_0}^{x}\frac{e^{t}}{t}\,dt,$$

and so $$y = y_c + y_p = c_1e^x + c_2e^{-x} + \frac{1}{2}e^x\int_{x_0}^{x}\frac{e^{-t}}{t}\,dt - \frac{1}{2}e^{-x}\int_{x_0}^{x}\frac{e^{t}}{t}\,dt. \qquad (8)$$ ■

In Example 3 we can integrate on any interval $[x_0, x]$ that does not contain the origin.

HIGHER-ORDER EQUATIONS The method that we have just examined for nonhomogeneous second-order differential equations can be generalized to linear nth-order equations that have been put into the standard form

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_0(x)y = f(x). \qquad (9)$$

If $y_c = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is the complementary function for (9), then a particular solution is

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n(x)y_n(x),$$

where the u_k', $k = 1, 2, \ldots, n$ are determined by the n equations

$$\begin{aligned} y_1u_1' + y_2u_2' + \cdots + y_nu_n' &= 0 \\ y_1'u_1' + y_2'u_2' + \cdots + y_n'u_n' &= 0 \\ \vdots \qquad\qquad\qquad & \ \ \vdots \\ y_1^{(n-1)}u_1' + y_2^{(n-1)}u_2' + \cdots + y_n^{(n-1)}u_n' &= f(x). \end{aligned} \qquad (10)$$

The first $n - 1$ equations in this system, like $y_1u'_1 + y_2u'_2 = 0$ in (4), are assumptions that are made to simplify the resulting equation after $y_p = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x)$ is substituted in (9). In this case Cramer's rule gives

$$u'_k = \frac{W_k}{W}, \quad k = 1, 2, \ldots, n,$$

where W is the Wronskian of $y_1, y_2, \ldots, y_n$ and W_k is the determinant obtained by replacing the kth column of the Wronskian by the column consisting of the right-hand side of (10)—that is, the column consisting of $(0, 0, \ldots, f(x))$. When $n = 2$, we get (5). When $n = 3$, the particular solution is $y_p = u_1y_1 + u_2y_2 + u_3y_3$, where y_1, y_2, and y_3 constitute a linearly independent set of solutions of the associated homogeneous DE and u_1, u_2, u_3 are determined from

$$u'_1 = \frac{W_1}{W}, \qquad u'_2 = \frac{W_2}{W}, \qquad u'_3 = \frac{W_3}{W}, \tag{11}$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y'_2 & y'_3 \\ f(x) & y''_2 & y''_3 \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y'_1 & 0 & y'_3 \\ y''_1 & f(x) & y''_3 \end{vmatrix}, \quad W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y'_1 & y'_2 & 0 \\ y''_1 & y''_2 & f(x) \end{vmatrix}, \quad \text{and} \quad W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix}.$$

See Problems 25 and 26 in Exercises 4.6.

REMARKS

(*i*) Variation of parameters has a distinct advantage over the method of undetermined coefficients in that it will *always* yield a particular solution y_p provided that the associated homogeneous equation can be solved. The present method is not limited to a function $f(x)$ that is a combination of the four types listed on page 141. As we shall see in the next section, variation of parameters, unlike undetermined coefficients, is applicable to linear DEs with variable coefficients.

(*ii*) In the problems that follow, do not hesitate to simplify the form of y_p. Depending on how the antiderivatives of u'_1 and u'_2 are found, you might not obtain the same y_p as given in the answer section. For example, in Problem 3 in Exercises 4.6 both $y_p = \frac{1}{2}\sin x - \frac{1}{2}x\cos x$ and $y_p = \frac{1}{4}\sin x - \frac{1}{2}x\cos x$ are valid answers. In either case the general solution $y = y_c + y_p$ simplifies to $y = c_1\cos x + c_2\sin x - \frac{1}{2}x\cos x$. Why?

EXERCISES 4.6

Answers to selected odd-numbered problems begin on page ANS-5.

In Problems 1–18 solve each differential equation by variation of parameters.

1. $y'' + y = \sec x$

2. $y'' + y = \tan x$

3. $y'' + y = \sin x$

4. $y'' + y = \sec\theta\tan\theta$

5. $y'' + y = \cos^2 x$

6. $y'' + y = \sec^2 x$

7. $y'' - y = \cosh x$

8. $y'' - y = \sinh 2x$

9. $y'' - 4y = \dfrac{e^{2x}}{x}$

10. $y'' - 9y = \dfrac{9x}{e^{3x}}$

11. $y'' + 3y' + 2y = \dfrac{1}{1 + e^x}$

12. $y'' - 2y' + y = \dfrac{e^x}{1 + x^2}$

13. $y'' + 3y' + 2y = \sin e^x$

14. $y'' - 2y' + y = e^t\arctan t$

15. $y'' + 2y' + y = e^{-t}\ln t$

16. $2y'' + 2y' + y = 4\sqrt{x}$

17. $3y'' - 6y' + 6y = e^x\sec x$

18. $4y'' - 4y' + y = e^{x/2}\sqrt{1 - x^2}$

In Problems 19–22 solve each differential equation by variation of parameters, subject to the initial conditions $y(0) = 1, y'(0) = 0$.

19. $4y'' - y = xe^{x/2}$

20. $2y'' + y' - y = x + 1$

21. $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$

22. $y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}$

In Problems 23 and 24 the indicated functions are known linearly independent solutions of the associated homogeneous differential equation on $(0, \infty)$. Find the general solution of the given nonhomogeneous equation.

23. $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = x^{3/2}$;
$y_1 = x^{-1/2}\cos x,\ y_2 = x^{-1/2}\sin x$

24. $x^2y'' + xy' + y = \sec(\ln x)$;
$y_1 = \cos(\ln x),\ y_2 = \sin(\ln x)$

In Problems 25 and 26 solve the given third-order differential equation by variation of parameters.

25. $y''' + y' = \tan x$

26. $y''' + 4y' = \sec 2x$

Discussion Problems

In Problems 27 and 28 discuss how the methods of undetermined coefficients and variation of parameters can be combined to solve the given differential equation. Carry out your ideas.

27. $3y'' - 6y' + 30y = 15\sin x + e^x \tan 3x$

28. $y'' - 2y' + y = 4x^2 - 3 + x^{-1}e^x$

29. What are the intervals of definition of the general solutions in Problems 1, 7, 9, and 18? Discuss why the interval of definition of the general solution in Problem 24 is *not* $(0, \infty)$.

30. Find the general solution of $x^4y'' + x^3y' - 4x^2y = 1$ given that $y_1 = x^2$ is a solution of the associated homogeneous equation.

31. Suppose $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$, where u_1 and u_2 are defined by (5) is a particular solution of (2) on an interval I for which P, Q, and f are continuous. Show that y_p can be written as

$$y_p(x) = \int_{x_0}^{x} G(x, t)f(t)\,dt, \tag{12}$$

where x and x_0 are in I,

$$G(x, t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}, \tag{13}$$

and $W(t) = W(y_1(t), y_2(t))$ is the Wronskian. The function $G(x, t)$ in (13) is called the **Green's function** for the differential equation (2).

32. Use (13) to construct the Green's function for the differential equation in Example 3. Express the general solution given in (8) in terms of the particular solution (12).

33. Verify that (12) is a solution of the initial-value problem

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = f(x), \quad y(x_0) = 0, \quad y'(x_0) = 0.$$

on the interval I. [*Hint*: Look up Leibniz's Rule for differentiation under an integral sign.]

34. Use the results of Problems 31 and 33 and the Green's function found in Problem 32 to find a solution of the initial-value problem

$$y'' - y = e^{2x}, \quad y(0) = 0, \quad y'(0) = 0$$

using (12). Evaluate the integral.

4.7 CAUCHY-EULER EQUATION

REVIEW MATERIAL

- Review the concept of the auxiliary equation in Section 4.3.

INTRODUCTION The same relative ease with which we were able to find explicit solutions of higher-order linear differential equations with constant coefficients in the preceding sections does not, in general, carry over to linear equations with variable coefficients. We shall see in Chapter 6 that when a linear DE has variable coefficients, the best that we can *usually* expect is to find a solution in the form of an infinite series. However, the type of differential equation that we consider in this section is an exception to this rule; it is a linear equation with variable coefficients whose general solution can always be expressed in terms of powers of x, sines, cosines, and logarithmic functions. Moreover, its method of solution is quite similar to that for constant-coefficient equations in that an auxiliary equation must be solved.

CAUCHY-EULER EQUATION A linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1 x\frac{dy}{dx} + a_0 y = g(x),$$

where the coefficients $a_n, a_{n-1}, \ldots, a_0$ are constants, is known as a **Cauchy-Euler equation.** The observable characteristic of this type of equation is that the degree $k = n, n-1, \ldots, 1, 0$ of the monomial coefficients x^k matches the order k of differentiation $d^k y/dx^k$:

$$\overset{\text{same}}{a_n x^{n} \frac{d^{n}y}{dx^{n}}} + \overset{\text{same}}{a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}} + \cdots.$$

As in Section 4.3, we start the discussion with a detailed examination of the forms of the general solutions of the homogeneous second-order equation

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0.$$

The solution of higher-order equations follows analogously. Also, we can solve the nonhomogeneous equation $ax^2y'' + bxy' + cy = g(x)$ by variation of parameters, once we have determined the complementary function y_c.

