SEVENTH EDITION

DIFFERENTIAL EQUATIONS

with Boundary-Value Problems

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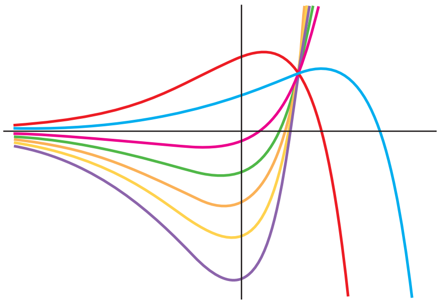

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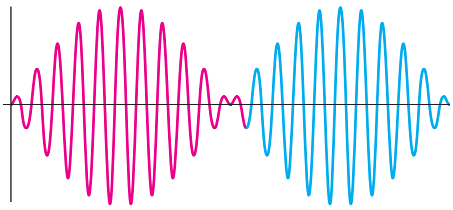

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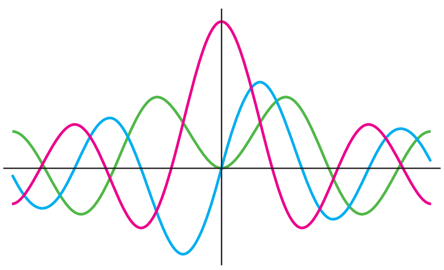

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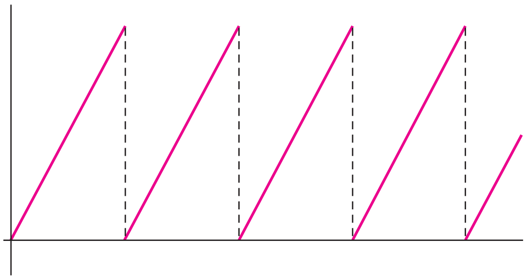

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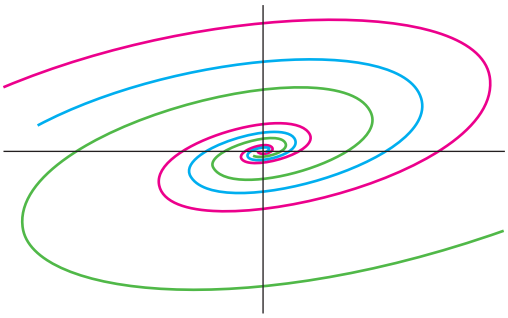

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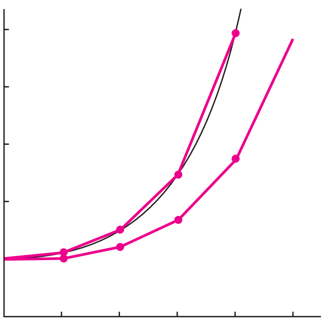

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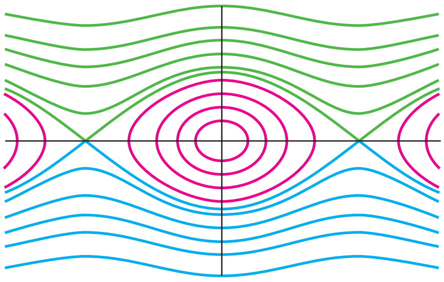

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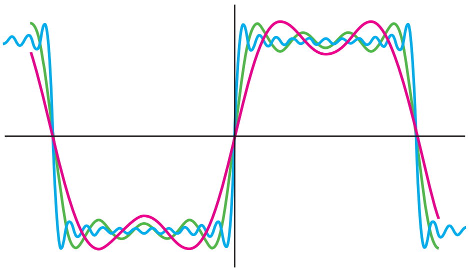

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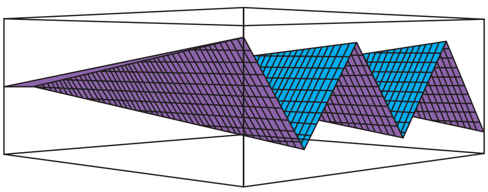
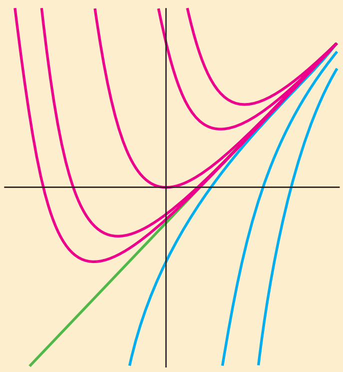


2 FIRST-ORDER DIFFERENTIAL EQUATIONS



The history of mathematics is rife with stories of people who devoted much of their lives to solving equations—algebraic equations at first and then eventually differential equations. In Sections 2.2–2.5 we will study some of the more important analytical methods for solving first-order DEs. However, before we start solving anything, you should be aware of two facts: It is possible for a differential equation to have no solutions, and a differential equation can possess a solution yet there might not exist any analytical method for finding it. In Sections 2.1 and 2.6 we do not solve any DEs but show how to glean information directly from the equation itself. In Section 2.1 we see how the DE yields qualitative information about graphs that enables us to sketch renditions of solutions curves. In Section 2.6 we use the differential equation to construct a numerical procedure for approximating solutions.

2.1 SOLUTION CURVES WITHOUT A SOLUTION

REVIEW MATERIAL

- The first derivative as slope of a tangent line
- The algebraic sign of the first derivative indicates increasing or decreasing

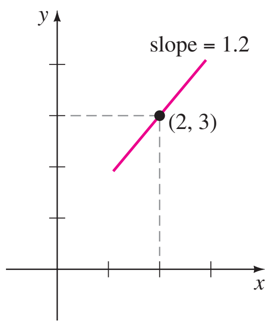
INTRODUCTION Let us imagine for the moment that we have in front of us a first-order differential equation $dy/dx = f(x, y)$, and let us further imagine that we can neither find nor invent a method for solving it analytically. This is not as bad a predicament as one might think, since the differential equation itself can sometimes "tell" us specifics about how its solutions "behave."

We begin our study of first-order differential equations with two ways of analyzing a DE qualitatively. Both these ways enable us to determine, in an approximate sense, what a solution curve must look like without actually solving the equation.

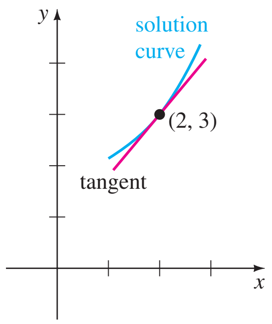
2.1.1 DIRECTION FIELDS

SOME FUNDAMENTAL QUESTIONS We saw in Section 1.2 that whenever $f(x, y)$ and $\partial f/\partial y$ satisfy certain continuity conditions, qualitative questions about existence and uniqueness of solutions can be answered. In this section we shall see that other qualitative questions about properties of solutions—How does a solution behave near a certain point? How does a solution behave as $x \to \infty$?—can often be answered when the function f depends solely on the variable y. We begin, however, with a simple concept from calculus:

A derivative dy/dx of a differentiable function $y = y(x)$ gives slopes of tangent lines at points on its graph.



(a) lineal element at a point



(b) lineal element is tangent to solution curve that passes through the point

FIGURE 2.1.1 A solution curve is tangent to lineal element at (2, 3)

SLOPE Because a solution $y = y(x)$ of a first-order differential equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

is necessarily a differentiable function on its interval I of definition, it must also be continuous on I. Thus the corresponding solution curve on I must have no breaks and must possess a tangent line at each point $(x, y(x))$. The function f in the normal form (1) is called the **slope function** or **rate function.** The slope of the tangent line at $(x, y(x))$ on a solution curve is the value of the first derivative dy/dx at this point, and we know from (1) that this is the value of the slope function $f(x, y(x))$. Now suppose that (x, y) represents any point in a region of the xy-plane over which the function f is defined. The value $f(x, y)$ that the function f assigns to the point represents the slope of a line or, as we shall envision it, a line segment called a **lineal element.** For example, consider the equation $dy/dx = 0.2xy$, where $f(x, y) = 0.2xy$. At, say, the point (2, 3) the slope of a lineal element is $f(2, 3) = 0.2(2)(3) = 1.2$. Figure 2.1.1(a) shows a line segment with slope 1.2 passing though (2, 3). As shown in Figure 2.1.1(b), *if* a solution curve also passes through the point (2, 3), it does so tangent to this line segment; in other words, the lineal element is a miniature tangent line at that point.

DIRECTION FIELD If we systematically evaluate f over a rectangular grid of points in the xy-plane and draw a line element at each point (x, y) of the grid with slope $f(x, y)$, then the collection of all these line elements is called a **direction field** or a **slope field** of the differential equation $dy/dx = f(x, y)$. Visually, the direction field suggests the appearance or shape of a family of solution curves of the differential equation, and consequently, it may be possible to see at a glance certain qualitative aspects of the solutions—regions in the plane, for example, in which a solution exhibits an unusual behavior. A single solution curve that passes through a direction field must follow the flow pattern of the field; it is tangent to a line element when it intersects a point in the grid. Figure 2.1.2 shows a computer-generated direction field of the differential equation $dy/dx = \sin(x + y)$ over a region of the xy-plane. Note how the three solution curves shown in color follow the flow of the field.

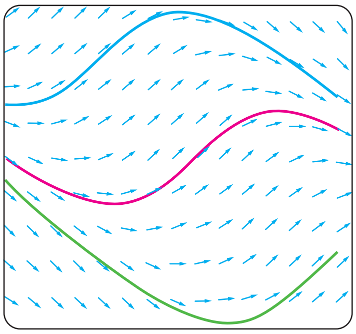
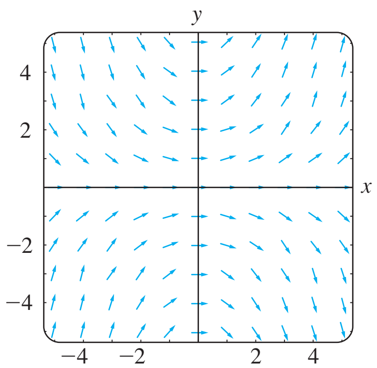


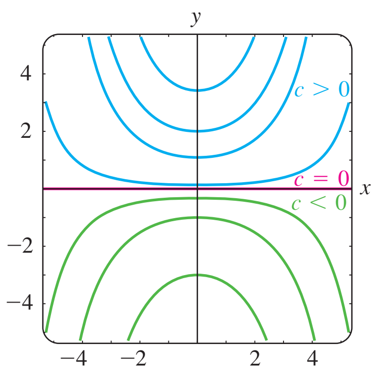
FIGURE 2.1.2 Solution curves following flow of a direction field

EXAMPLE 1 Direction Field

The direction field for the differential equation $dy/dx = 0.2xy$ shown in Figure 2.1.3(a) was obtained by using computer software in which a 5×5 grid of points (mh, nh), m and n integers, was defined by letting $-5 \le m \le 5$, $-5 \le n \le 5$, and $h = 1$. Notice in Figure 2.1.3(a) that at any point along the x-axis ($y = 0$) and the y-axis ($x = 0$), the slopes are $f(x, 0) = 0$ and $f(0, y) = 0$, respectively, so the lineal elements are horizontal. Moreover, observe in the first quadrant that for a fixed value of x the values of $f(x, y) = 0.2xy$ increase as y increases; similarly, for a fixed y the values of $f(x, y) = 0.2xy$ increase as x increases. This means that as both x and y increase, the lineal elements almost become vertical and have positive slope ($f(x, y) = 0.2xy > 0$ for $x > 0$, $y > 0$). In the second quadrant, $|f(x, y)|$ increases as $|x|$ and y increase, so the lineal elements again become almost vertical but this time have negative slope ($f(x, y) = 0.2xy < 0$ for $x < 0$, $y > 0$). Reading from left to right, imagine a solution curve that starts at a point in the second quadrant, moves steeply downward, becomes flat as it passes through the y-axis, and then, as it enters the first quadrant, moves steeply upward—in other words, its shape would be concave upward and similar to a horseshoe. From this it could be surmised that $y \to \infty$ as $x \to \pm\infty$. Now in the third and fourth quadrants, since $f(x, y) = 0.2xy > 0$ and $f(x, y) = 0.2xy < 0$, respectively, the situation is reversed: A solution curve increases and then decreases as we move from left to right. We saw in (1) of Section 1.1 that $y = e^{0.1x^2}$ is an explicit solution of the differential equation $dy/dx = 0.2xy$; you should verify that a one-parameter family of solutions of the same equation is given by $y = ce^{0.1x^2}$. For purposes of comparison with Figure 2.1.3(a) some representative graphs of members of this family are shown in Figure 2.1.3(b). ■



(a) direction field for $dy/dx = 0.2xy$



(b) some solution curves in the family $y = ce^{0.1x^2}$

FIGURE 2.1.3 Direction field and solution curves

EXAMPLE 2 Direction Field

Use a direction field to sketch an approximate solution curve for the initial-value problem $dy/dx = \sin y$, $y(0) = -\frac{3}{2}$.