NOTE The coefficient ax^2 of y'' is zero at $x = 0$. Hence to guarantee that the fundamental results of Theorem 4.1.1 are applicable to the Cauchy-Euler equation, we confine our attention to finding the general solutions defined on the interval $(0, \infty)$. Solutions on the interval $(-\infty, 0)$ can be obtained by substituting $t = -x$ into the differential equation. See Problems 37 and 38 in Exercises 4.7.

METHOD OF SOLUTION We try a solution of the form $y = x^m$, where m is to be determined. Analogous to what happened when we substituted e^{mx} into a linear equation with constant coefficients, when we substitute x^m, each term of a Cauchy-Euler equation becomes a polynomial in m times x^m, since

$$a_k x^k \frac{d^k y}{dx^k} = a_k x^k m(m-1)(m-2)\cdots(m-k+1)x^{m-k} = a_k m(m-1)(m-2)\cdots(m-k+1)x^m.$$

For example, when we substitute $y = x^m$, the second-order equation becomes

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = am(m-1)x^m + bmx^m + cx^m = (am(m-1) + bm + c)x^m.$$

Thus $y = x^m$ is a solution of the differential equation whenever m is a solution of the **auxiliary equation**

$$am(m-1) + bm + c = 0 \quad \text{or} \quad am^2 + (b-a)m + c = 0. \tag{1}$$

There are three different cases to be considered, depending on whether the roots of this quadratic equation are real and distinct, real and equal, or complex. In the last case the roots appear as a conjugate pair.

CASE I: DISTINCT REAL ROOTS Let m_1 and m_2 denote the real roots of (1) such that $m_1 \neq m_2$. Then $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$ form a fundamental set of solutions. Hence the general solution is

$$y = c_1 x^{m_1} + c_2 x^{m_2}. \tag{2}$$

EXAMPLE 1 Distinct Roots

Solve $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = 0$.

SOLUTION Rather than just memorizing equation (1), it is preferable to assume $y = x^m$ as the solution a few times to understand the origin and the difference between this new form of the auxiliary equation and that obtained in Section 4.3. Differentiate twice,

$$\frac{dy}{dx} = mx^{m-1}, \qquad \frac{d^2y}{dx^2} = m(m-1)x^{m-2},$$

and substitute back into the differential equation:

$$\begin{aligned} x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y &= x^2 \cdot m(m-1)x^{m-2} - 2x \cdot mx^{m-1} - 4x^m \\ &= x^m(m(m-1) - 2m - 4) = x^m(m^2 - 3m - 4) = 0 \end{aligned}$$

if $m^2 - 3m - 4 = 0$. Now $(m+1)(m-4) = 0$ implies $m_1 = -1$, $m_2 = 4$, so $y = c_1x^{-1} + c_2x^4$. ■

CASE II: REPEATED REAL ROOTS If the roots of (1) are repeated (that is, $m_1 = m_2$), then we obtain only one solution—namely, $y = x^{m_1}$. When the roots of the quadratic equation $am^2 + (b-a)m + c = 0$ are equal, the discriminant of the coefficients is necessarily zero. It follows from the quadratic formula that the root must be $m_1 = -(b-a)/2a$.

Now we can construct a second solution y_2, using (5) of Section 4.2. We first write the Cauchy-Euler equation in the standard form

$$\frac{d^2y}{dx^2} + \frac{b}{ax}\frac{dy}{dx} + \frac{c}{ax^2}y = 0$$

and make the identifications $P(x) = b/ax$ and $\int(b/ax)\,dx = (b/a)\ln x$. Thus

$$\begin{aligned} y_2 &= x^{m_1}\int \frac{e^{-(b/a)\ln x}}{x^{2m_1}}\,dx \\ &= x^{m_1}\int x^{-b/a} \cdot x^{-2m_1}\,dx && \leftarrow e^{-(b/a)\ln x} = e^{\ln x^{-b/a}} = x^{-b/a} \\ &= x^{m_1}\int x^{-b/a} \cdot x^{(b-a)/a}\,dx && \leftarrow -2m_1 = (b-a)/a \\ &= x^{m_1}\int \frac{dx}{x} = x^{m_1}\ln x. \end{aligned}$$

The general solution is then

$$y = c_1x^{m_1} + c_2x^{m_1}\ln x. \tag{3}$$

EXAMPLE 2 Repeated Roots

Solve $4x^2\dfrac{d^2y}{dx^2} + 8x\dfrac{dy}{dx} + y = 0$.

SOLUTION The substitution $y = x^m$ yields

$$4x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + y = x^m(4m(m-1) + 8m + 1) = x^m(4m^2 + 4m + 1) = 0$$

when $4m^2 + 4m + 1 = 0$ or $(2m + 1)^2 = 0$. Since $m_1 = -\frac{1}{2}$, the general solution is $y = c_1x^{-1/2} + c_2x^{-1/2}\ln x$. ■

For higher-order equations, if m_1 is a root of multiplicity k, then it can be shown that

$$x^{m_1}, \quad x^{m_1}\ln x, \quad x^{m_1}(\ln x)^2, \ldots, \quad x^{m_1}(\ln x)^{k-1}$$

are k linearly independent solutions. Correspondingly, the general solution of the differential equation must then contain a linear combination of these k solutions.

CASE III: CONJUGATE COMPLEX ROOTS If the roots of (1) are the conjugate pair $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, where α and $\beta > 0$ are real, then a solution is

$$y = C_1x^{\alpha+i\beta} + C_2x^{\alpha-i\beta}.$$

But when the roots of the auxiliary equation are complex, as in the case of equations with constant coefficients, we wish to write the solution in terms of real functions only. We note the identity

$$x^{i\beta} = (e^{\ln x})^{i\beta} = e^{i\beta\ln x},$$

which, by Euler's formula, is the same as

$$x^{i\beta} = \cos(\beta\ln x) + i\sin(\beta\ln x).$$

Similarly, $$x^{-i\beta} = \cos(\beta\ln x) - i\sin(\beta\ln x).$$

Adding and subtracting the last two results yields

$$x^{i\beta} + x^{-i\beta} = 2\cos(\beta\ln x) \qquad \text{and} \qquad x^{i\beta} - x^{-i\beta} = 2i\sin(\beta\ln x),$$

respectively. From the fact that $y = C_1x^{\alpha+i\beta} + C_2x^{\alpha-i\beta}$ is a solution for any values of the constants, we see, in turn, for $C_1 = C_2 = 1$ and $C_1 = 1$, $C_2 = -1$ that

$$y_1 = x^{\alpha}(x^{i\beta} + x^{-i\beta}) \qquad \text{and} \qquad y_2 = x^{\alpha}(x^{i\beta} - x^{-i\beta})$$

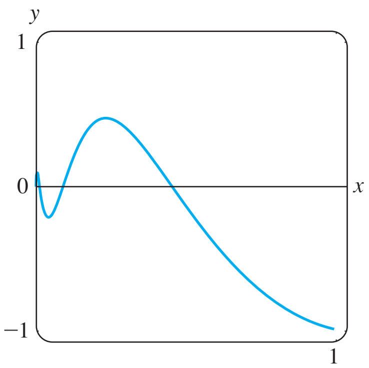
or $$y_1 = 2x^{\alpha}\cos(\beta\ln x) \qquad \text{and} \qquad y_2 = 2ix^{\alpha}\sin(\beta\ln x)$$

are also solutions. Since $W(x^{\alpha}\cos(\beta\ln x), x^{\alpha}\sin(\beta\ln x)) = \beta x^{2\alpha-1} \neq 0$, $\beta > 0$ on the interval $(0, \infty)$, we conclude that

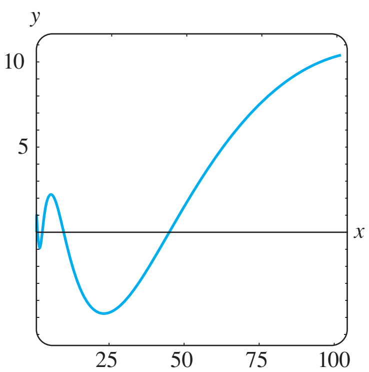
$$y_1 = x^{\alpha}\cos(\beta\ln x) \qquad \text{and} \qquad y_2 = x^{\alpha}\sin(\beta\ln x)$$

constitute a fundamental set of real solutions of the differential equation. Hence the general solution is

$$y = x^{\alpha}[c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x)]. \tag{4}$$



(a) solution for $0 < x \leq 1$



(b) solution for $0 < x \leq 100$

FIGURE 4.7.1 Solution curve of IVP in Example 3

EXAMPLE 3 An Initial-Value Problem

Solve $4x^2y'' + 17y = 0$, $y(1) = -1$, $y'(1) = -\frac{1}{2}$.

SOLUTION The y' term is missing in the given Cauchy-Euler equation; nevertheless, the substitution $y = x^m$ yields

$$4x^2y'' + 17y = x^m(4m(m-1) + 17) = x^m(4m^2 - 4m + 17) = 0$$

when $4m^2 - 4m + 17 = 0$. From the quadratic formula we find that the roots are $m_1 = \frac{1}{2} + 2i$ and $m_2 = \frac{1}{2} - 2i$. With the identifications $\alpha = \frac{1}{2}$ and $\beta = 2$ we see from (4) that the general solution of the differential equation is

$$y = x^{1/2}[c_1\cos(2\ln x) + c_2\sin(2\ln x)].$$

By applying the initial conditions $y(1) = -1$, $y'(1) = -\frac{1}{2}$ to the foregoing solution and using $\ln 1 = 0$, we then find, in turn, that $c_1 = -1$ and $c_2 = 0$. Hence the solution of the initial-value problem is $y = -x^{1/2} \cos(2 \ln x)$. The graph of this function, obtained with the aid of computer software, is given in Figure 4.7.1. The particular solution is seen to be oscillatory and unbounded as $x \to \infty$. ■

The next example illustrates the solution of a third-order Cauchy-Euler equation.

EXAMPLE 4 Third-Order Equation

Solve $x^3 \dfrac{d^3y}{dx^3} + 5x^2 \dfrac{d^2y}{dx^2} + 7x \dfrac{dy}{dx} + 8y = 0$.

SOLUTION The first three derivatives of $y = x^m$ are

$$\frac{dy}{dx} = mx^{m-1}, \qquad \frac{d^2y}{dx^2} = m(m-1)x^{m-2}, \qquad \frac{d^3y}{dx^3} = m(m-1)(m-2)x^{m-3},$$

so the given differential equation becomes

$$\begin{aligned} x^3 \frac{d^3y}{dx^3} + 5x^2 \frac{d^2y}{dx^2} + 7x \frac{dy}{dx} + 8y &= x^3 m(m-1)(m-2)x^{m-3} + 5x^2 m(m-1)x^{m-2} + 7xmx^{m-1} + 8x^m \\ &= x^m(m(m-1)(m-2) + 5m(m-1) + 7m + 8) \\ &= x^m(m^3 + 2m^2 + 4m + 8) = x^m(m+2)(m^2+4) = 0. \end{aligned}$$

In this case we see that $y = x^m$ will be a solution of the differential equation for $m_1 = -2$, $m_2 = 2i$, and $m_3 = -2i$. Hence the general solution is $y = c_1x^{-2} + c_2 \cos(2 \ln x) + c_3 \sin(2 \ln x)$. ■

The method of undetermined coefficients described in Sections 4.5 and 4.6 does not carry over, *in general,* to linear differential equations with variable coefficients. Consequently, in our next example the method of variation of parameters is employed.

EXAMPLE 5 Variation of Parameters

Solve $x^2y'' - 3xy' + 3y = 2x^4e^x$.

SOLUTION Since the equation is nonhomogeneous, we first solve the associated homogeneous equation. From the auxiliary equation $(m-1)(m-3) = 0$ we find $y_c = c_1x + c_2x^3$. Now before using variation of parameters to find a particular solution $y_p = u_1y_1 + u_2y_2$, recall that the formulas $u_1' = W_1/W$ and $u_2' = W_2/W$, where W_1, W_2, and W are the determinants defined on page 158, were derived under the assumption that the differential equation has been put into the standard form $y'' + P(x)y' + Q(x)y = f(x)$. Therefore we divide the given equation by x^2, and from

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x$$

we make the identification $f(x) = 2x^2e^x$. Now with $y_1 = x$, $y_2 = x^3$, and

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3, \quad W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = -2x^5e^x, \quad W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix} = 2x^3e^x,$$

we find $\qquad u_1' = -\dfrac{2x^5e^x}{2x^3} = -x^2e^x \qquad$ and $\qquad u_2' = \dfrac{2x^3e^x}{2x^3} = e^x.$

The integral of the last function is immediate, but in the case of u_1' we integrate by parts twice. The results are $u_1 = -x^2e^x + 2xe^x - 2e^x$ and $u_2 = e^x$. Hence $y_p = u_1y_1 + u_2y_2$ is

$$y_p = (-x^2e^x + 2xe^x - 2e^x)x + e^xx^3 = 2x^2e^x - 2xe^x.$$

Finally, $$y = y_c + y_p = c_1x + c_2x^3 + 2x^2e^x - 2xe^x.$$ ■

REDUCTION TO CONSTANT COEFFICIENTS The similarities between the forms of solutions of Cauchy-Euler equations and solutions of linear equations with constant coefficients are not just a coincidence. For example, when the roots of the auxiliary equations for $ay'' + by' + cy = 0$ and $ax^2y'' + bxy' + cy = 0$ are distinct and real, the respective general solutions are

$$y = c_1e^{m_1x} + c_2e^{m_2x} \quad \text{and} \quad y = c_1x^{m_1} + c_2x^{m_2}, \quad x > 0. \tag{5}$$

In view of the identity $e^{\ln x} = x$, $x > 0$, the second solution given in (5) can be expressed in the same form as the first solution:

$$y = c_1e^{m_1 \ln x} + c_2e^{m_2 \ln x} = c_1e^{m_1t} + c_2e^{m_2t},$$

where $t = \ln x$. This last result illustrates the fact that any Cauchy-Euler equation can *always* be rewritten as a linear differential equation with constant coefficients by means of the substitution $x = e^t$. The idea is to solve the new differential equation in terms of the variable t, using the methods of the previous sections, and, once the general solution is obtained, resubstitute $t = \ln x$. This method, illustrated in the last example, requires the use of the Chain Rule of differentiation.

EXAMPLE 6 Changing to Constant Coefficients

Solve $x^2y'' - xy' + y = \ln x$.

SOLUTION With the substitution $x = e^t$ or $t = \ln x$, it follows that

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt} \qquad \leftarrow \text{Chain Rule}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\left(-\frac{1}{x^2}\right) \qquad \leftarrow \text{Product Rule and Chain Rule}$$

$$= \frac{1}{x}\left(\frac{d^2y}{dt^2}\frac{1}{x}\right) + \frac{dy}{dt}\left(-\frac{1}{x^2}\right) = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

Substituting in the given differential equation and simplifying yields

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t.$$

Since this last equation has constant coefficients, its auxiliary equation is $m^2 - 2m + 1 = 0$, or $(m - 1)^2 = 0$. Thus we obtain $y_c = c_1e^t + c_2te^t$.

By undetermined coefficients we try a particular solution of the form $y_p = A + Bt$. This assumption leads to $-2B + A + Bt = t$, so $A = 2$ and $B = 1$. Using $y = y_c + y_p$, we get

$$y = c_1e^t + c_2te^t + 2 + t,$$

so the general solution of the original differential equation on the interval $(0, \infty)$ is $y = c_1x + c_2x \ln x + 2 + \ln x$. ■

EXERCISES 4.7

Answers to selected odd-numbered problems begin on page ANS-5.

In Problems 1–18 solve the given differential equation.

1. $x^2y'' - 2y = 0$

2. $4x^2y'' + y = 0$

3. $xy'' + y' = 0$

4. $xy'' - 3y' = 0$

5. $x^2y'' + xy' + 4y = 0$

6. $x^2y'' + 5xy' + 3y = 0$

7. $x^2y'' - 3xy' - 2y = 0$

8. $x^2y'' + 3xy' - 4y = 0$

9. $25x^2y'' + 25xy' + y = 0$

10. $4x^2y'' + 4xy' - y = 0$

11. $x^2y'' + 5xy' + 4y = 0$

12. $x^2y'' + 8xy' + 6y = 0$

13. $3x^2y'' + 6xy' + y = 0$

14. $x^2y'' - 7xy' + 41y = 0$

15. $x^3y''' - 6y = 0$

16. $x^3y''' + xy' - y = 0$

17. $xy^{(4)} + 6y''' = 0$

18. $x^4y^{(4)} + 6x^3y''' + 9x^2y'' + 3xy' + y = 0$

In Problems 19–24 solve the given differential equation by variation of parameters.

19. $xy'' - 4y' = x^4$

20. $2x^2y'' + 5xy' + y = x^2 - x$

21. $x^2y'' - xy' + y = 2x$

22. $x^2y'' - 2xy' + 2y = x^4e^x$

23. $x^2y'' + xy' - y = \ln x$

24. $x^2y'' + xy' - y = \dfrac{1}{x+1}$

In Problems 25–30 solve the given initial-value problem. Use a graphing utility to graph the solution curve.

25. $x^2y'' + 3xy' = 0, \quad y(1) = 0, y'(1) = 4$

26. $x^2y'' - 5xy' + 8y = 0, \quad y(2) = 32, y'(2) = 0$

27. $x^2y'' + xy' + y = 0, \quad y(1) = 1, y'(1) = 2$

28. $x^2y'' - 3xy' + 4y = 0, \quad y(1) = 5, y'(1) = 3$

29. $xy'' + y' = x, \quad y(1) = 1, y'(1) = -\frac{1}{2}$

30. $x^2y'' - 5xy' + 8y = 8x^6, \quad y\left(\frac{1}{2}\right) = 0, y'\left(\frac{1}{2}\right) = 0$

In Problems 31–36 use the substitution $x = e^t$ to transform the given Cauchy-Euler equation to a differential equation with constant coefficients. Solve the original equation by solving the new equation using the procedures in Sections 4.3–4.5.