SOLUTION Before proceeding, recall that from the continuity of $f(x, y) = \sin y$ and $\partial f/\partial y = \cos y$, Theorem 1.2.1 guarantees the existence of a unique solution curve passing through any specified point (x_0, y_0) in the plane. Now we set our computer software again for a 5×5 rectangular region and specify (because of the initial condition) points in that region with vertical and horizontal separation of $\frac{1}{2}$ unit—that is, at points (mh, nh), $h = \frac{1}{2}$, m and n integers such that $-10 \le m \le 10$, $-10 \le n \le 10$. The result is shown in Figure 2.1.4. Because the right-hand side of $dy/dx = \sin y$ is 0 at $y = 0$, and at $y = -\pi$, the lineal elements are horizontal at all points whose second coordinates are $y = 0$ or $y = -\pi$. It makes sense then that a solution curve passing through the initial point $(0, -\frac{3}{2})$ has the shape shown in the figure. ■

INCREASING/DECREASING Interpretation of the derivative dy/dx as a function that gives slope plays the key role in the construction of a direction field. Another telling property of the first derivative will be used next, namely, if $dy/dx > 0$ (or $dy/dx < 0$) for all x in an interval I, then a differentiable function $y = y(x)$ is increasing (or decreasing) on I.

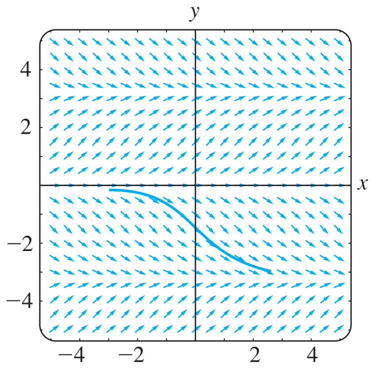


FIGURE 2.1.4 Direction field for Example 2

REMARKS

Sketching a direction field by hand is straightforward but time consuming; it is probably one of those tasks about which an argument can be made for doing it once or twice in a lifetime, but it is overall most efficiently carried out by means of computer software. Before calculators, PCs, and software the **method of isoclines** was used to facilitate sketching a direction field by hand. For the DE $dy/dx = f(x, y)$, any member of the family of curves $f(x, y) = c$, c a constant, is called an **isocline.** Lineal elements drawn through points on a specific isocline, say, $f(x, y) = c_1$ all have the same slope c_1. In Problem 15 in Exercises 2.1 you have your two opportunities to sketch a direction field by hand.

2.1.2 AUTONOMOUS FIRST-ORDER DEs

AUTONOMOUS FIRST-ORDER DEs In Section 1.1 we divided the class of ordinary differential equations into two types: linear and nonlinear. We now consider briefly another kind of classification of ordinary differential equations, a classification that is of particular importance in the qualitative investigation of differential equations. An ordinary differential equation in which the independent variable does not appear explicitly is said to be **autonomous.** If the symbol x denotes the independent variable, then an autonomous first-order differential equation can be written as $f(y, y') = 0$ or in normal form as

$$\frac{dy}{dx} = f(y). \tag{2}$$

We shall assume throughout that the function f in (2) and its derivative f' are continuous functions of y on some interval I. The first-order equations

$$\frac{dy}{dx} = \overset{\overset{f(y)}{\downarrow}}{1 + y^2} \qquad \text{and} \qquad \frac{dy}{dx} = \overset{\overset{f(x,\,y)}{\downarrow}}{0.2xy}$$

are autonomous and nonautonomous, respectively.

Many differential equations encountered in applications or equations that are models of physical laws that do not change over time are autonomous. As we have already seen in Section 1.3, in an applied context, symbols other than y and x are routinely used to represent the dependent and independent variables. For example, if t represents time then inspection of

$$\frac{dA}{dt} = kA, \qquad \frac{dx}{dt} = kx(n + 1 - x), \qquad \frac{dT}{dt} = k(T - T_m), \qquad \frac{dA}{dt} = 6 - \frac{1}{100}A,$$

where k, n, and T_m are constants, shows that each equation is time independent. Indeed, *all* of the first-order differential equations introduced in Section 1.3 are time independent and so are autonomous.

CRITICAL POINTS The zeros of the function f in (2) are of special importance. We say that a real number c is a **critical point** of the autonomous differential equation (2) if it is a zero of f—that is, $f(c) = 0$. A critical point is also called an **equilibrium point** or **stationary point.** Now observe that if we substitute the constant function $y(x) = c$ into (2), then both sides of the equation are zero. This means:

> *If c is a critical point of* (2), *then $y(x) = c$ is a constant solution of the autonomous differential equation.*

A constant solution $y(x) = c$ of (2) is called an **equilibrium solution;** equilibria are the *only* constant solutions of (2).

As was already mentioned, we can tell when a nonconstant solution $y = y(x)$ of (2) is increasing or decreasing by determining the algebraic sign of the derivative dy/dx; in the case of (2) we do this by identifying intervals on the y-axis over which the function $f(y)$ is positive or negative.

EXAMPLE 3 An Autonomous DE

The differential equation

$$\frac{dP}{dt} = P(a - bP),$$

where a and b are positive constants, has the normal form $dP/dt = f(P)$, which is (2) with t and P playing the parts of x and y, respectively, and hence is autonomous. From $f(P) = P(a - bP) = 0$ we see that 0 and a/b are critical points of the equation, so the equilibrium solutions are $P(t) = 0$ and $P(t) = a/b$. By putting the critical points on a vertical line, we divide the line into three intervals defined by $-\infty < P < 0$, $0 < P < a/b$, $a/b < P < \infty$. The arrows on the line shown in Figure 2.1.5 indicate the algebraic sign of $f(P) = P(a - bP)$ on these intervals and whether a nonconstant solution $P(t)$ is increasing or decreasing on an interval. The following table explains the figure.

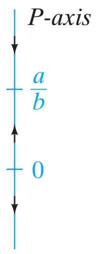


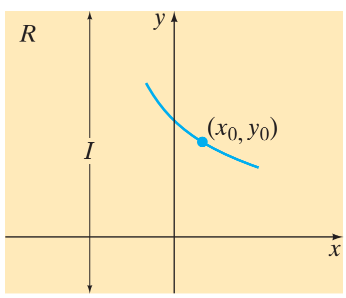
FIGURE 2.1.5 Phase portrait of $dP/dt = P(a - bP)$

Interval	Sign of $f(P)$	$P(t)$	Arrow
$(-\infty, 0)$	minus	decreasing	points down
$(0, a/b)$	plus	increasing	points up
$(a/b, \infty)$	minus	decreasing	points down

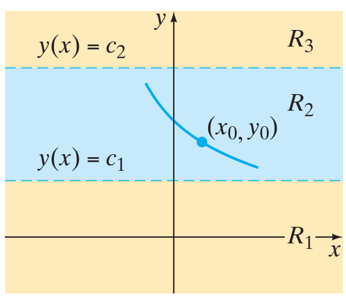
■

Figure 2.1.5 is called a **one-dimensional phase portrait,** or simply **phase portrait,** of the differential equation $dP/dt = P(a - bP)$. The vertical line is called a **phase line.**

SOLUTION CURVES Without solving an autonomous differential equation, we can usually say a great deal about its solution curves. Since the function f in (2) is independent of the variable x, we may consider f defined for $-\infty < x < \infty$ or for $0 \le x < \infty$. Also, since f and its derivative f' are continuous functions of y on some interval I of the y-axis, the fundamental results of Theorem 1.2.1 hold in some horizontal strip or region R in the xy-plane corresponding to I, and so through any point (x_0, y_0) in R there passes only one solution curve of (2). See Figure 2.1.6(a). For the sake of discussion, let us suppose that (2) possesses exactly two critical points c_1 and c_2 and that $c_1 < c_2$. The graphs of the equilibrium solutions $y(x) = c_1$ and $y(x) = c_2$ are horizontal lines, and these lines partition the region R into three subregions R_1, R_2, and R_3, as illustrated in Figure 2.1.6(b). Without proof here are some conclusions that we can draw about a nonconstant solution $y(x)$ of (2):



(a) region R



(b) subregions R_1, R_2, and R_3 of R

FIGURE 2.1.6 Lines $y(x) = c_1$ and $y(x) = c_2$ partition R into three horizontal subregions

- If (x_0, y_0) is in a subregion R_i, $i = 1, 2, 3$, and $y(x)$ is a solution whose graph passes through this point, then $y(x)$ remains in the subregion R_i for all x. As illustrated in Figure 2.1.6(b), the solution $y(x)$ in R_2 is bounded below by c_1 and above by c_2, that is, $c_1 < y(x) < c_2$ for all x. The solution curve stays within R_2 for all x because the graph of a nonconstant solution of (2) cannot cross the graph of either equilibrium solution $y(x) = c_1$ or $y(x) = c_2$. See Problem 33 in Exercises 2.1.
- By continuity of f we must then have either $f(y) > 0$ or $f(y) < 0$ for all x in a subregion R_i, $i = 1, 2, 3$. In other words, $f(y)$ cannot change signs in a subregion. See Problem 33 in Exercises 2.1.

- Since $dy/dx = f(y(x))$ is either positive or negative in a subregion R_i, $i = 1, 2, 3$, a solution $y(x)$ is strictly monotonic—that is, $y(x)$ is either increasing or decreasing in the subregion R_i. Therefore $y(x)$ cannot be oscillatory, nor can it have a relative extremum (maximum or minimum). See Problem 33 in Exercises 2.1.
- If $y(x)$ is *bounded above* by a critical point c_1 (as in subregion R_1 where $y(x) < c_1$ for all x), then the graph of $y(x)$ must approach the graph of the equilibrium solution $y(x) = c_1$ either as $x \to \infty$ or as $x \to -\infty$. If $y(x)$ is *bounded*—that is, bounded above and below by two consecutive critical points (as in subregion R_2 where $c_1 < y(x) < c_2$ for all x)—then the graph of $y(x)$ must approach the graphs of the equilibrium solutions $y(x) = c_1$ and $y(x) = c_2$, one as $x \to \infty$ and the other as $x \to -\infty$. If $y(x)$ is *bounded below* by a critical point (as in subregion R_3 where $c_2 < y(x)$ for all x), then the graph of $y(x)$ must approach the graph of the equilibrium solution $y(x) = c_2$ either as $x \to \infty$ or as $x \to -\infty$. See Problem 34 in Exercises 2.1.

With the foregoing facts in mind, let us reexamine the differential equation in Example 3.

EXAMPLE 4 Example 3 Revisited

The three intervals determined on the P-axis or phase line by the critical points $P = 0$ and $P = a/b$ now correspond in the tP-plane to three subregions defined by:

$$R_1\colon -\infty < P < 0, \qquad R_2\colon 0 < P < a/b, \qquad \text{and} \qquad R_3\colon a/b < P < \infty,$$

where $-\infty < t < \infty$. The phase portrait in Figure 2.1.7 tells us that $P(t)$ is decreasing in R_1, increasing in R_2, and decreasing in R_3. If $P(0) = P_0$ is an initial value, then in R_1, R_2, and R_3 we have, respectively, the following:

(*i*) For $P_0 < 0$, $P(t)$ is bounded above. Since $P(t)$ is decreasing, $P(t)$ decreases without bound for increasing t, and so $P(t) \to 0$ as $t \to -\infty$. This means that the negative t-axis, the graph of the equilibrium solution $P(t) = 0$, is a horizontal asymptote for a solution curve.

(*ii*) For $0 < P_0 < a/b$, $P(t)$ is bounded. Since $P(t)$ is increasing, $P(t) \to a/b$ as $t \to \infty$ and $P(t) \to 0$ as $t \to -\infty$. The graphs of the two equilibrium solutions, $P(t) = 0$ and $P(t) = a/b$, are horizontal lines that are horizontal asymptotes for any solution curve starting in this subregion.

(*iii*) For $P_0 > a/b$, $P(t)$ is bounded below. Since $P(t)$ is decreasing, $P(t) \to a/b$ as $t \to \infty$. The graph of the equilibrium solution $P(t) = a/b$ is a horizontal asymptote for a solution curve.

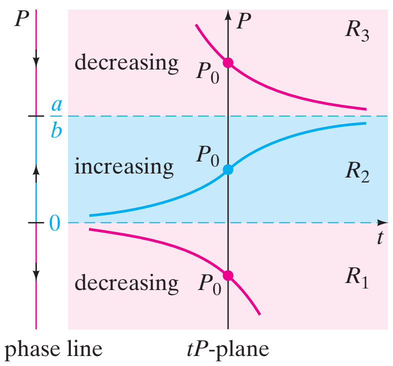


FIGURE 2.1.7 Phase portrait and solution curves in each of the three subregions

In Figure 2.1.7 the phase line is the P-axis in the tP-plane. For clarity the original phase line from Figure 2.1.5 is reproduced to the left of the plane in which the subregions R_1, R_2, and R_3 are shaded. The graphs of the equilibrium solutions $P(t) = a/b$ and $P(t) = 0$ (the t-axis) are shown in the figure as blue dashed lines; the solid graphs represent typical graphs of $P(t)$ illustrating the three cases just discussed. ■

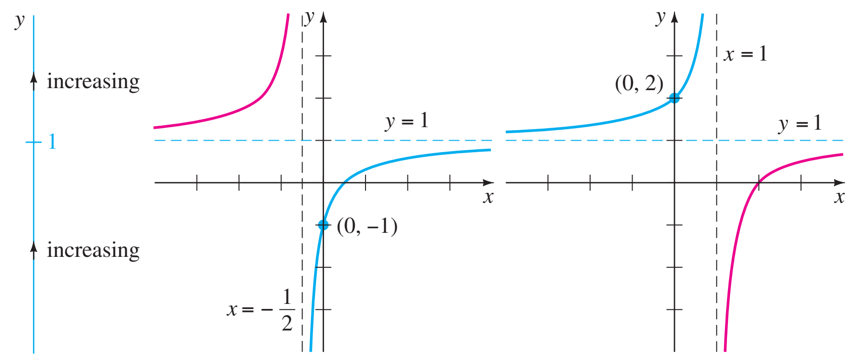
In a subregion such as R_1 in Example 4, where $P(t)$ is decreasing and unbounded below, we must necessarily have $P(t) \to -\infty$. Do *not* interpret this last statement to mean $P(t) \to -\infty$ as $t \to \infty$; we could have $P(t) \to -\infty$ as $t \to T$, where $T > 0$ is a finite number that depends on the initial condition $P(t_0) = P_0$. Thinking in dynamic terms, $P(t)$ could "blow up" in finite time; thinking graphically, $P(t)$ could have a vertical asymptote at $t = T > 0$. A similar remark holds for the subregion R_3.

The differential equation $dy/dx = \sin y$ in Example 2 is autonomous and has an infinite number of critical points, since $\sin y = 0$ at $y = n\pi$, n an integer. Moreover, we now know that because the solution $y(x)$ that passes through $(0, -\frac{3}{2})$ is bounded above and below by two consecutive critical points ($-\pi < y(x) < 0$) and is decreasing ($\sin y < 0$ for $-\pi < y < 0$), the graph of $y(x)$ must approach the graphs of the equilibrium solutions as horizontal asymptotes: $y(x) \to -\pi$ as $x \to \infty$ and $y(x) \to 0$ as $x \to -\infty$.

EXAMPLE 5 Solution Curves of an Autonomous DE

The autonomous equation $dy/dx = (y - 1)^2$ possesses the single critical point 1. From the phase portrait in Figure 2.1.8(a) we conclude that a solution $y(x)$ is an increasing function in the subregions defined by $-\infty < y < 1$ and $1 < y < \infty$, where $-\infty < x < \infty$. For an initial condition $y(0) = y_0 < 1$, a solution $y(x)$ is increasing and bounded above by 1, and so $y(x) \to 1$ as $x \to \infty$; for $y(0) = y_0 > 1$ a solution $y(x)$ is increasing and unbounded.

Now $y(x) = 1 - 1/(x + c)$ is a one-parameter family of solutions of the differential equation. (See Problem 4 in Exercises 2.2) A given initial condition determines a value for c. For the initial conditions, say, $y(0) = -1 < 1$ and $y(0) = 2 > 1$, we find, in turn, that $y(x) = 1 - 1/(x + \frac{1}{2})$, and $y(x) = 1 - 1/(x - 1)$. As shown in Figures 2.1.8(b) and 2.1.8(c), the graph of each of these rational functions possesses



(a) phase line **(b)** xy-plane $y(0) < 1$ **(c)** xy-plane $y(0) > 1$

FIGURE 2.1.8 Behavior of solutions near $y = 1$

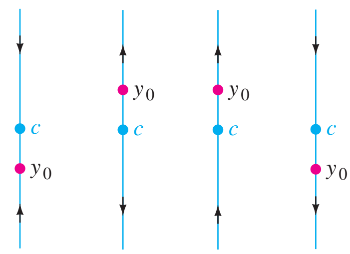
a vertical asymptote. But bear in mind that the solutions of the IVPs

$$\frac{dy}{dx} = (y - 1)^2, \quad y(0) = -1 \qquad \text{and} \qquad \frac{dy}{dx} = (y - 1)^2, \quad y(0) = 2$$

are defined on special intervals. They are, respectively,

$$y(x) = 1 - \frac{1}{x + \frac{1}{2}}, \quad -\tfrac{1}{2} < x < \infty \qquad \text{and} \qquad y(x) = 1 - \frac{1}{x - 1}, \quad -\infty < x < 1.$$

The solution curves are the portions of the graphs in Figures 2.1.8(b) and 2.1.8(c) shown in blue. As predicted by the phase portrait, for the solution curve in Figure 2.1.8(b), $y(x) \to 1$ as $x \to \infty$; for the solution curve in Figure 2.1.8(c), $y(x) \to \infty$ as $x \to 1$ from the left. ■



(a) **(b)** **(c)** **(d)**

FIGURE 2.1.9 Critical point c is an attractor in (a), a repeller in (b), and semi-stable in (c) and (d).

ATTRACTORS AND REPELLERS Suppose that $y(x)$ is a nonconstant solution of the autonomous differential equation given in (1) and that c is a critical point of the DE. There are basically three types of behavior that $y(x)$ can exhibit near c. In Figure 2.1.9 we have placed c on four vertical phase lines. When both arrowheads on either side of the dot labeled c point *toward* c, as in Figure 2.1.9(a), all solutions $y(x)$ of (1) that start from an initial point (x_0, y_0) sufficiently near c exhibit the asymptotic behavior $\lim_{x\to\infty} y(x) = c$. For this reason the critical point c is said to be

asymptotically stable. Using a physical analogy, a solution that starts near c is like a charged particle that, over time, is drawn to a particle of opposite charge, and so c is also referred to as an **attractor.** When both arrowheads on either side of the dot labeled c point *away* from c, as in Figure 2.1.9(b), all solutions $y(x)$ of (1) that start from an initial point (x_0, y_0) move away from c as x increases. In this case the critical point c is said to be **unstable.** An unstable critical point is also called a **repeller,** for obvious reasons. The critical point c illustrated in Figures 2.1.9(c) and 2.1.9(d) is neither an attractor nor a repeller. But since c exhibits characteristics of both an attractor and a repeller—that is, a solution starting from an initial point (x_0, y_0) sufficiently near c is attracted to c from one side and repelled from the other side—we say that the critical point c is **semi-stable.** In Example 3 the critical point a/b is asymptotically stable (an attractor) and the critical point 0 is unstable (a repeller). The critical point 1 in Example 5 is semi-stable.

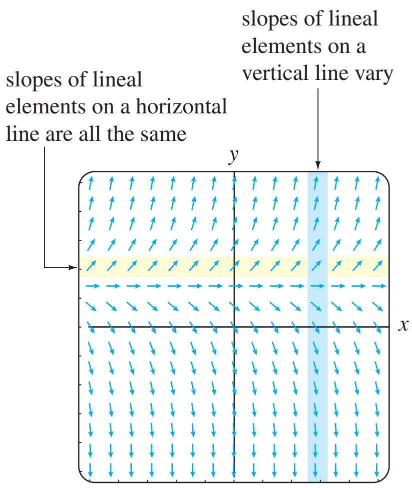


FIGURE 2.1.10 Direction field for an autonomous DE

AUTONOMOUS DEs AND DIRECTION FIELDS If a first-order differential equation is autonomous, then we see from the right-hand side of its normal form $dy/dx = f(y)$ that slopes of lineal elements through points in the rectangular grid used to construct a direction field for the DE depend solely on the y-coordinate of the points. Put another way, lineal elements passing through points on any *horizontal* line must all have the same slope; slopes of lineal elements along any *vertical* line will, of course, vary. These facts are apparent from inspection of the horizontal gold strip and vertical blue strip in Figure 2.1.10. The figure exhibits a direction field for the autonomous equation $dy/dx = 2y - 2$. With these facts in mind, reexamine Figure 2.1.4.

EXERCISES 2.1

Answers to selected odd-numbered problems begin on page ANS-1.

2.1.1 DIRECTION FIELDS

In Problems 1–4 reproduce the given computer-generated direction field. Then sketch, by hand, an approximate solution curve that passes through each of the indicated points. Use different colored pencils for each solution curve.

1. $\dfrac{dy}{dx} = x^2 - y^2$

(a) $y(-2) = 1$ **(b)** $y(3) = 0$

(c) $y(0) = 2$ **(d)** $y(0) = 0$

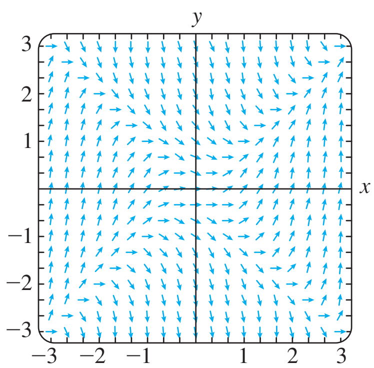


FIGURE 2.1.11 Direction field for Problem 1

2. $\dfrac{dy}{dx} = e^{-0.01xy^2}$

(a) $y(-6) = 0$ **(b)** $y(0) = 1$

(c) $y(0) = -4$ **(d)** $y(8) = -4$

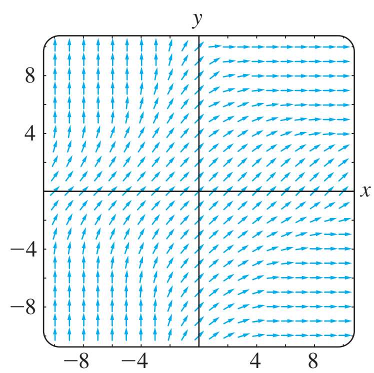


FIGURE 2.1.12 Direction field for Problem 2

3. $\dfrac{dy}{dx} = 1 - xy$

(a) $y(0) = 0$ **(b)** $y(-1) = 0$

(c) $y(2) = 2$ **(d)** $y(0) = -4$

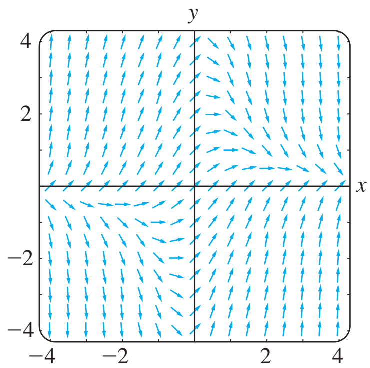


FIGURE 2.1.13 Direction field for Problem 3

4. $\dfrac{dy}{dx} = (\sin x)\cos y$

(a) $y(0) = 1$ **(b)** $y(1) = 0$

(c) $y(3) = 3$ **(d)** $y(0) = -\frac{5}{2}$

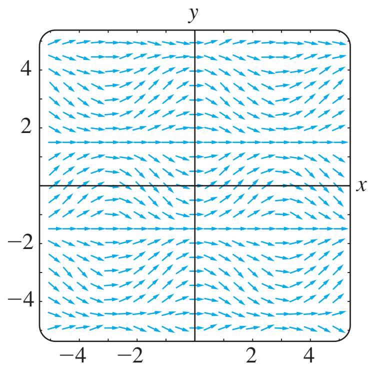


FIGURE 2.1.14 Direction field for Problem 4

In Problems 5–12 use computer software to obtain a direction field for the given differential equation. By hand, sketch an approximate solution curve passing through each of the given points.