31. $x^2y'' + 9xy' - 20y = 0$

32. $x^2y'' - 9xy' + 25y = 0$

33. $x^2y'' + 10xy' + 8y = x^2$

34. $x^2y'' - 4xy' + 6y = \ln x^2$

35. $x^2y'' - 3xy' + 13y = 4 + 3x$

36. $x^3y''' - 3x^2y'' + 6xy' - 6y = 3 + \ln x^3$

In Problems 37 and 38 solve the given initial-value problem on the interval $(-\infty, 0)$.

37. $4x^2y'' + y = 0, \quad y(-1) = 2, y'(-1) = 4$

38. $x^2y'' - 4xy' + 6y = 0, \quad y(-2) = 8, y'(-2) = 0$

Discussion Problems

39. How would you use the method of this section to solve

$$(x + 2)^2y'' + (x + 2)y' + y = 0?$$

Carry out your ideas. State an interval over which the solution is defined.

40. Can a Cauchy-Euler differential equation of lowest order with real coefficients be found if it is known that 2 and $1 - i$ are roots of its auxiliary equation? Carry out your ideas.

41. The initial-conditions $y(0) = y_0$, $y'(0) = y_1$ apply to each of the following differential equations:

$$x^2y'' = 0,$$
$$x^2y'' - 2xy' + 2y = 0,$$
$$x^2y'' - 4xy' + 6y = 0.$$

For what values of y_0 and y_1 does each initial-value problem have a solution?

42. What are the x-intercepts of the solution curve shown in Figure 4.7.1? How many x-intercepts are there for $0 < x < \frac{1}{2}$?

Computer Lab Assignments

In Problems 43–46 solve the given differential equation by using a CAS to find the (approximate) roots of the auxiliary equation.

43. $2x^3y''' - 10.98x^2y'' + 8.5xy' + 1.3y = 0$

44. $x^3y''' + 4x^2y'' + 5xy' - 9y = 0$

45. $x^4y^{(4)} + 6x^3y''' + 3x^2y'' - 3xy' + 4y = 0$

46. $x^4y^{(4)} - 6x^3y''' + 33x^2y'' - 105xy' + 169y = 0$

47. Solve $x^3y''' - x^2y'' - 2xy' + 6y = x^2$ by variation of parameters. Use a CAS as an aid in computing roots of the auxiliary equation and the determinants given in (10) of Section 4.6.

4.8 SOLVING SYSTEMS OF LINEAR DEs BY ELIMINATION

REVIEW MATERIAL

- Because the method of systematic elimination uncouples a system into distinct linear ODEs in each dependent variable, this section gives you an opportunity to practice what you learned in Sections 4.3, 4.4 (or 4.5), and 4.6.

INTRODUCTION Simultaneous ordinary differential equations involve two or more equations that contain derivatives of two or more dependent variables—the unknown functions—with respect to a single independent variable. The method of **systematic elimination** for solving systems of differential equations with constant coefficients is based on the algebraic principle of elimination of variables. We shall see that the analogue of *multiplying* an algebraic equation by a constant is *operating* on an ODE with some combination of derivatives.

SYSTEMATIC ELIMINATION The elimination of an unknown in a system of linear differential equations is expedited by rewriting each equation in the system in differential operator notation. Recall from Section 4.1 that a single linear equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(t),$$

where the a_i, $i = 0, 1, \ldots, n$ are constants, can be written as

$$(a_n D^n + a_{n-1} D^{(n-1)} + \cdots + a_1 D + a_0) y = g(t).$$

If the nth-order differential operator $a_n D^n + a_{n-1} D^{(n-1)} + \cdots + a_1 D + a_0$ factors into differential operators of lower order, then the factors commute. Now, for example, to rewrite the system

$$\begin{aligned} x'' + 2x' + y'' &= x + 3y + \sin t \\ x' + y' &= -4x + 2y + e^{-t} \end{aligned}$$

in terms of the operator D, we first bring all terms involving the dependent variables to one side and group the same variables:

$$\begin{aligned} x'' + 2x' - x + y'' - 3y &= \sin t \\ x' - 4x + y' - 2y &= e^{-t} \end{aligned} \qquad \text{is the same as} \qquad \begin{aligned} (D^2 + 2D - 1)x + (D^2 - 3)y &= \sin t \\ (D - 4)x + (D - 2)y &= e^{-t}. \end{aligned}$$

SOLUTION OF A SYSTEM A **solution** of a system of differential equations is a set of sufficiently differentiable functions $x = \phi_1(t)$, $y = \phi_2(t)$, $z = \phi_3(t)$, and so on that satisfies each equation in the system on some common interval I.

METHOD OF SOLUTION Consider the simple system of linear first-order equations

$$\begin{aligned} \frac{dx}{dt} &= 3y \\ \frac{dy}{dt} &= 2x \end{aligned} \qquad \text{or, equivalently,} \qquad \begin{aligned} Dx - 3y &= 0 \\ 2x - Dy &= 0. \end{aligned} \tag{1}$$

Operating on the first equation in (1) by D while multiplying the second by -3 and then adding eliminates y from the system and gives $D^2x - 6x = 0$. Since the roots of the auxiliary equation of the last DE are $m_1 = \sqrt{6}$ and $m_2 = -\sqrt{6}$, we obtain

$$x(t) = c_1 e^{-\sqrt{6}t} + c_2 e^{\sqrt{6}t}. \tag{2}$$

Multiplying the first equation in (1) by 2 while operating on the second by D and then subtracting gives the differential equation for y, $D^2y - 6y = 0$. It follows immediately that

$$y(t) = c_3e^{-\sqrt{6}t} + c_4e^{\sqrt{6}t}. \tag{3}$$

Now (2) and (3) do not satisfy the system (1) for every choice of c_1, c_2, c_3, and c_4 because the system itself puts a constraint on the number of parameters in a solution that can be chosen arbitrarily. To see this, observe that substituting $x(t)$ and $y(t)$ into the first equation of the original system (1) gives, after simplification,

$$\left(-\sqrt{6}c_1 - 3c_3\right)e^{-\sqrt{6}t} + \left(\sqrt{6}c_2 - 3c_4\right)e^{\sqrt{6}t} = 0.$$

Since the latter expression is to be zero for all values of t, we must have $-\sqrt{6}c_1 - 3c_3 = 0$ and $\sqrt{6}c_2 - 3c_4 = 0$. These two equations enable us to write c_3 as a multiple of c_1 and c_4 as a multiple of c_2:

$$c_3 = -\frac{\sqrt{6}}{3}c_1 \quad \text{and} \quad c_4 = \frac{\sqrt{6}}{3}c_2. \tag{4}$$

Hence we conclude that a solution of the system must be

$$x(t) = c_1e^{-\sqrt{6}t} + c_2e^{\sqrt{6}t}, \qquad y(t) = -\frac{\sqrt{6}}{3}c_1e^{-\sqrt{6}t} + \frac{\sqrt{6}}{3}c_2e^{\sqrt{6}t}.$$

You are urged to substitute (2) and (3) into the second equation of (1) and verify that the same relationship (4) holds between the constants.

EXAMPLE 1 Solution by Elimination

Solve

$$\begin{aligned} Dx + (D + 2)y &= 0 \\ (D - 3)x - \quad 2y &= 0. \end{aligned} \tag{5}$$

SOLUTION Operating on the first equation by $D - 3$ and on the second by D and then subtracting eliminates x from the system. It follows that the differential equation for y is

$$[(D - 3)(D + 2) + 2D]y = 0 \quad \text{or} \quad (D^2 + D - 6)y = 0.$$

Since the characteristic equation of this last differential equation is $m^2 + m - 6 = (m - 2)(m + 3) = 0$, we obtain the solution

$$y(t) = c_1e^{2t} + c_2e^{-3t}. \tag{6}$$

Eliminating y in a similar manner yields $(D^2 + D - 6)x = 0$, from which we find

$$x(t) = c_3e^{2t} + c_4e^{-3t}. \tag{7}$$

As we noted in the foregoing discussion, a solution of (5) does not contain four independent constants. Substituting (6) and (7) into the first equation of (5) gives

$$(4c_1 + 2c_3)e^{2t} + (-c_2 - 3c_4)e^{-3t} = 0.$$

From $4c_1 + 2c_3 = 0$ and $-c_2 - 3c_4 = 0$ we get $c_3 = -2c_1$ and $c_4 = -\frac{1}{3}c_2$. Accordingly, a solution of the system is

$$x(t) = -2c_1e^{2t} - \frac{1}{3}c_2e^{-3t}, \qquad y(t) = c_1e^{2t} + c_2e^{-3t}.$$ ■

Because we could just as easily solve for c_3 and c_4 in terms of c_1 and c_2, the solution in Example 1 can be written in the alternative form

$$x(t) = c_3e^{2t} + c_4e^{-3t}, \qquad y(t) = -\frac{1}{2}c_3e^{2t} - 3c_4e^{-3t}.$$

This might save you some time.