5. $y' = x$

(a) $y(0) = 0$

(b) $y(0) = -3$

6. $y' = x + y$

(a) $y(-2) = 2$

(b) $y(1) = -3$

7. $y\dfrac{dy}{dx} = -x$

(a) $y(1) = 1$

(b) $y(0) = 4$

8. $\dfrac{dy}{dx} = \dfrac{1}{y}$

(a) $y(0) = 1$

(b) $y(-2) = -1$

9. $\dfrac{dy}{dx} = 0.2x^2 + y$

(a) $y(0) = \frac{1}{2}$

(b) $y(2) = -1$

10. $\dfrac{dy}{dx} = xe^y$

(a) $y(0) = -2$

(b) $y(1) = 2.5$

11. $y' = y - \cos\dfrac{\pi}{2}x$

(a) $y(2) = 2$

(b) $y(-1) = 0$

12. $\dfrac{dy}{dx} = 1 - \dfrac{y}{x}$

(a) $y\left(-\frac{1}{2}\right) = 2$

(b) $y\left(\frac{3}{2}\right) = 0$

In Problems 13 and 14 the given figure represents the graph of $f(y)$ and $f(x)$, respectively. By hand, sketch a direction field over an appropriate grid for $dy/dx = f(y)$ (Problem 13) and then for $dy/dx = f(x)$ (Problem 14).

13.

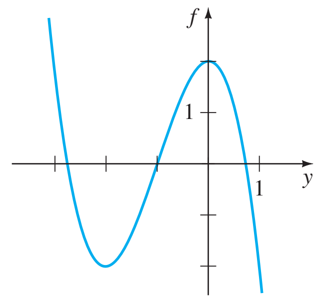


FIGURE 2.1.15 Graph for Problem 13

14.

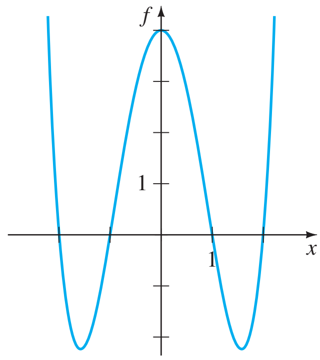


FIGURE 2.1.16 Graph for Problem 14

15. In parts (a) and (b) sketch **isoclines** $f(x, y) = c$ (see the *Remarks* on page 37) for the given differential equation using the indicated values of c. Construct a direction field over a grid by carefully drawing lineal elements with the appropriate slope at chosen points on each isocline. In each case, use this rough direction field to sketch an approximate solution curve for the IVP consisting of the DE and the initial condition $y(0) = 1$.

(a) $dy/dx = x + y$; c an integer satisfying $-5 \le c \le 5$

(b) $dy/dx = x^2 + y^2$; $c = \frac{1}{4}$, $c = 1$, $c = \frac{9}{4}$, $c = 4$

Discussion Problems

16. **(a)** Consider the direction field of the differential equation $dy/dx = x(y - 4)^2 - 2$, but do not use technology to obtain it. Describe the slopes of the lineal elements on the lines $x = 0$, $y = 3$, $y = 4$, and $y = 5$.

(b) Consider the IVP $dy/dx = x(y - 4)^2 - 2$, $y(0) = y_0$, where $y_0 < 4$. Can a solution $y(x) \to \infty$ as $x \to \infty$? Based on the information in part (a), discuss.

17. For a first-order DE $dy/dx = f(x, y)$ a curve in the plane defined by $f(x, y) = 0$ is called a **nullcline** of the equation, since a lineal element at a point on the curve has zero slope. Use computer software to obtain a direction field over a rectangular grid of points for $dy/dx = x^2 - 2y$, and then superimpose the graph of the nullcline $y = \frac{1}{2}x^2$ over the direction field. Discuss the behavior of solution curves in regions of the plane defined by $y < \frac{1}{2}x^2$ and by $y > \frac{1}{2}x^2$. Sketch some approximate solution curves. Try to generalize your observations.

18. **(a)** Identify the nullclines (see Problem 17) in Problems 1, 3, and 4. With a colored pencil, circle any lineal elements in Figures 2.1.11, 2.1.13, and 2.1.14 that you think may be a lineal element at a point on a nullcline.

(b) What are the nullclines of an autonomous first-order DE?

2.1.2 AUTONOMOUS FIRST-ORDER DEs

19. Consider the autonomous first-order differential equation $dy/dx = y - y^3$ and the initial condition $y(0) = y_0$. By hand, sketch the graph of a typical solution $y(x)$ when y_0 has the given values.

(a) $y_0 > 1$ **(b)** $0 < y_0 < 1$
(c) $-1 < y_0 < 0$ **(d)** $y_0 < -1$

20. Consider the autonomous first-order differential equation $dy/dx = y^2 - y^4$ and the initial condition $y(0) = y_0$. By hand, sketch the graph of a typical solution $y(x)$ when y_0 has the given values.

(a) $y_0 > 1$ **(b)** $0 < y_0 < 1$
(c) $-1 < y_0 < 0$ **(d)** $y_0 < -1$

In Problems 21–28 find the critical points and phase portrait of the given autonomous first-order differential equation. Classify each critical point as asymptotically stable, unstable, or semi-stable. By hand, sketch typical solution curves in the regions in the xy-plane determined by the graphs of the equilibrium solutions.

21. $\dfrac{dy}{dx} = y^2 - 3y$ **22.** $\dfrac{dy}{dx} = y^2 - y^3$

23. $\dfrac{dy}{dx} = (y - 2)^4$ **24.** $\dfrac{dy}{dx} = 10 + 3y - y^2$

25. $\dfrac{dy}{dx} = y^2(4 - y^2)$ **26.** $\dfrac{dy}{dx} = y(2 - y)(4 - y)$

27. $\dfrac{dy}{dx} = y \ln(y + 2)$ **28.** $\dfrac{dy}{dx} = \dfrac{ye^y - 9y}{e^y}$

In Problems 29 and 30 consider the autonomous differential equation $dy/dx = f(y)$, where the graph of f is given. Use the graph to locate the critical points of each differential equation. Sketch a phase portrait of each differential equation. By hand, sketch typical solution curves in the subregions in the xy-plane determined by the graphs of the equilibrium solutions.

29.

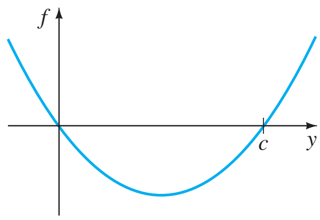


FIGURE 2.1.17 Graph for Problem 29

30.

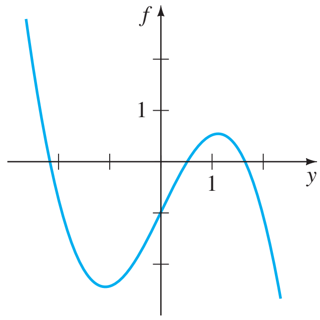


FIGURE 2.1.18 Graph for Problem 30

Discussion Problems

31. Consider the autonomous DE $dy/dx = (2/\pi)y - \sin y$. Determine the critical points of the equation. Discuss a way of obtaining a phase portrait of the equation. Classify the critical points as asymptotically stable, unstable, or semi-stable.

32. A critical point c of an autonomous first-order DE is said to be **isolated** if there exists some open interval that contains c but no other critical point. Can there exist an autonomous DE of the form given in (1) for which *every* critical point is nonisolated? Discuss; do not think profound thoughts.

33. Suppose that $y(x)$ is a nonconstant solution of the autonomous equation $dy/dx = f(y)$ and that c is a critical point of the DE. Discuss. Why can't the graph of $y(x)$ cross the graph of the equilibrium solution $y = c$? Why can't $f(y)$ change signs in one of the subregions discussed on page 38? Why can't $y(x)$ be oscillatory or have a relative extremum (maximum or minimum)?

34. Suppose that $y(x)$ is a solution of the autonomous equation $dy/dx = f(y)$ and is bounded above and below by two consecutive critical points $c_1 < c_2$, as in subregion R_2 of Figure 2.1.6(b). If $f(y) > 0$ in the region, then $\lim_{x\to\infty} y(x) = c_2$. Discuss why there cannot exist a number $L < c_2$ such that $\lim_{x\to\infty} y(x) = L$. As part of your discussion, consider what happens to $y'(x)$ as $x \to \infty$.

35. Using the autonomous equation (1), discuss how it is possible to obtain information about the location of points of inflection of a solution curve.

36. Consider the autonomous DE $dy/dx = y^2 - y - 6$. Use your ideas from Problem 35 to find intervals on the y-axis for which solution curves are concave up and intervals for which solution curves are concave down. Discuss why *each* solution curve of an initial-value problem of the form $dy/dx = y^2 - y - 6$, $y(0) = y_0$, where $-2 < y_0 < 3$, has a point of inflection with the same y-coordinate. What is that y-coordinate? Carefully sketch the solution curve for which $y(0) = -1$. Repeat for $y(2) = 2$.

37. Suppose the autonomous DE in (1) has no critical points. Discuss the behavior of the solutions.

Mathematical Models

38. Population Model The differential equation in Example 3 is a well-known population model. Suppose the DE is changed to

$$\frac{dP}{dt} = P(aP - b),$$

where a and b are positive constants. Discuss what happens to the population P as time t increases.

39. Population Model Another population model is given by

$$\frac{dP}{dt} = kP - h,$$

where h and k are positive constants. For what initial values $P(0) = P_0$ does this model predict that the population will go extinct?

40. Terminal Velocity In Section 1.3 we saw that the autonomous differential equation

$$m\frac{dv}{dt} = mg - kv,$$

where k is a positive constant and g is the acceleration due to gravity, is a model for the velocity v of a body of mass m that is falling under the influence of gravity. Because the term $-kv$ represents air resistance, the velocity of a body falling from a great height does not increase without bound as time t increases. Use a phase portrait of the differential equation to find the limiting, or terminal, velocity of the body. Explain your reasoning.

41. Suppose the model in Problem 40 is modified so that air resistance is proportional to v^2, that is,

$$m\frac{dv}{dt} = mg - kv^2.$$

See Problem 17 in Exercises 1.3. Use a phase portrait to find the terminal velocity of the body. Explain your reasoning.

42. Chemical Reactions When certain kinds of chemicals are combined, the rate at which the new compound is formed is modeled by the autonomous differential equation

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X),$$

where $k > 0$ is a constant of proportionality and $\beta > \alpha > 0$. Here $X(t)$ denotes the number of grams of the new compound formed in time t.

(a) Use a phase portrait of the differential equation to predict the behavior of $X(t)$ as $t \to \infty$.

(b) Consider the case when $\alpha = \beta$. Use a phase portrait of the differential equation to predict the behavior of $X(t)$ as $t \to \infty$ when $X(0) < \alpha$. When $X(0) > \alpha$.

(c) Verify that an explicit solution of the DE in the case when $k = 1$ and $\alpha = \beta$ is $X(t) = \alpha - 1/(t + c)$. Find a solution that satisfies $X(0) = \alpha/2$. Then find a solution that satisfies $X(0) = 2\alpha$. Graph these two solutions. Does the behavior of the solutions as $t \to \infty$ agree with your answers to part (b)?

2.2 SEPARABLE VARIABLES

REVIEW MATERIAL

- Basic integration formulas (See inside front cover)
- Techniques of integration: integration by parts and partial fraction decomposition
- See also the *Student Resource and Solutions Manual.*

INTRODUCTION We begin our study of how to solve differential equations with the simplest of all differential equations: first-order equations with separable variables. Because the method in this section and many techniques for solving differential equations involve integration, you are urged to refresh your memory on important formulas (such as $\int du/u$) and techniques (such as integration by parts) by consulting a calculus text.

SOLUTION BY INTEGRATION Consider the first-order differential equation $dy/dx = f(x, y)$. When f does not depend on the variable y, that is, $f(x, y) = g(x)$, the differential equation

$$\frac{dy}{dx} = g(x) \tag{1}$$

can be solved by integration. If $g(x)$ is a continuous function, then integrating both sides of (1) gives $y = \int g(x)\, dx = G(x) + c$, where $G(x)$ is an antiderivative (indefinite integral) of $g(x)$. For example, if $dy/dx = 1 + e^{2x}$, then its solution is $y = \int(1 + e^{2x})\, dx$ or $y = x + \frac{1}{2}e^{2x} + c$.

A DEFINITION Equation (1), as well as its method of solution, is just a special case when the function f in the normal form $dy/dx = f(x, y)$ can be factored into a function of x times a function of y.

DEFINITION 2.2.1 Separable Equation

A first-order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be **separable** or to have **separable variables.**

For example, the equations

$$\frac{dy}{dx} = y^2xe^{3x+4y} \qquad \text{and} \qquad \frac{dy}{dx} = y + \sin x$$

are separable and nonseparable, respectively. In the first equation we can factor $f(x, y) = y^2xe^{3x+4y}$ as

$$f(x, y) = y^2xe^{3x+4y} = \underbrace{(xe^{3x})}_{g(x)}\underbrace{(y^2e^{4y})}_{h(y)},$$

but in the second equation there is no way of expressing $y + \sin x$ as a product of a function of x times a function of y.