It sometimes pays to keep one's eyes open when solving systems. Had we solved for x first in Example 1, then y could be found, along with the relationship between the constants, using the last equation in the system (5). You should verify that substituting $x(t)$ into $y = \frac{1}{2}(Dx - 3x)$ yields $y = -\frac{1}{2}c_3e^{2t} - 3c_4e^{-3t}$. Also note in the initial discussion that the relationship given in (4) and the solution $y(t)$ of (1) could also have been obtained by using $x(t)$ in (2) and the first equation of (1) in the form

$$y = \tfrac{1}{3}Dx = -\tfrac{1}{3}\sqrt{6}c_1e^{-\sqrt{6}t} + \tfrac{1}{3}\sqrt{6}c_2e^{\sqrt{6}t}.$$

EXAMPLE 2 Solution by Elimination

Solve

$$\begin{aligned} x' - 4x + y'' &= t^2 \\ x' + x + y' &= 0. \end{aligned} \tag{8}$$

SOLUTION First we write the system in differential operator notation:

$$\begin{aligned} (D - 4)x + D^2y &= t^2 \\ (D + 1)x + Dy &= 0. \end{aligned} \tag{9}$$

Then, by eliminating x, we obtain

$$[(D + 1)D^2 - (D - 4)D]y = (D + 1)t^2 - (D - 4)0$$

or

$$(D^3 + 4D)y = t^2 + 2t.$$

Since the roots of the auxiliary equation $m(m^2 + 4) = 0$ are $m_1 = 0$, $m_2 = 2i$, and $m_3 = -2i$, the complementary function is $y_c = c_1 + c_2 \cos 2t + c_3 \sin 2t$. To determine the particular solution y_p, we use undetermined coefficients by assuming that $y_p = At^3 + Bt^2 + Ct$. Therefore $y_p' = 3At^2 + 2Bt + C$, $y_p'' = 6At + 2B$, $y_p''' = 6A$,

$$y_p''' + 4y_p' = 12At^2 + 8Bt + 6A + 4C = t^2 + 2t.$$

The last equality implies that $12A = 1$, $8B = 2$, and $6A + 4C = 0$; hence $A = \frac{1}{12}$, $B = \frac{1}{4}$, and $C = -\frac{1}{8}$. Thus

$$y = y_c + y_p = c_1 + c_2 \cos 2t + c_3 \sin 2t + \frac{1}{12}t^3 + \frac{1}{4}t^2 - \frac{1}{8}t. \tag{10}$$

Eliminating y from the system (9) leads to

$$[(D - 4) - D(D + 1)]x = t^2 \qquad \text{or} \qquad (D^2 + 4)x = -t^2.$$

It should be obvious that $x_c = c_4 \cos 2t + c_5 \sin 2t$ and that undetermined coefficients can be applied to obtain a particular solution of the form $x_p = At^2 + Bt + C$. In this case the usual differentiations and algebra yield $x_p = -\frac{1}{4}t^2 + \frac{1}{8}$, and so

$$x = x_c + x_p = c_4 \cos 2t + c_5 \sin 2t - \frac{1}{4}t^2 + \frac{1}{8}. \tag{11}$$

Now c_4 and c_5 can be expressed in terms of c_2 and c_3 by substituting (10) and (11) into either equation of (8). By using the second equation, we find, after combining terms,

$$(c_5 - 2c_4 - 2c_2) \sin 2t + (2c_5 + c_4 + 2c_3) \cos 2t = 0,$$

so $c_5 - 2c_4 - 2c_2 = 0$ and $2c_5 + c_4 + 2c_3 = 0$. Solving for c_4 and c_5 in terms of c_2 and c_3 gives $c_4 = -\frac{1}{5}(4c_2 + 2c_3)$ and $c_5 = \frac{1}{5}(2c_2 - 4c_3)$. Finally, a solution of (8) is found to be

$$x(t) = -\frac{1}{5}(4c_2 + 2c_3) \cos 2t + \frac{1}{5}(2c_2 - 4c_3) \sin 2t - \frac{1}{4}t^2 + \frac{1}{8},$$

$$y(t) = c_1 + c_2 \cos 2t + c_3 \sin 2t + \frac{1}{12}t^3 + \frac{1}{4}t^2 - \frac{1}{8}t.$$

■

EXAMPLE 3 A Mixture Problem Revisited

In (3) of Section 3.3 we saw that the system of linear first-order differential equations

$$\frac{dx_1}{dt} = -\frac{2}{25}x_1 + \frac{1}{50}x_2$$
$$\frac{dx_2}{dt} = \frac{2}{25}x_1 - \frac{2}{25}x_2$$

is a model for the number of pounds of salt $x_1(t)$ and $x_2(t)$ in brine mixtures in tanks A and B, respectively, shown in Figure 3.3.1. At that time we were not able to solve the system. But now, in terms of differential operators, the foregoing system can be written as

$$\left(D + \frac{2}{25}\right)x_1 - \frac{1}{50}x_2 = 0$$
$$-\frac{2}{25}x_1 + \left(D + \frac{2}{25}\right)x_2 = 0.$$

Operating on the first equation by $D + \frac{2}{25}$, multiplying the second equation by $\frac{1}{50}$, adding, and then simplifying gives $(625D^2 + 100D + 3)x_1 = 0$. From the auxiliary equation

$$625m^2 + 100m + 3 = (25m + 1)(25m + 3) = 0$$

we see immediately that $x_1(t) = c_1e^{-t/25} + c_2e^{-3t/25}$. We can now obtain $x_2(t)$ by using the first DE of the system in the form $x_2 = 50(D + \frac{2}{25})x_1$. In this manner we find the solution of the system to be

$$x_1(t) = c_1e^{-t/25} + c_2e^{-3t/25}, \qquad x_2(t) = 2c_1e^{-t/25} - 2c_2e^{-3t/25}.$$

In the original discussion on page 107 we assumed that the initial conditions were $x_1(0) = 25$ and $x_2(0) = 0$. Applying these conditions to the solution yields $c_1 + c_2 = 25$ and $2c_1 - 2c_2 = 0$. Solving these equations simultaneously gives $c_1 = c_2 = \frac{25}{2}$. Finally, a solution of the initial-value problem is

$$x_1(t) = \frac{25}{2}e^{-t/25} + \frac{25}{2}e^{-3t/25}, \qquad x_2(t) = 25e^{-t/25} - 25e^{-3t/25}.$$

The graphs of both of these equations are given in Figure 4.8.1. Consistent with the fact that pure water is being pumped into tank A we see in the figure that $x_1(t) \to 0$ and $x_2(t) \to 0$ as $t \to \infty$. ■

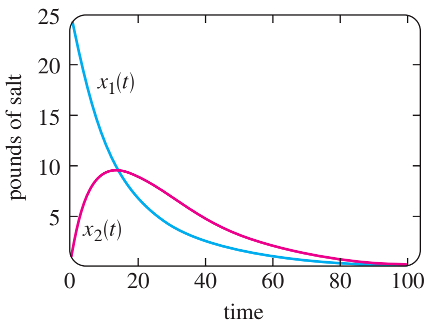


FIGURE 4.8.1 Pounds of salt in tanks A and B

EXERCISES 4.8

Answers to selected odd-numbered problems begin on page ANS-6.

In Problems 1–20 solve the given system of differential equations by systematic elimination.

1. $\dfrac{dx}{dt} = 2x - y$, $\dfrac{dy}{dt} = x$

2. $\dfrac{dx}{dt} = 4x + 7y$, $\dfrac{dy}{dt} = x - 2y$

3. $\dfrac{dx}{dt} = -y + t$, $\dfrac{dy}{dt} = x - t$

4. $\dfrac{dx}{dt} - 4y = 1$, $\dfrac{dy}{dt} + x = 2$

5. $(D^2 + 5)x - 2y = 0$
$-2x + (D^2 + 2)y = 0$

6. $(D + 1)x + (D - 1)y = 2$
$3x + (D + 2)y = -1$

7. $\dfrac{d^2x}{dt^2} = 4y + e^t$, $\dfrac{d^2y}{dt^2} = 4x - e^t$

8. $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} = -5x$, $\dfrac{dx}{dt} + \dfrac{dy}{dt} = -x + 4y$

9. $Dx + D^2y = e^{3t}$
$(D + 1)x + (D - 1)y = 4e^{3t}$

10. $$\begin{aligned} D^2x - \quad Dy &= t \\ (D+3)x + (D+3)y &= 2 \end{aligned}$$

11. $$\begin{aligned} (D^2 - 1)x - \quad y &= 0 \\ (D-1)x + Dy &= 0 \end{aligned}$$

12. $$\begin{aligned} (2D^2 - D - 1)x - (2D+1)y &= 1 \\ (D-1)x + \quad Dy &= -1 \end{aligned}$$

13. $$\begin{aligned} 2\frac{dx}{dt} - 5x + \frac{dy}{dt} &= e^t \\ \frac{dx}{dt} - \quad x + \frac{dy}{dt} &= 5e^t \end{aligned}$$

14. $$\begin{aligned} \frac{dx}{dt} + \frac{dy}{dt} \qquad &= e^t \\ -\frac{d^2x}{dt^2} + \frac{dx}{dt} + x + y &= 0 \end{aligned}$$

15. $$\begin{aligned} (D-1)x + (D^2+1)y &= 1 \\ (D^2-1)x + \quad (D+1)y &= 2 \end{aligned}$$

16. $$\begin{aligned} D^2x - 2(D^2 + D)y &= \sin t \\ x + \qquad Dy &= 0 \end{aligned}$$

17. $$\begin{aligned} Dx &= y \\ Dy &= z \\ Dz &= x \end{aligned}$$

18. $$\begin{aligned} Dx + \qquad\qquad z &= e^t \\ (D-1)x + Dy + Dz &= 0 \\ x + 2y + Dz &= e^t \end{aligned}$$

19. $$\begin{aligned} \frac{dx}{dt} &= 6y \\ \frac{dy}{dt} &= x + z \\ \frac{dz}{dt} &= x + y \end{aligned}$$

20. $$\begin{aligned} \frac{dx}{dt} &= -x + z \\ \frac{dy}{dt} &= -y + z \\ \frac{dz}{dt} &= -x + y \end{aligned}$$

In Problems 21 and 22 solve the given initial-value problem.

21. $$\begin{aligned} \frac{dx}{dt} &= -5x - y \\ \frac{dy}{dt} &= 4x - y \end{aligned}$$
$x(1) = 0,\ y(1) = 1$

22. $$\begin{aligned} \frac{dx}{dt} &= y - 1 \\ \frac{dy}{dt} &= -3x + 2y \end{aligned}$$
$x(0) = 0,\ y(0) = 0$

Mathematical Models

23. Projectile Motion A projectile shot from a gun has weight $w = mg$ and velocity $\mathbf{v}$ tangent to its path of motion. Ignoring air resistance and all other forces acting on the projectile except its weight, determine a system of differential equations that describes its path of motion. See Figure 4.8.2. Solve the system. [*Hint*: Use Newton's second law of motion in the x and y directions.]

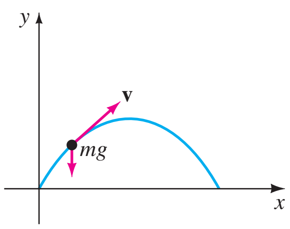


FIGURE 4.8.2 Path of projectile in Problem 23

24. Projectile Motion with Air Resistance Determine a system of differential equations that describes the path of motion in Problem 23 if air resistance is a retarding force $\mathbf{k}$ (of magnitude k) acting tangent to the path of the projectile but opposite to its motion. See Figure 4.8.3. Solve the system. [*Hint*: $\mathbf{k}$ is a multiple of velocity, say, $c\mathbf{v}$.]

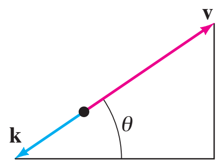


FIGURE 4.8.3 Forces in Problem 24

Discussion Problems

25. Examine and discuss the following system:

$$\begin{aligned} Dx \qquad - 2Dy &= t^2 \\ (D+1)x - 2(D+1)y &= 1. \end{aligned}$$

Computer Lab Assignments

26. Reexamine Figure 4.8.1 in Example 3. Then use a root-finding application to determine when tank B contains more salt than tank A.

27. (a) Reread Problem 8 of Exercises 3.3. In that problem you were asked to show that the system of differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{1}{50}x_1 \\ \frac{dx_2}{dt} &= \frac{1}{50}x_1 - \frac{2}{75}x_2 \\ \frac{dx_3}{dt} &= \frac{2}{75}x_2 - \frac{1}{25}x_3 \end{aligned}$$

is a model for the amounts of salt in the connected mixing tanks A, B, and C shown in Figure 3.3.7. Solve the system subject to $x_1(0) = 15$, $x_2(t) = 10$, $x_3(t) = 5$.

(b) Use a CAS to graph $x_1(t)$, $x_2(t)$, and $x_3(t)$ in the same coordinate plane (as in Figure 4.8.1) on the interval [0, 200].

(c) Because only pure water is pumped into Tank A, it stands to reason that the salt will eventually be flushed out of all three tanks. Use a root-finding application of a CAS to determine the time when the amount of salt in each tank is less than or equal to 0.5 pound. When will the amounts of salt $x_1(t)$, $x_2(t)$, and $x_3(t)$ be simultaneously less than or equal to 0.5 pound?

4.9 NONLINEAR DIFFERENTIAL EQUATIONS

REVIEW MATERIAL

- Sections 2.2 and 2.5
- Section 4.2
- A review of Taylor series from calculus is also recommended.

INTRODUCTION The difficulties that surround higher-order *nonlinear* differential equations and the few methods that yield analytic solutions are examined next. Two of the solution methods considered in this section employ a change of variable to reduce a second-order DE to a first-order DE. In that sense these methods are analogous to the material in Section 4.2.

SOME DIFFERENCES There are several significant differences between linear and nonlinear differential equations. We saw in Section 4.1 that homogeneous linear equations of order two or higher have the property that a linear combination of solutions is also a solution (Theorem 4.1.2). Nonlinear equations do not possess this property of superposability. See Problems 1 and 18 in Exercises 4.9. We can find general solutions of linear first-order DEs and higher-order equations with constant coefficients. Even when we can solve a nonlinear first-order differential equation in the form of a one-parameter family, this family does not, as a rule, represent a general solution. Stated another way, nonlinear first-order DEs can possess singular solutions, whereas linear equations cannot. But the major difference between linear and nonlinear equations of order two or higher lies in the realm of solvability. Given a linear equation, there is a chance that we can find some form of a solution that we can look at—an explicit solution or perhaps a solution in the form of an infinite series (see Chapter 6). On the other hand, nonlinear higher-order differential equations virtually defy solution by analytical methods. Although this might sound disheartening, there are still things that can be done. As was pointed out at the end of Section 1.3, we can always analyze a nonlinear DE qualitatively and numerically.

Let us make it clear at the outset that nonlinear higher-order differential equations are important—dare we say even more important than linear equations?—because as we fine-tune the mathematical model of, say, a physical system, we also increase the likelihood that this higher-resolution model will be nonlinear.

We begin by illustrating an analytical method that *occasionally* enables us to find explicit/implicit solutions of special kinds of nonlinear second-order differential equations.

REDUCTION OF ORDER Nonlinear second-order differential equations $F(x, y', y'') = 0$, where the dependent variable y is missing, and $F(y, y', y'') = 0$, where the independent variable x is missing, can sometimes be solved by using first-order methods. Each equation can be reduced to a first-order equation by means of the substitution $u = y'$.

The next example illustrates the substitution technique for an equation of the form $F(x, y', y'') = 0$. If $u = y'$, then the differential equation becomes $F(x, u, u') = 0$. If we can solve this last equation for u, we can find y by integration. Note that since we are solving a second-order equation, its solution will contain two arbitrary constants.

EXAMPLE 1 **Dependent Variable *y* Is Missing**

Solve $y'' = 2x(y')^2$.

SOLUTION If we let $u = y'$, then $du/dx = y''$. After substituting, the second-order equation reduces to a first-order equation with separable variables; the independent variable is x and the dependent variable is u:

$$\frac{du}{dx} = 2xu^2 \quad \text{or} \quad \frac{du}{u^2} = 2x\,dx$$

$$\int u^{-2}\,du = \int 2x\,dx$$

$$-u^{-1} = x^2 + c_1^2.$$

The constant of integration is written as c_1^2 for convenience. The reason should be obvious in the next few steps. Because $u^{-1} = 1/y'$, it follows that

$$\frac{dy}{dx} = -\frac{1}{x^2 + c_1^2},$$

and so $$y = -\int \frac{dx}{x^2 + c_1^2} \quad \text{or} \quad y = -\frac{1}{c_1}\tan^{-1}\frac{x}{c_1} + c_2.$$ ■

Next we show how to solve an equation that has the form $F(y, y', y'') = 0$. Once more we let $u = y'$, but because the independent variable x is missing, we use this substitution to transform the differential equation into one in which the independent variable is y and the dependent variable is u. To this end we use the Chain Rule to compute the second derivative of y:

$$y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy}.$$

In this case the first-order equation that we must now solve is

$$F\left(y, u, u\frac{du}{dy}\right) = 0.$$

EXAMPLE 2 Independent Variable *x* Is Missing

Solve $yy'' = (y')^2$.

SOLUTION With the aid of $u = y'$, the Chain Rule shown above, and separation of variables, the given differential equation becomes

$$y\left(u\frac{du}{dy}\right) = u^2 \quad \text{or} \quad \frac{du}{u} = \frac{dy}{y}.$$

Integrating the last equation then yields $\ln|u| = \ln|y| + c_1$, which, in turn, gives $u = c_2 y$, where the constant $\pm e^{c_1}$ has been relabeled as c_2. We now resubstitute $u = dy/dx$, separate variables once again, integrate, and relabel constants a second time:

$$\int \frac{dy}{y} = c_2 \int dx \quad \text{or} \quad \ln|y| = c_2 x + c_3 \quad \text{or} \quad y = c_4 e^{c_2 x}.$$ ■

USE OF TAYLOR SERIES In some instances a solution of a nonlinear initial-value problem, in which the initial conditions are specified at x_0, can be approximated by a Taylor series centered at x_0.