Observe that by dividing by the function $h(y)$, we can write a separable equation $dy/dx = g(x)h(y)$ as

$$p(y)\frac{dy}{dx} = g(x), \tag{2}$$

where, for convenience, we have denoted $1/h(y)$ by $p(y)$. From this last form we can see immediately that (2) reduces to (1) when $h(y) = 1$.

Now if $y = \phi(x)$ represents a solution of (2), we must have $p(\phi(x))\phi'(x) = g(x)$, and therefore

$$\int p(\phi(x))\phi'(x)\, dx = \int g(x)\, dx. \tag{3}$$

But $dy = \phi'(x)\, dx$, and so (3) is the same as

$$\int p(y)\, dy = \int g(x)\, dx \qquad \text{or} \qquad H(y) = G(x) + c, \tag{4}$$

where $H(y)$ and $G(x)$ are antiderivatives of $p(y) = 1/h(y)$ and $g(x)$, respectively.

METHOD OF SOLUTION Equation (4) indicates the procedure for solving separable equations. A one-parameter family of solutions, usually given implicitly, is obtained by integrating both sides of $p(y)\,dy = g(x)\,dx$.

NOTE There is no need to use two constants in the integration of a separable equation, because if we write $H(y) + c_1 = G(x) + c_2$, then the difference $c_2 - c_1$ can be replaced by a single constant c, as in (4). In many instances throughout the chapters that follow, we will relabel constants in a manner convenient to a given equation. For example, multiples of constants or combinations of constants can sometimes be replaced by a single constant.

EXAMPLE 1 Solving a Separable DE

Solve $(1 + x)\,dy - y\,dx = 0$.

SOLUTION Dividing by $(1 + x)y$, we can write $dy/y = dx/(1 + x)$, from which it follows that

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\ln|y| = \ln|1 + x| + c_1$$

$$y = e^{\ln|1+x|+c_1} = e^{\ln|1+x|} \cdot e^{c_1} \qquad \leftarrow \text{laws of exponents}$$

$$= |1 + x|\,e^{c_1}$$

$$= \pm e^{c_1}(1 + x). \qquad \leftarrow \begin{cases} |1+x| = 1 + x, & x \geq -1 \\ |1+x| = -(1+x), & x < -1 \end{cases}$$

Relabeling $\pm e^{c_1}$ as c then gives $y = c(1 + x)$.

ALTERNATIVE SOLUTION Because each integral results in a logarithm, a judicious choice for the constant of integration is $\ln|c|$ rather than c. Rewriting the second line of the solution as $\ln|y| = \ln|1 + x| + \ln|c|$ enables us to combine the terms on the right-hand side by the properties of logarithms. From $\ln|y| = \ln|c(1 + x)|$ we immediately get $y = c(1 + x)$. Even if the indefinite integrals are not *all* logarithms, it may still be advantageous to use $\ln|c|$. However, no firm rule can be given. ■

In Section 1.1 we saw that a solution curve may be only a segment or an arc of the graph of an implicit solution $G(x, y) = 0$.

EXAMPLE 2 Solution Curve

Solve the initial-value problem $\dfrac{dy}{dx} = -\dfrac{x}{y}$, $y(4) = -3$.

SOLUTION Rewriting the equation as $y\,dy = -x\,dx$, we get

$$\int y\,dy = -\int x\,dx \qquad \text{and} \qquad \frac{y^2}{2} = -\frac{x^2}{2} + c_1.$$

We can write the result of the integration as $x^2 + y^2 = c^2$ by replacing the constant $2c_1$ by c^2. This solution of the differential equation represents a family of concentric circles centered at the origin.

Now when $x = 4$, $y = -3$, so $16 + 9 = 25 = c^2$. Thus the initial-value problem determines the circle $x^2 + y^2 = 25$ with radius 5. Because of its simplicity we can solve this implicit solution for an explicit solution that satisfies the initial condition.

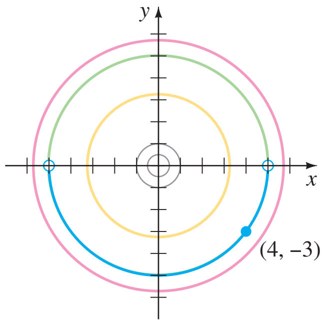


FIGURE 2.2.1 Solution curve for the IVP in Example 2

We saw this solution as $y = \phi_2(x)$ or $y = -\sqrt{25 - x^2}$, $-5 < x < 5$ in Example 3 of Section 1.1. A solution curve is the graph of a differentiable function. In this case the solution curve is the lower semicircle, shown in dark blue in Figure 2.2.1 containing the point $(4, -3)$. ■

LOSING A SOLUTION Some care should be exercised in separating variables, since the variable divisors could be zero at a point. Specifically, if r is a zero of the function $h(y)$, then substituting $y = r$ into $dy/dx = g(x)h(y)$ makes both sides zero; in other words, $y = r$ is a constant solution of the differential equation. But after variables are separated, the left-hand side of $\dfrac{dy}{h(y)} = g(x)\,dx$ is undefined at r. As a consequence, $y = r$ might not show up in the family of solutions that are obtained after integration and simplification. Recall that such a solution is called a singular solution.

EXAMPLE 3 Losing a Solution

Solve $\dfrac{dy}{dx} = y^2 - 4$.

SOLUTION We put the equation in the form

$$\frac{dy}{y^2 - 4} = dx \qquad \text{or} \qquad \left[\frac{\frac{1}{4}}{y - 2} - \frac{\frac{1}{4}}{y + 2}\right] dy = dx. \tag{5}$$

The second equation in (5) is the result of using partial fractions on the left-hand side of the first equation. Integrating and using the laws of logarithms gives

$$\frac{1}{4}\ln|y - 2| - \frac{1}{4}\ln|y + 2| = x + c_1$$

$$\text{or} \qquad \ln\left|\frac{y - 2}{y + 2}\right| = 4x + c_2 \qquad \text{or} \qquad \frac{y - 2}{y + 2} = \pm e^{4x + c_2}.$$

Here we have replaced $4c_1$ by c_2. Finally, after replacing $\pm e^{c_2}$ by c and solving the last equation for y, we get the one-parameter family of solutions

$$y = 2\,\frac{1 + ce^{4x}}{1 - ce^{4x}}. \tag{6}$$

Now if we factor the right-hand side of the differential equation as $dy/dx = (y - 2)(y + 2)$, we know from the discussion of critical points in Section 2.1 that $y = 2$ and $y = -2$ are two constant (equilibrium) solutions. The solution $y = 2$ is a member of the family of solutions defined by (6) corresponding to the value $c = 0$. However, $y = -2$ is a singular solution; it cannot be obtained from (6) for any choice of the parameter c. This latter solution was lost early on in the solution process. Inspection of (5) clearly indicates that we must preclude $y = \pm 2$ in these steps. ■

EXAMPLE 4 An Initial-Value Problem

Solve $(e^{2y} - y)\cos x \dfrac{dy}{dx} = e^y \sin 2x, \quad y(0) = 0.$

SOLUTION Dividing the equation by $e^y \cos x$ gives

$$\frac{e^{2y} - y}{e^y}\,dy = \frac{\sin 2x}{\cos x}\,dx.$$

Before integrating, we use termwise division on the left-hand side and the trigonometric identity $\sin 2x = 2 \sin x \cos x$ on the right-hand side. Then

integration by parts →
$$\int (e^y - ye^{-y})\,dy = 2\int \sin x\,dx$$

yields
$$e^y + ye^{-y} + e^{-y} = -2 \cos x + c. \tag{7}$$

The initial condition $y = 0$ when $x = 0$ implies $c = 4$. Thus a solution of the initial-value problem is

$$e^y + ye^{-y} + e^{-y} = 4 - 2 \cos x. \tag{8}$$ ■

USE OF COMPUTERS The *Remarks* at the end of Section 1.1 mentioned that it may be difficult to use an implicit solution $G(x, y) = 0$ to find an explicit solution $y = \phi(x)$. Equation (8) shows that the task of solving for y in terms of x may present more problems than just the drudgery of symbol pushing—sometimes it simply cannot be done! Implicit solutions such as (8) are somewhat frustrating; neither the graph of the equation nor an interval over which a solution satisfying $y(0) = 0$ is defined is apparent. The problem of "seeing" what an implicit solution looks like can be overcome in some cases by means of technology. One way* of proceeding is to use the contour plot application of a computer algebra system (CAS). Recall from multivariate calculus that for a function of two variables $z = G(x, y)$ the *two-dimensional* curves defined by $G(x, y) = c$, where c is constant, are called the *level curves* of the function. With the aid of a CAS, some of the level curves of the function $G(x, y) = e^y + ye^{-y} + e^{-y} + 2 \cos x$ have been reproduced in Figure 2.2.2. The family of solutions defined by (7) is the level curves $G(x, y) = c$. Figure 2.2.3 illustrates the level curve $G(x, y) = 4$, which is the particular solution (8), in blue color. The other curve in Figure 2.2.3 is the level curve $G(x, y) = 2$, which is the member of the family $G(x, y) = c$ that satisfies $y(\pi/2) = 0$.

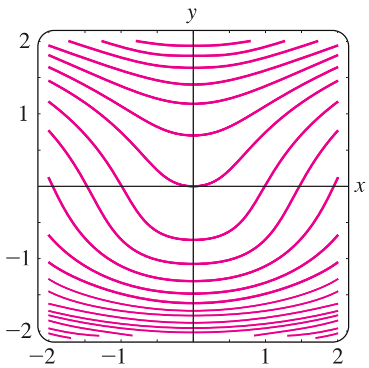


FIGURE 2.2.2 Level curves $G(x, y) = c$, where $G(x, y) = e^y + ye^{-y} + e^{-y} + 2 \cos x$

If an initial condition leads to a particular solution by yielding a specific value of the parameter c in a family of solutions for a first-order differential equation, there is a natural inclination for most students (and instructors) to relax and be content. However, a solution of an initial-value problem might not be unique. We saw in Example 4 of Section 1.2 that the initial-value problem

$$\frac{dy}{dx} = xy^{1/2}, \quad y(0) = 0 \tag{9}$$

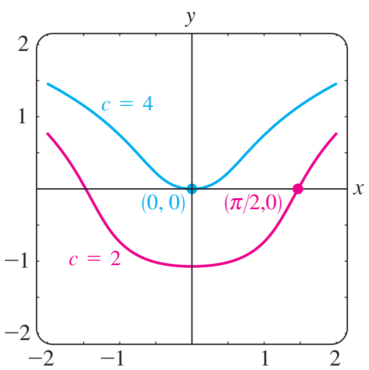


FIGURE 2.2.3 Level curves $c = 2$ and $c = 4$

has at least two solutions, $y = 0$ and $y = \frac{1}{16}x^4$. We are now in a position to solve the equation. Separating variables and integrating $y^{-1/2}\,dy = x\,dx$ gives

$$2y^{1/2} = \frac{x^2}{2} + c_1 \qquad \text{or} \qquad y = \left(\frac{x^2}{4} + c\right)^2.$$

When $x = 0$, then $y = 0$, so necessarily, $c = 0$. Therefore $y = \frac{1}{16}x^4$. The trivial solution $y = 0$ was lost by dividing by $y^{1/2}$. In addition, the initial-value problem (9) possesses infinitely many more solutions, since for any choice of the parameter $a \geq 0$ the

*In Section 2.6 we will discuss several other ways of proceeding that are based on the concept of a numerical solver.

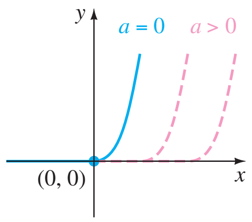


FIGURE 2.2.4 Piecewise-defined solutions of (9)

piecewise-defined function

$$y = \begin{cases} 0, & x < a \\ \frac{1}{16}(x^2 - a^2)^2, & x \geq a \end{cases}$$

satisfies both the differential equation and the initial condition. See Figure 2.2.4.

SOLUTIONS DEFINED BY INTEGRALS If g is a function continuous on an open interval I containing a, then for every x in I,

$$\frac{d}{dx}\int_a^x g(t)\,dt = g(x).$$

You might recall that the foregoing result is one of the two forms of the fundamental theorem of calculus. In other words, $\int_a^x g(t)\,dt$ is an antiderivative of the function g. There are times when this form is convenient in solving DEs. For example, if g is continuous on an interval I containing x_0 and x, then a solution of the simple initial-value problem $dy/dx = g(x)$, $y(x_0) = y_0$, that is defined on I is given by

$$y(x) = y_0 + \int_{x_0}^x g(t)\,dt$$

You should verify that $y(x)$ defined in this manner satisfies the initial condition. Since an antiderivative of a continuous function g cannot always be expressed in terms of elementary functions, this might be the best we can do in obtaining an explicit solution of an IVP. The next example illustrates this idea.

EXAMPLE 5 An Initial-Value Problem

Solve $\dfrac{dy}{dx} = e^{-x^2}$, $y(3) = 5$.

SOLUTION The function $g(x) = e^{-x^2}$ is continuous on $(-\infty, \infty)$, but its antiderivative is not an elementary function. Using t as dummy variable of integration, we can write

$$\int_3^x \frac{dy}{dt}\,dt = \int_3^x e^{-t^2}\,dt$$

$$y(t)\Big]_3^x = \int_3^x e^{-t^2}\,dt$$

$$y(x) - y(3) = \int_3^x e^{-t^2}\,dt$$

$$y(x) = y(3) + \int_3^x e^{-t^2}\,dt.$$

Using the initial condition $y(3) = 5$, we obtain the solution

$$y(x) = 5 + \int_3^x e^{-t^2}\,dt.$$ ■

The procedure demonstrated in Example 5 works equally well on separable equations $dy/dx = g(x)f(y)$ where, say, $f(y)$ possesses an elementary antiderivative but $g(x)$ does not possess an elementary antiderivative. See Problems 29 and 30 in Exercises 2.2.

REMARKS

(i) As we have just seen in Example 5, some simple functions do not possess an antiderivative that is an elementary function. Integrals of these kinds of functions are called **nonelementary.** For example, $\int_3^x e^{-t^2}\,dt$ and $\int \sin x^2\,dx$ are nonelementary integrals. We will run into this concept again in Section 2.3.

(*ii*) In some of the preceding examples we saw that the constant in the one-parameter family of solutions for a first-order differential equation can be relabeled when convenient. Also, it can easily happen that two individuals solving the same equation correctly arrive at dissimilar expressions for their answers. For example, by separation of variables we can show that one-parameter families of solutions for the DE $(1 + y^2)\,dx + (1 + x^2)\,dy = 0$ are

$$\arctan x + \arctan y = c \qquad \text{or} \qquad \frac{x + y}{1 - xy} = c.$$

As you work your way through the next several sections, bear in mind that families of solutions may be equivalent in the sense that one family may be obtained from another by either relabeling the constant or applying algebra and trigonometry. See Problems 27 and 28 in Exercises 2.2.

EXERCISES 2.2

Answers to selected odd-numbered problems begin on page ANS-1.

In Problems 1–22 solve the given differential equation by separation of variables.

1. $\dfrac{dy}{dx} = \sin 5x$

2. $\dfrac{dy}{dx} = (x + 1)^2$

3. $dx + e^{3x}dy = 0$

4. $dy - (y - 1)^2dx = 0$

5. $x\dfrac{dy}{dx} = 4y$

6. $\dfrac{dy}{dx} + 2xy^2 = 0$

7. $\dfrac{dy}{dx} = e^{3x+2y}$

8. $e^xy\dfrac{dy}{dx} = e^{-y} + e^{-2x-y}$

9. $y \ln x\dfrac{dx}{dy} = \left(\dfrac{y + 1}{x}\right)^2$

10. $\dfrac{dy}{dx} = \left(\dfrac{2y + 3}{4x + 5}\right)^2$

11. $\csc y\,dx + \sec^2 x\,dy = 0$

12. $\sin 3x\,dx + 2y\cos^3 3x\,dy = 0$

13. $(e^y + 1)^2e^{-y}\,dx + (e^x + 1)^3e^{-x}\,dy = 0$

14. $x(1 + y^2)^{1/2}\,dx = y(1 + x^2)^{1/2}\,dy$

15. $\dfrac{dS}{dr} = kS$

16. $\dfrac{dQ}{dt} = k(Q - 70)$

17. $\dfrac{dP}{dt} = P - P^2$

18. $\dfrac{dN}{dt} + N = Nte^{t+2}$

19. $\dfrac{dy}{dx} = \dfrac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$

20. $\dfrac{dy}{dx} = \dfrac{xy + 2y - x - 2}{xy - 3y + x - 3}$

21. $\dfrac{dy}{dx} = x\sqrt{1 - y^2}$

22. $(e^x + e^{-x})\dfrac{dy}{dx} = y^2$

In Problems 23–28 find an explicit solution of the given initial-value problem.

23. $\dfrac{dx}{dt} = 4(x^2 + 1), \quad x(\pi/4) = 1$

24. $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{x^2 - 1}, \quad y(2) = 2$

25. $x^2\dfrac{dy}{dx} = y - xy, \quad y(-1) = -1$

26. $\dfrac{dy}{dt} + 2y = 1, \quad y(0) = \frac{5}{2}$

27. $\sqrt{1 - y^2}\,dx - \sqrt{1 - x^2}\,dy = 0, \quad y(0) = \dfrac{\sqrt{3}}{2}$

28. $(1 + x^4)\,dy + x(1 + 4y^2)\,dx = 0, \quad y(1) = 0$

In Problems 29 and 30 proceed as in Example 5 and find an explicit solution of the given initial-value problem.

29. $\dfrac{dy}{dx} = ye^{-x^2}, \quad y(4) = 1$

30. $\dfrac{dy}{dx} = y^2\sin x^2, \quad y(-2) = \frac{1}{3}$

31. **(a)** Find a solution of the initial-value problem consisting of the differential equation in Example 3 and the initial conditions $y(0) = 2$, $y(0) = -2$, and $y\left(\frac{1}{4}\right) = 1$.

(b) Find the solution of the differential equation in Example 4 when $\ln c_1$ is used as the constant of integration on the *left-hand* side in the solution and $4 \ln c_1$ is replaced by $\ln c$. Then solve the same initial-value problems in part (a).

32. Find a solution of $x\dfrac{dy}{dx} = y^2 - y$ that passes through the indicated points.

(a) $(0, 1)$ **(b)** $(0, 0)$ **(c)** $\left(\frac{1}{2}, \frac{1}{2}\right)$ **(d)** $\left(2, \frac{1}{4}\right)$

33. Find a singular solution of Problem 21. Of Problem 22.

34. Show that an implicit solution of

$$2x \sin^2 y\, dx - (x^2 + 10) \cos y\, dy = 0$$

is given by $\ln(x^2 + 10) + \csc y = c$. Find the constant solutions, if any, that were lost in the solution of the differential equation.

Often a radical change in the form of the solution of a differential equation corresponds to a very small change in either the initial condition or the equation itself. In Problems 35–38 find an explicit solution of the given initial-value problem. Use a graphing utility to plot the graph of each solution. Compare each solution curve in a neighborhood of $(0, 1)$.

35. $\dfrac{dy}{dx} = (y - 1)^2, \quad y(0) = 1$

36. $\dfrac{dy}{dx} = (y - 1)^2, \quad y(0) = 1.01$

37. $\dfrac{dy}{dx} = (y - 1)^2 + 0.01, \quad y(0) = 1$

38. $\dfrac{dy}{dx} = (y - 1)^2 - 0.01, \quad y(0) = 1$

39. Every autonomous first-order equation $dy/dx = f(y)$ is separable. Find explicit solutions $y_1(x)$, $y_2(x)$, $y_3(x)$, and $y_4(x)$ of the differential equation $dy/dx = y - y^3$ that satisfy, in turn, the initial conditions $y_1(0) = 2$, $y_2(0) = \frac{1}{2}$, $y_3(0) = -\frac{1}{2}$, and $y_4(0) = -2$. Use a graphing utility to plot the graphs of each solution. Compare these graphs with those predicted in Problem 19 of Exercises 2.1. Give the exact interval of definition for each solution.

40. (a) The autonomous first-order differential equation $dy/dx = 1/(y - 3)$ has no critical points. Nevertheless, place 3 on the phase line and obtain a phase portrait of the equation. Compute d^2y/dx^2 to determine where solution curves are concave up and where they are concave down (see Problems 35 and 36 in Exercises 2.1). Use the phase portrait and concavity to sketch, by hand, some typical solution curves.

(b) Find explicit solutions $y_1(x)$, $y_2(x)$, $y_3(x)$, and $y_4(x)$ of the differential equation in part (a) that satisfy, in turn, the initial conditions $y_1(0) = 4$, $y_2(0) = 2$, $y_3(1) = 2$, and $y_4(-1) = 4$. Graph each solution and compare with your sketches in part (a). Give the exact interval of definition for each solution.

41. (a) Find an explicit solution of the initial-value problem

$$\frac{dy}{dx} = \frac{2x + 1}{2y}, \quad y(-2) = -1.$$

(b) Use a graphing utility to plot the graph of the solution in part (a). Use the graph to estimate the interval I of definition of the solution.

(c) Determine the exact interval I of definition by analytical methods.

42. Repeat parts (a)–(c) of Problem 41 for the IVP consisting of the differential equation in Problem 7 and the initial condition $y(0) = 0$.

Discussion Problems

43. (a) Explain why the interval of definition of the explicit solution $y = \phi_2(x)$ of the initial-value problem in Example 2 is the *open* interval $(-5, 5)$.

(b) Can any solution of the differential equation cross the x-axis? Do you think that $x^2 + y^2 = 1$ is an implicit solution of the initial-value problem $dy/dx = -x/y$, $y(1) = 0$?

44. (a) If $a > 0$, discuss the differences, if any, between the solutions of the initial-value problems consisting of the differential equation $dy/dx = x/y$ and each of the initial conditions $y(a) = a$, $y(a) = -a$, $y(-a) = a$, and $y(-a) = -a$.

(b) Does the initial-value problem $dy/dx = x/y$, $y(0) = 0$ have a solution?

(c) Solve $dy/dx = x/y$, $y(1) = 2$ and give the exact interval I of definition of its solution.

45. In Problems 39 and 40 we saw that every autonomous first-order differential equation $dy/dx = f(y)$ is separable. Does this fact help in the solution of the initial-value problem $\dfrac{dy}{dx} = \sqrt{1 + y^2} \sin^2 y, \quad y(0) = \frac{1}{2}$? Discuss. Sketch, by hand, a plausible solution curve of the problem.

46. Without the use of technology, how would you solve

$$\left(\sqrt{x} + x\right)\frac{dy}{dx} = \sqrt{y} + y?$$

Carry out your ideas.

47. Find a function whose square plus the square of its derivative is 1.

48. (a) The differential equation in Problem 27 is equivalent to the normal form

$$\frac{dy}{dx} = \sqrt{\frac{1 - y^2}{1 - x^2}}$$

in the square region in the xy-plane defined by $|x| < 1, |y| < 1$. But the quantity under the radical is nonnegative also in the regions defined by $|x| > 1$, $|y| > 1$. Sketch all regions in the xy-plane for which this differential equation possesses real solutions.

(b) Solve the DE in part (a) in the regions defined by $|x| > 1, |y| > 1$. Then find an implicit and an explicit solution of the differential equation subject to $y(2) = 2$.

Mathematical Model

49. Suspension Bridge In (16) of Section 1.3 we saw that a mathematical model for the shape of a flexible cable strung between two vertical supports is

$$\frac{dy}{dx} = \frac{W}{T_1}, \tag{10}$$

where W denotes the portion of the total vertical load between the points P_1 and P_2 shown in Figure 1.3.7. The DE (10) is separable under the following conditions that describe a suspension bridge.

Let us assume that the x- and y-axes are as shown in Figure 2.2.5—that is, the x-axis runs along the horizontal roadbed, and the y-axis passes through $(0, a)$, which is the lowest point on one cable over the span of the bridge, coinciding with the interval $[-L/2, L/2]$. In the case of a suspension bridge, the usual assumption is that the vertical load in (10) is only a uniform roadbed distributed along the horizontal axis. In other words, it is assumed that the weight of all cables is negligible in comparison to the weight of the roadbed and that the weight per unit length of the roadbed (say, pounds per horizontal foot) is a constant ρ. Use this information to set up and solve an appropriate initial-value problem from which the shape (a curve with equation $y = \phi(x)$) of each of the two cables in a suspension bridge is determined. Express your solution of the IVP in terms of the sag h and span L. See Figure 2.2.5.