EXAMPLE 3 Taylor Series Solution of an IVP

Let us assume that a solution of the initial-value problem

$$y'' = x + y - y^2, \quad y(0) = -1, \quad y'(0) = 1 \tag{1}$$

exists. If we further assume that the solution $y(x)$ of the problem is analytic at 0, then $y(x)$ possesses a Taylor series expansion centered at 0:

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 + \cdots. \tag{2}$$

Note that the values of the first and second terms in the series (2) are known since those values are the specified initial conditions $y(0) = -1$, $y'(0) = 1$. Moreover, the differential equation itself defines the value of the second derivative at 0: $y''(0) = 0 + y(0) - y(0)^2 = 0 + (-1) - (-1)^2 = -2$. We can then find expressions for the higher derivatives y''', $y^{(4)}$, . . . by calculating the successive derivatives of the differential equation:

$$y'''(x) = \frac{d}{dx}(x + y - y^2) = 1 + y' - 2yy' \tag{3}$$

$$y^{(4)}(x) = \frac{d}{dx}(1 + y' - 2yy') = y'' - 2yy'' - 2(y')^2 \tag{4}$$

$$y^{(5)}(x) = \frac{d}{dx}(y'' - 2yy'' - 2(y')^2) = y''' - 2yy''' - 6y'y'', \tag{5}$$

and so on. Now using $y(0) = -1$ and $y'(0) = 1$, we find from (3) that $y'''(0) = 4$. From the values $y(0) = -1$, $y'(0) = 1$, and $y''(0) = -2$ we find $y^{(4)}(0) = -8$ from (4). With the additional information that $y'''(0) = 4$, we then see from (5) that $y^{(5)}(0) = 24$. Hence from (2) the first six terms of a series solution of the initial-value problem (1) are

$$y(x) = -1 + x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{5}x^5 + \cdots.$$ ■

USE OF A NUMERICAL SOLVER Numerical methods, such as Euler's method or the Runge-Kutta method, are developed solely for first-order differential equations and then are extended to systems of first-order equations. To analyze an nth-order initial-value problem numerically, we express the nth-order ODE as a system of n first-order equations. In brief, here is how it is done for a second-order initial-value problem: First, solve for y''—that is, put the DE into normal form $y'' = f(x, y, y')$—and then let $y' = u$. For example, if we substitute $y' = u$ in

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = u_0, \tag{6}$$

then $y'' = u'$ and $y'(x_0) = u(x_0)$, so the initial-value problem (6) becomes

Solve: $$\begin{cases} y' = u \\ u' = f(x, y, u) \end{cases}$$

Subject to: $y(x_0) = y_0,\ u(x_0) = u_0.$

However, it should be noted that a commercial numerical solver *might not* require* that you supply the system.

*Some numerical solvers require only that a second-order differential equation be expressed in normal form $y'' = f(x, y, y')$. The translation of the single equation into a system of two equations is then built into the computer program, since the first equation of the system is always $y' = u$ and the second equation is $u' = f(x, y, u)$.

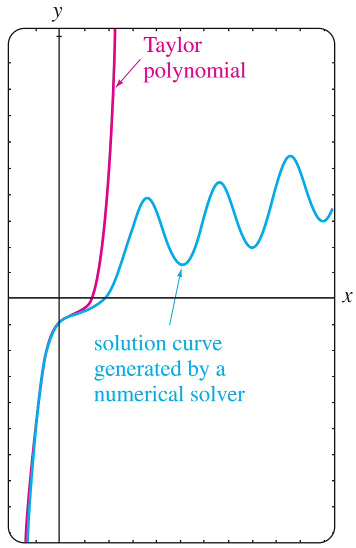


FIGURE 4.9.1 Comparison of two approximate solutions

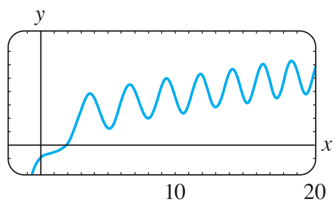


FIGURE 4.9.2 Numerical solution curve for the IVP in (1)

EXAMPLE 4 Graphical Analysis of Example 3

Following the foregoing procedure, we find that the second-order initial-value problem in Example 3 is equivalent to

$$\frac{dy}{dx} = u$$
$$\frac{du}{dx} = x + y - y^2$$

with initial conditions $y(0) = -1$, $u(0) = 1$. With the aid of a numerical solver we get the solution curve shown in blue in Figure 4.9.1. For comparison the graph of the fifth-degree Taylor polynomial $T_5(x) = -1 + x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{5}x^5$ is shown in red. Although we do not know the interval of convergence of the Taylor series obtained in Example 3, the closeness of the two curves in a neighborhood of the origin suggests that the power series may converge on the interval $(-1, 1)$. ■

QUALITATIVE QUESTIONS The blue graph in Figure 4.9.1 raises some questions of a qualitative nature: Is the solution of the original initial-value problem oscillatory as $x \to \infty$? The graph generated by a numerical solver on the larger interval shown in Figure 4.9.2 would seem to *suggest* that the answer is yes. But this single example—or even an assortment of examples—does not answer the basic question as to whether *all* solutions of the differential equation $y'' = x + y - y^2$ are oscillatory in nature. Also, what is happening to the solution curve in Figure 4.9.2 when x is near -1? What is the behavior of solutions of the differential equation as $x \to -\infty$? Are solutions bounded as $x \to \infty$? Questions such as these are not easily answered, in general, for nonlinear second-order differential equations. But certain kinds of second-order equations lend themselves to a systematic qualitative analysis, and these, like their first-order relatives encountered in Section 2.1, are the kind that have no explicit dependence on the independent variable. Second-order ODEs of the form

$$F(y, y', y'') = 0 \qquad \text{or} \qquad \frac{d^2y}{dx^2} = f(y, y'),$$

equations free of the independent variable x, are called **autonomous.** The differential equation in Example 2 is autonomous, and because of the presence of the x term on its right-hand side, the equation in Example 3 is nonautonomous. For an in-depth treatment of the topic of stability of autonomous second-order differential equations and autonomous systems of differential equations, refer to Chapter 10 in *Differential Equations with Boundary-Value Problems.*

EXERCISES 4.9

Answers to selected odd-numbered problems begin on page ANS-6.

In Problems 1 and 2 verify that y_1 and y_2 are solutions of the given differential equation but that $y = c_1y_1 + c_2y_2$ is, in general, not a solution.

1. $(y'')^2 = y^2$; $y_1 = e^x$, $y_2 = \cos x$

2. $yy'' = \frac{1}{2}(y')^2$; $y_1 = 1$, $y_2 = x^2$

In Problems 3–8 solve the given differential equation by using the substitution $u = y'$.

3. $y'' + (y')^2 + 1 = 0$ **4.** $y'' = 1 + (y')^2$

5. $x^2y'' + (y')^2 = 0$ **6.** $(y + 1)y'' = (y')^2$

7. $y'' + 2y(y')^3 = 0$ **8.** $y^2y'' = y'$

9. Consider the initial-value problem

$$y'' + yy' = 0, \quad y(0) = 1, y'(0) = -1.$$

(a) Use the DE and a numerical solver to graph the solution curve.

(b) Find an explicit solution of the IVP. Use a graphing utility to graph this solution.

(c) Find an interval of definition for the solution in part (b).

10. Find two solutions of the initial-value problem

$$(y'')^2 + (y')^2 = 1, \quad y\left(\frac{\pi}{2}\right) = \frac{1}{2}, \quad y'\left(\frac{\pi}{2}\right) = \frac{\sqrt{3}}{2}.$$

Use a numerical solver to graph the solution curves.

In Problems 11 and 12 show that the substitution $u = y'$ leads to a Bernoulli equation. Solve this equation (see Section 2.5).

11. $xy'' = y' + (y')^3$ **12.** $xy'' = y' + x(y')^2$

In Problems 13–16 proceed as in Example 3 and obtain the first six nonzero terms of a Taylor series solution, centered at 0, of the given initial-value problem. Use a numerical solver and a graphing utility to compare the solution curve with the graph of the Taylor polynomial.

13. $y'' = x + y^2, \quad y(0) = 1, y'(0) = 1$

14. $y'' + y^2 = 1, \quad y(0) = 2, y'(0) = 3$

15. $y'' = x^2 + y^2 - 2y', \quad y(0) = 1, y'(0) = 1$

16. $y'' = e^y, \quad y(0) = 0, y'(0) = -1$

17. In calculus the curvature of a curve that is defined by a function $y = f(x)$ is defined as

$$\kappa = \frac{y''}{[1 + (y')^2]^{3/2}}.$$

Find $y = f(x)$ for which $\kappa = 1$. [*Hint*: For simplicity, ignore constants of integration.]