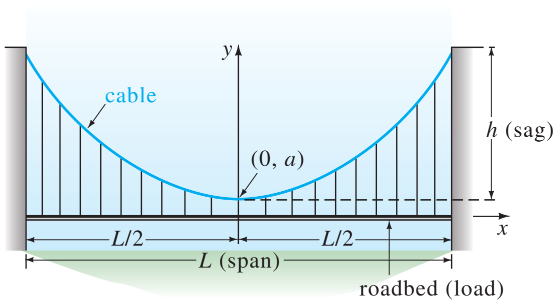


FIGURE 2.2.5 Shape of a cable in Problem 49

Computer Lab Assignments

50. (a) Use a CAS and the concept of level curves to plot representative graphs of members of the family of solutions of the differential equation $\frac{dy}{dx} = -\frac{8x + 5}{3y^2 + 1}$. Experiment with different numbers of level curves as well as various rectangular regions defined by $a \le x \le b, c \le y \le d$.

(b) On separate coordinate axes plot the graphs of the particular solutions corresponding to the initial conditions: $y(0) = -1$; $y(0) = 2$; $y(-1) = 4$; $y(-1) = -3$.

51. (a) Find an implicit solution of the IVP

$$(2y + 2)\, dy - (4x^3 + 6x)\, dx = 0, \quad y(0) = -3.$$

(b) Use part (a) to find an explicit solution $y = \phi(x)$ of the IVP.

(c) Consider your answer to part (b) as a *function* only. Use a graphing utility or a CAS to graph this function, and then use the graph to estimate its domain.

(d) With the aid of a root-finding application of a CAS, determine the approximate largest interval I of definition of the *solution* $y = \phi(x)$ in part (b). Use a graphing utility or a CAS to graph the solution curve for the IVP on this interval.

52. (a) Use a CAS and the concept of level curves to plot representative graphs of members of the family of solutions of the differential equation $\frac{dy}{dx} = \frac{x(1 - x)}{y(-2 + y)}$. Experiment with different numbers of level curves as well as various rectangular regions in the xy-plane until your result resembles Figure 2.2.6.

(b) On separate coordinate axes, plot the graph of the implicit solution corresponding to the initial condition $y(0) = \frac{3}{2}$. Use a colored pencil to mark off that segment of the graph that corresponds to the solution curve of a solution ϕ that satisfies the initial condition. With the aid of a root-finding application of a CAS, determine the approximate largest interval I of definition of the solution ϕ. [*Hint*: First find the points on the curve in part (a) where the tangent is vertical.]

(c) Repeat part (b) for the initial condition $y(0) = -2$.

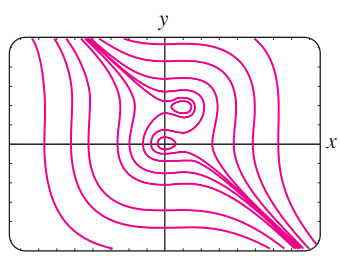


FIGURE 2.2.6 Level curves in Problem 52

2.3 LINEAR EQUATIONS

REVIEW MATERIAL

- Review the definition of linear DEs in (6) and (7) of Section 1.1

INTRODUCTION We continue our quest for solutions of first-order DEs by next examining linear equations. Linear differential equations are an especially "friendly" family of differential equations in that, given a linear equation, whether first order or a higher-order kin, there is always a good possibility that we can find some sort of solution of the equation that we can examine.

A DEFINITION The form of a linear first-order DE was given in (7) of Section 1.1. This form, the case when $n = 1$ in (6) of that section, is reproduced here for convenience.

DEFINITION 2.3.1 Linear Equation

A first-order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1}$$

is said to be a **linear equation** in the dependent variable y.

When $g(x) = 0$, the linear equation (1) is said to be **homogeneous;** otherwise, it is **nonhomogeneous.**

STANDARD FORM By dividing both sides of (1) by the lead coefficient $a_1(x)$, we obtain a more useful form, the **standard form,** of a linear equation:

$$\frac{dy}{dx} + P(x)y = f(x). \tag{2}$$

We seek a solution of (2) on an interval I for which both coefficient functions P and f are continuous.

In the discussion that follows we illustrate a property and a procedure and end up with a formula representing the form that every solution of (2) must have. But more than the formula, the property and the procedure are important, because these two concepts carry over to linear equations of higher order.

THE PROPERTY The differential equation (2) has the property that its solution is the **sum** of the two solutions: $y = y_c + y_p$, where y_c is a solution of the associated homogeneous equation

$$\frac{dy}{dx} + P(x)y = 0 \tag{3}$$

and y_p is a particular solution of the nonhomogeneous equation (2). To see this, observe that

$$\frac{d}{dx}[y_c + y_p] + P(x)[y_c + y_p] = \underbrace{\left[\frac{dy_c}{dx} + P(x)y_c\right]}_{0} + \underbrace{\left[\frac{dy_p}{dx} + P(x)y_p\right]}_{f(x)} = f(x).$$

Now the homogeneous equation (3) is also separable. This fact enables us to find y_c by writing (3) as

$$\frac{dy}{y} + P(x)\,dx = 0$$

and integrating. Solving for y gives $y_c = ce^{-\int P(x)dx}$. For convenience let us write $y_c = cy_1(x)$, where $y_1 = e^{-\int P(x)dx}$. The fact that $dy_1/dx + P(x)y_1 = 0$ will be used next to determine y_p.

THE PROCEDURE We can now find a particular solution of equation (2) by a procedure known as **variation of parameters.** The basic idea here is to find a function u so that $y_p = u(x)y_1(x) = u(x)e^{-\int P(x)dx}$ is a solution of (2). In other words, our assumption for y_p is the same as $y_c = cy_1(x)$ except that c is replaced by the "variable parameter" u. Substituting $y_p = uy_1$ into (2) gives

$$\underbrace{u\frac{dy_1}{dx} + y_1\frac{du}{dx}}_{\text{Product Rule}} + P(x)uy_1 = f(x) \quad \text{or} \quad u\underbrace{\left[\frac{dy_1}{dx} + P(x)y_1\right]}_{\text{zero}} + y_1\frac{du}{dx} = f(x)$$

so

$$y_1\frac{du}{dx} = f(x).$$

Separating variables and integrating then gives

$$du = \frac{f(x)}{y_1(x)}\,dx \quad \text{and} \quad u = \int \frac{f(x)}{y_1(x)}\,dx.$$

Since $y_1(x) = e^{-\int P(x)dx}$, we see that $1/y_1(x) = e^{\int P(x)dx}$. Therefore

$$y_p = uy_1 = \left(\int \frac{f(x)}{y_1(x)}\,dx\right)e^{-\int P(x)dx} = e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx,$$

and

$$y = \underbrace{ce^{-\int P(x)dx}}_{y_c} + \underbrace{e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx}_{y_p}. \tag{4}$$

Hence if (2) has a solution, it must be of form (4). Conversely, it is a straightforward exercise in differentiation to verify that (4) constitutes a one-parameter family of solutions of equation (2).

You should not memorize the formula given in (4). However, you should remember the special term

$$e^{\int P(x)dx} \tag{5}$$

because it is used in an equivalent but easier way of solving (2). If equation (4) is multiplied by (5),

$$e^{\int P(x)dx}y = c + \int e^{\int P(x)dx}f(x)\,dx, \tag{6}$$

and then (6) is differentiated,

$$\frac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x), \tag{7}$$

we get

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = e^{\int P(x)dx}f(x). \tag{8}$$

Dividing the last result by $e^{\int P(x)dx}$ gives (2).

METHOD OF SOLUTION The recommended method of solving (2) actually consists of (6)–(8) worked in reverse order. In other words, if (2) is multiplied by (5), we get (8). The left-hand side of (8) is recognized as the derivative of the product of $e^{\int P(x)dx}$ and y. This gets us to (7). We then integrate both sides of (7) to get the solution (6). Because we can solve (2) by integration after multiplication by $e^{\int P(x)dx}$, we call this function an **integrating factor** for the differential equation. For convenience we summarize these results. We again emphasize that you should not memorize formula (4) but work through the following procedure each time.

SOLVING A LINEAR FIRST-ORDER EQUATION

(*i*) Put a linear equation of form (1) into the standard form (2).

(*ii*) From the standard form identify $P(x)$ and then find the integrating factor $e^{\int P(x)dx}$.

(*iii*) Multiply the standard form of the equation by the integrating factor. The left-hand side of the resulting equation is automatically the derivative of the integrating factor and y:

$$\frac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x).$$

(*iv*) Integrate both sides of this last equation.

EXAMPLE 1 Solving a Homogeneous Linear DE

Solve $\dfrac{dy}{dx} - 3y = 0$.

SOLUTION This linear equation can be solved by separation of variables. Alternatively, since the equation is already in the standard form (2), we see that $P(x) = -3$, and so the integrating factor is $e^{\int(-3)dx} = e^{-3x}$. We multiply the equation by this factor and recognize that

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 0 \qquad \text{is the same as} \qquad \frac{d}{dx}[e^{-3x}y] = 0.$$

Integrating both sides of the last equation gives $e^{-3x}y = c$. Solving for y gives us the explicit solution $y = ce^{3x}$, $-\infty < x < \infty$. ■

EXAMPLE 2 Solving a Nonhomogeneous Linear DE

Solve $\dfrac{dy}{dx} - 3y = 6$.

SOLUTION The associated homogeneous equation for this DE was solved in Example 1. Again the equation is already in the standard form (2), and the integrating factor is still $e^{\int(-3)dx} = e^{-3x}$. This time multiplying the given equation by this factor gives

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x}, \qquad \text{which is the same as} \qquad \frac{d}{dx}[e^{-3x}y] = 6e^{-3x}.$$

Integrating both sides of the last equation gives $e^{-3x}y = -2e^{-3x} + c$ or $y = -2 + ce^{3x}$, $-\infty < x < \infty$. ■

The final solution in Example 2 is the sum of two solutions: $y = y_c + y_p$, where $y_c = ce^{3x}$ is the solution of the homogeneous equation in Example 1 and $y_p = -2$ is a particular solution of the nonhomogeneous equation $y' - 3y = 6$. You need not be concerned about whether a linear first-order equation is homogeneous or nonhomogeneous; when you follow the solution procedure outlined above, a solution of a nonhomogeneous equation necessarily turns out to be $y = y_c + y_p$. However, the distinction between solving a homogeneous DE and solving a nonhomogeneous DE becomes more important in Chapter 4, where we solve linear higher-order equations.

When a_1, a_0, and g in (1) are constants, the differential equation is autonomous. In Example 2 you can verify from the normal form $dy/dx = 3(y + 2)$ that -2 is a critical point and that it is unstable (a repeller). Thus a solution curve with an initial point either above or below the graph of the equilibrium solution $y = -2$ pushes away from this horizontal line as x increases. Figure 2.3.1, obtained with the aid of a graphing utility, shows the graph of $y = -2$ along with some additional solution curves.

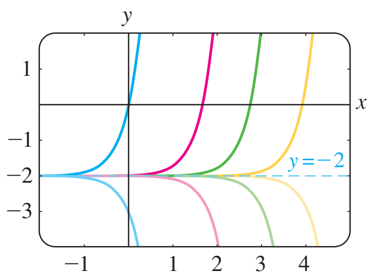


FIGURE 2.3.1 Some solutions of $y' - 3y = 6$

CONSTANT OF INTEGRATION Notice that in the general discussion and in Examples 1 and 2 we disregarded a constant of integration in the evaluation of the indefinite integral in the exponent of $e^{\int P(x)dx}$. If you think about the laws of exponents and the fact that the integrating factor multiplies both sides of the differential equation, you should be able to explain why writing $\int P(x)\,dx + c$ is unnecessary. See Problem 44 in Exercises 2.3.

GENERAL SOLUTION Suppose again that the functions P and f in (2) are continuous on a common interval I. In the steps leading to (4) we showed that *if* (2) has a solution on I, then it must be of the form given in (4). Conversely, it is a straightforward exercise in differentiation to verify that any function of the form given in (4) is a solution of the differential equation (2) on I. In other words, (4) is a one-parameter family of solutions of equation (2) and *every solution of* (2) *defined on I is a member of this family.* Therefore we call (4) the **general solution** of the differential equation on the interval I. (See the *Remarks* at the end of Section 1.1.) Now by writing (2) in the normal form $y' = F(x, y)$, we can identify $F(x, y) = -P(x)y + f(x)$ and $\partial F/\partial y = -P(x)$. From the continuity of P and f on the interval I we see that F and $\partial F/\partial y$ are also continuous on I. With Theorem 1.2.1 as our justification, we conclude that there exists one and only one solution of the initial-value problem

$$\frac{dy}{dx} + P(x)y = f(x), \quad y(x_0) = y_0 \tag{9}$$

defined on *some* interval I_0 containing x_0. But when x_0 is in I, finding a solution of (9) is just a matter of finding an appropriate value of c in (4)—that is, to each x_0 in I there corresponds a distinct c. In other words, the interval I_0 of existence and uniqueness in Theorem 1.2.1 for the initial-value problem (9) is the entire interval I.