Discussion Problems

18. In Problem 1 we saw that $\cos x$ and e^x were solutions of the nonlinear equation $(y'')^2 - y^2 = 0$. Verify that $\sin x$ and e^{-x} are also solutions. Without attempting to solve the differential equation, discuss how these explicit solutions can be found by using knowledge about linear equations. Without attempting to verify, discuss why the linear combinations $y = c_1e^x + c_2e^{-x} + c_3 \cos x + c_4 \sin x$ and $y = c_2e^{-x} + c_4 \sin x$ are not, in general, solutions, but the two special linear combinations $y = c_1e^x + c_2e^{-x}$ and $y = c_3 \cos x + c_4 \sin x$ *must* satisfy the differential equation.

19. Discuss how the method of reduction of order considered in this section can be applied to the third-order differential equation $y''' = \sqrt{1 + (y'')^2}$. Carry out your ideas and solve the equation.

20. Discuss how to find an alternative two-parameter family of solutions for the nonlinear differential equation $y'' = 2x(y')^2$ in Example 1. [*Hint*: Suppose that $-c_1^2$ is used as the constant of integration instead of $+c_1^2$.]

Mathematical Models

21. Motion in a Force Field A mathematical model for the position $x(t)$ of a body moving rectilinearly on the x-axis in an inverse-square force field is given by

$$\frac{d^2x}{dt^2} = -\frac{k^2}{x^2}.$$

Suppose that at $t = 0$ the body starts from rest from the position $x = x_0$, $x_0 > 0$. Show that the velocity of the body at time t is given by $v^2 = 2k^2(1/x - 1/x_0)$. Use the last expression and a CAS to carry out the integration to express time t in terms of x.

22. A mathematical model for the position $x(t)$ of a moving object is

$$\frac{d^2x}{dt^2} + \sin x = 0.$$

Use a numerical solver to graphically investigate the solutions of the equation subject to $x(0) = 0$, $x'(0) = x_1$, $x_1 \geq 0$. Discuss the motion of the object for $t \geq 0$ and for various choices of x_1. Investigate the equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + \sin x = 0$$

in the same manner. Give a possible physical interpretation of the dx/dt term.

CHAPTER 4 IN REVIEW

Answers to selected odd-numbered problems begin on page ANS-6.

Answer Problems 1–4 without referring back to the text. Fill in the blank or answer true or false.

1. The only solution of the initial-value problem $y'' + x^2y = 0$, $y(0) = 0$, $y'(0) = 0$ is __________.

2. For the method of undetermined coefficients, the assumed form of the particular solution y_p for $y'' - y = 1 + e^x$ is __________.

3. A constant multiple of a solution of a linear differential equation is also a solution. __________

4. If the set consisting of two functions f_1 and f_2 is linearly independent on an interval I, then the Wronskian $W(f_1, f_2) \neq 0$ for all x in I. __________

5. Give an interval over which the set of two functions $f_1(x) = x^2$ and $f_2(x) = x\,|x|$ is linearly independent.

Then give an interval over which the set consisting of f_1 and f_2 is linearly dependent.

6. Without the aid of the Wronskian, determine whether the given set of functions is linearly independent or linearly dependent on the indicated interval.

(a) $f_1(x) = \ln x, f_2(x) = \ln x^2, (0, \infty)$

(b) $f_1(x) = x^n, f_2(x) = x^{n+1}, n = 1, 2, \ldots, (-\infty, \infty)$

(c) $f_1(x) = x, f_2(x) = x + 1, (-\infty, \infty)$

(d) $f_1(x) = \cos\left(x + \dfrac{\pi}{2}\right), f_2(x) = \sin x, (-\infty, \infty)$

(e) $f_1(x) = 0, f_2(x) = x, (-5, 5)$

(f) $f_1(x) = 2, f_2(x) = 2x, (-\infty, \infty)$

(g) $f_1(x) = x^2, f_2(x) = 1 - x^2, f_3(x) = 2 + x^2, (-\infty, \infty)$

(h) $f_1(x) = xe^{x+1}, f_2(x) = (4x - 5)e^x,$
$f_3(x) = xe^x, (-\infty, \infty)$

7. Suppose $m_1 = 3$, $m_2 = -5$, and $m_3 = 1$ are roots of multiplicity one, two, and three, respectively, of an auxiliary equation. Write down the general solution of the corresponding homogeneous linear DE if it is

(a) an equation with constant coefficients,

(b) a Cauchy-Euler equation.

8. Consider the differential equation $ay'' + by' + cy = g(x)$, where a, b, and c are constants. Choose the input functions $g(x)$ for which the method of undetermined coefficients is applicable and the input functions for which the method of variation of parameters is applicable.

(a) $g(x) = e^x \ln x$ **(b)** $g(x) = x^3 \cos x$

(c) $g(x) = \dfrac{\sin x}{e^x}$ **(d)** $g(x) = 2x^{-2}e^x$

(e) $g(x) = \sin^2 x$ **(f)** $g(x) = \dfrac{e^x}{\sin x}$

In Problems 9–24 use the procedures developed in this chapter to find the general solution of each differential equation.

9. $y'' - 2y' - 2y = 0$

10. $2y'' + 2y' + 3y = 0$

11. $y''' + 10y'' + 25y' = 0$

12. $2y''' + 9y'' + 12y' + 5y = 0$

13. $3y''' + 10y'' + 15y' + 4y = 0$

14. $2y^{(4)} + 3y''' + 2y'' + 6y' - 4y = 0$

15. $y'' - 3y' + 5y = 4x^3 - 2x$

16. $y'' - 2y' + y = x^2e^x$

17. $y''' - 5y'' + 6y' = 8 + 2\sin x$

18. $y''' - y'' = 6$

19. $y'' - 2y' + 2y = e^x \tan x$

20. $y'' - y = \dfrac{2e^x}{e^x + e^{-x}}$

21. $6x^2y'' + 5xy' - y = 0$

22. $2x^3y''' + 19x^2y'' + 39xy' + 9y = 0$

23. $x^2y'' - 4xy' + 6y = 2x^4 + x^2$

24. $x^2y'' - xy' + y = x^3$

25. Write down the form of the general solution $y = y_c + y_p$ of the given differential equation in the two cases $\omega \neq \alpha$ and $\omega = \alpha$. Do not determine the coefficients in y_p.

(a) $y'' + \omega^2 y = \sin \alpha x$ **(b)** $y'' - \omega^2 y = e^{\alpha x}$

26. (a) Given that $y = \sin x$ is a solution of

$$y^{(4)} + 2y''' + 11y'' + 2y' + 10y = 0,$$

find the general solution of the DE *without the aid of a calculator or a computer.*

(b) Find a linear second-order differential equation with constant coefficients for which $y_1 = 1$ and $y_2 = e^{-x}$ are solutions of the associated homogeneous equation and $y_p = \frac{1}{2}x^2 - x$ is a particular solution of the nonhomogeneous equation.

27. (a) Write the general solution of the fourth-order DE $y^{(4)} - 2y'' + y = 0$ entirely in terms of hyperbolic functions.

(b) Write down the form of a particular solution of $y^{(4)} - 2y'' + y = \sinh x$.

28. Consider the differential equation

$$x^2y'' - (x^2 + 2x)y' + (x + 2)y = x^3.$$

Verify that $y_1 = x$ is one solution of the associated homogeneous equation. Then show that the method of reduction of order discussed in Section 4.2 leads to a second solution y_2 of the homogeneous equation as well as a particular solution y_p of the nonhomogeneous equation. Form the general solution of the DE on the interval $(0, \infty)$.

In Problems 29–34 solve the given differential equation subject to the indicated conditions.

29. $y'' - 2y' + 2y = 0,\quad y\left(\dfrac{\pi}{2}\right) = 0, y(\pi) = -1$

30. $y'' + 2y' + y = 0,\quad y(-1) = 0, y'(0) = 0$

31. $y'' - y = x + \sin x,\quad y(0) = 2, y'(0) = 3$

32. $y'' + y = \sec^3 x,\quad y(0) = 1, y'(0) = \dfrac{1}{2}$

33. $y'y'' = 4x, \quad y(1) = 5, y'(1) = 2$

34. $2y'' = 3y^2, \quad y(0) = 1, y'(0) = 1$

35. **(a)** Use a CAS as an aid in finding the roots of the auxiliary equation for

$$12y^{(4)} + 64y''' + 59y'' - 23y' - 12y = 0.$$

Give the general solution of the equation.

(b) Solve the DE in part (a) subject to the initial conditions $y(0) = -1$, $y'(0) = 2$, $y''(0) = 5$, $y'''(0) = 0$. Use a CAS as an aid in solving the resulting systems of four equations in four unknowns.

36. Find a member of the family of solutions of $xy'' + y' + \sqrt{x} = 0$ whose graph is tangent to the x-axis at $x = 1$. Use a graphing utility to graph the solution curve.

In Problems 37–40 use systematic elimination to solve the given system.

37.
$$\frac{dx}{dt} + \frac{dy}{dt} = 2x + 2y + 1$$
$$\frac{dx}{dt} + 2\frac{dy}{dt} = y + 3$$

38.
$$\frac{dx}{dt} = 2x + y + t - 2$$
$$\frac{dy}{dt} = 3x + 4y - 4t$$

39.
$$(D - 2)x - y = -e^t$$
$$-3x + (D - 4)y = -7e^t$$

40.
$$(D + 2)x + (D + 1)y = \sin 2t$$
$$5x + (D + 3)y = \cos 2t$$