EXAMPLE 3 General Solution

Solve $x\dfrac{dy}{dx} - 4y = x^6e^x$.

SOLUTION Dividing by x, we get the standard form

$$\frac{dy}{dx} - \frac{4}{x}y = x^5e^x. \tag{10}$$

From this form we identify $P(x) = -4/x$ and $f(x) = x^5e^x$ and further observe that P and f are continuous on $(0, \infty)$. Hence the integrating factor is

we can use $\ln x$ instead of $\ln |x|$ since $x > 0$

$$e^{-4\int dx/x} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}.$$

Here we have used the basic identity $b^{\log_b N} = N, N > 0$. Now we multiply (10) by x^{-4} and rewrite

$$x^{-4}\frac{dy}{dx} - 4x^{-5}y = xe^x \quad \text{as} \quad \frac{d}{dx}[x^{-4}y] = xe^x.$$

It follows from integration by parts that the general solution defined on the interval $(0, \infty)$ is $x^{-4}y = xe^x - e^x + c$ or $y = x^5e^x - x^4e^x + cx^4$. ■

Except in the case in which the lead coefficient is 1, the recasting of equation (1) into the standard form (2) requires division by $a_1(x)$. Values of x for which $a_1(x) = 0$ are called **singular points** of the equation. Singular points are potentially troublesome. Specifically, in (2), if $P(x)$ (formed by dividing $a_0(x)$ by $a_1(x)$) is discontinuous at a point, the discontinuity may carry over to solutions of the differential equation.

EXAMPLE 4 General Solution

Find the general solution of $(x^2 - 9)\dfrac{dy}{dx} + xy = 0$.

SOLUTION We write the differential equation in standard form

$$\frac{dy}{dx} + \frac{x}{x^2 - 9}y = 0 \tag{11}$$

and identify $P(x) = x/(x^2 - 9)$. Although P is continuous on $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$, we shall solve the equation on the first and third intervals. On these intervals the integrating factor is

$$e^{\int x\,dx/(x^2-9)} = e^{\frac{1}{2}\int 2x\,dx/(x^2-9)} = e^{\frac{1}{2}\ln|x^2-9|} = \sqrt{x^2 - 9}.$$

After multiplying the standard form (11) by this factor, we get

$$\frac{d}{dx}\left[\sqrt{x^2 - 9}\,y\right] = 0.$$

Integrating both sides of the last equation gives $\sqrt{x^2 - 9}\,y = c$. Thus for either $x > 3$ or $x < -3$ the general solution of the equation is $y = \dfrac{c}{\sqrt{x^2 - 9}}$. ■

Notice in Example 4 that $x = 3$ and $x = -3$ are singular points of the equation and that every function in the general solution $y = c/\sqrt{x^2 - 9}$ is discontinuous at these points. On the other hand, $x = 0$ is a singular point of the differential equation in Example 3, but the general solution $y = x^5e^x - x^4e^x + cx^4$ is noteworthy in that every function in this one-parameter family is continuous at $x = 0$ and is defined on the interval $(-\infty, \infty)$ and not just on $(0, \infty)$, as stated in the solution. However, the family $y = x^5e^x - x^4e^x + cx^4$ defined on $(-\infty, \infty)$ cannot be considered the general solution of the DE, since the singular point $x = 0$ still causes a problem. See Problem 39 in Exercises 2.3.

EXAMPLE 5 An Initial-Value Problem

Solve $\dfrac{dy}{dx} + y = x, \quad y(0) = 4.$

SOLUTION The equation is in standard form, and $P(x) = 1$ and $f(x) = x$ are continuous on $(-\infty, \infty)$. The integrating factor is $e^{\int dx} = e^x$, so integrating

$$\frac{d}{dx}[e^x y] = xe^x$$

gives $e^x y = xe^x - e^x + c$. Solving this last equation for y yields the general solution $y = x - 1 + ce^{-x}$. But from the initial condition we know that $y = 4$ when $x = 0$. Substituting these values into the general solution implies that $c = 5$. Hence the solution of the problem is

$$y = x - 1 + 5e^{-x}, \quad -\infty < x < \infty. \tag{12}$$ ■

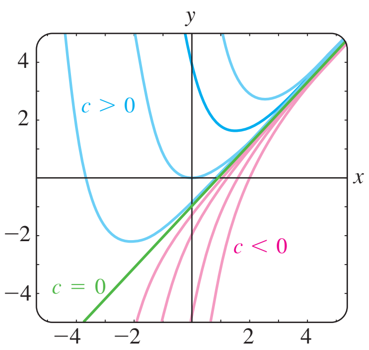


FIGURE 2.3.2 Some solutions of $y' + y = x$

Figure 2.3.2, obtained with the aid of a graphing utility, shows the graph of (12) in dark blue, along with the graphs of other representative solutions in the one-parameter family $y = x - 1 + ce^{-x}$. In this general solution we identify $y_c = ce^{-x}$ and $y_p = x - 1$. It is interesting to observe that as x increases, the graphs of *all* members of the family are close to the graph of the particular solution $y_p = x - 1$, which is shown in solid green in Figure 2.3.2. This is because the contribution of $y_c = ce^{-x}$ to the values of a solution becomes negligible for increasing values of x. We say that $y_c = ce^{-x}$ is a **transient term,** since $y_c \to 0$ as $x \to \infty$. While this behavior is not a characteristic of all general solutions of linear equations (see Example 2), the notion of a transient is often important in applied problems.

DISCONTINUOUS COEFFICIENTS In applications the coefficients $P(x)$ and $f(x)$ in (2) may be piecewise continuous. In the next example $f(x)$ is piecewise continuous on $[0, \infty)$ with a single discontinuity, namely, a (finite) jump discontinuity at $x = 1$. We solve the problem in two parts corresponding to the two intervals over which f is defined. It is then possible to piece together the two solutions at $x = 1$ so that $y(x)$ is continuous on $[0, \infty)$.

EXAMPLE 6 An Initial-Value Problem

Solve $\dfrac{dy}{dx} + y = f(x), \quad y(0) = 0 \quad \text{where} \quad f(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & x > 1. \end{cases}$

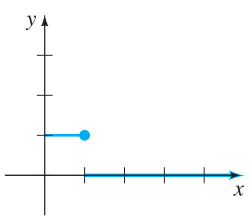


FIGURE 2.3.3 Discontinuous $f(x)$

SOLUTION The graph of the discontinuous function f is shown in Figure 2.3.3. We solve the DE for $y(x)$ first on the interval $[0, 1]$ and then on the interval $(1, \infty)$. For $0 \le x \le 1$ we have

$$\frac{dy}{dx} + y = 1 \qquad \text{or, equivalently,} \qquad \frac{d}{dx}[e^x y] = e^x.$$

Integrating this last equation and solving for y gives $y = 1 + c_1 e^{-x}$. Since $y(0) = 0$, we must have $c_1 = -1$, and therefore $y = 1 - e^{-x}$, $0 \le x \le 1$. Then for $x > 1$ the equation

$$\frac{dy}{dx} + y = 0$$

leads to $y = c_2e^{-x}$. Hence we can write

$$y = \begin{cases} 1 - e^{-x}, & 0 \le x \le 1, \\ c_2e^{-x}, & x > 1. \end{cases}$$

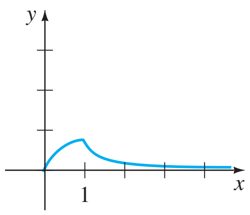


FIGURE 2.3.4 Graph of function in (13)

By appealing to the definition of continuity at a point, it is possible to determine c_2 so that the foregoing function is continuous at $x = 1$. The requirement that $\lim_{x\to 1^+} y(x) = y(1)$ implies that $c_2e^{-1} = 1 - e^{-1}$ or $c_2 = e - 1$. As seen in Figure 2.3.4, the function

$$y = \begin{cases} 1 - e^{-x}, & 0 \le x \le 1, \\ (e - 1)e^{-x}, & x > 1 \end{cases} \tag{13}$$

is continuous on $(0, \infty)$. ■

It is worthwhile to think about (13) and Figure 2.3.4 a little bit; you are urged to read and answer Problem 42 in Exercises 2.3.

FUNCTIONS DEFINED BY INTEGRALS At the end of Section 2.2 we discussed the fact that some simple continuous functions do not possess antiderivatives that are elementary functions and that integrals of these kinds of functions are called **nonelementary.** For example, you may have seen in calculus that $\int e^{-x^2}\,dx$ and $\int \sin x^2\,dx$ are nonelementary integrals. In applied mathematics some important functions are *defined* in terms of nonelementary integrals. Two such **special functions** are the **error function** and **complementary error function:**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \quad \text{and} \quad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt. \tag{14}$$

From the known result $\int_0^\infty e^{-t^2}\,dt = \sqrt{\pi}/2$* we can write $(2/\sqrt{\pi})\int_0^\infty e^{-t^2}\,dt = 1$. Then from $\int_0^\infty = \int_0^x + \int_x^\infty$ it is seen from (14) that the complementary error function $\operatorname{erfc}(x)$ is related to $\operatorname{erf}(x)$ by $\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$. Because of its importance in probability, statistics, and applied partial differential equations, the error function has been extensively tabulated. Note that $\operatorname{erf}(0) = 0$ is one obvious function value. Values of $\operatorname{erf}(x)$ can also be found by using a CAS.

EXAMPLE 7 The Error Function

Solve the initial-value problem $\dfrac{dy}{dx} - 2xy = 2, \quad y(0) = 1.$

SOLUTION Since the equation is already in standard form, we see that the integrating factor is $e^{-x^2}\,dx$, so from

$$\frac{d}{dx}[e^{-x^2}y] = 2e^{-x^2} \quad \text{we get} \quad y = 2e^{x^2}\int_0^x e^{-t^2}\,dt + ce^{x^2}. \tag{15}$$

Applying $y(0) = 1$ to the last expression then gives $c = 1$. Hence the solution of the problem is

$$y = 2e^{x^2}\int_0^x e^{-t^2}\,dt + e^{x^2} \text{ or } y = e^{x^2}[1 + \sqrt{\pi}\operatorname{erf}(x)].$$

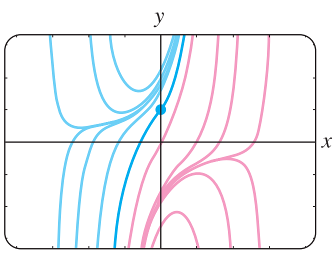


FIGURE 2.3.5 Some solutions of $y' - 2xy = 2$

The graph of this solution on the interval $(-\infty, \infty)$, shown in dark blue in Figure 2.3.5 among other members of the family defined in (15), was obtained with the aid of a computer algebra system. ■

*This result is usually proved in the third semester of calculus.

USE OF COMPUTERS The computer algebra systems *Mathematica* and *Maple* are capable of producing implicit or explicit solutions for some kinds of differential equations using their *dsolve* commands.*

REMARKS

(*i*) In general, a linear DE of any order is said to be homogeneous when $g(x) = 0$ in (6) of Section 1.1. For example, the linear second-order DE $y'' - 2y' + 6y = 0$ is homogeneous. As can be seen in this example and in the special case (3) of this section, the trivial solution $y = 0$ is always a solution of a homogeneous linear DE.

(*ii*) Occasionally, a first-order differential equation is not linear in one variable but is linear in the other variable. For example, the differential equation

$$\frac{dy}{dx} = \frac{1}{x + y^2}$$

is not linear in the variable y. But its reciprocal

$$\frac{dx}{dy} = x + y^2 \qquad \text{or} \qquad \frac{dx}{dy} - x = y^2$$

is recognized as linear in the variable x. You should verify that the integrating factor $e^{\int(-1)dy} = e^{-y}$ and integration by parts yield the explicit solution $x = -y^2 - 2y - 2 + ce^y$ for the second equation. This expression is, then, an implicit solution of the first equation.

(*iii*) Mathematicians have adopted as their own certain words from engineering, which they found appropriately descriptive. The word *transient,* used earlier, is one of these terms. In future discussions the words *input* and *output* will occasionally pop up. The function f in (2) is called the **input** or **driving function;** a solution $y(x)$ of the differential equation for a given input is called the **output** or **response.**

(*iv*) The term **special functions** mentioned in conjunction with the error function also applies to the **sine integral function** and the **Fresnel sine integral** introduced in Problems 49 and 50 in Exercises 2.3. "Special Functions" is actually a well-defined branch of mathematics. More special functions are studied in Section 6.3.

*Certain commands have the same spelling, but in *Mathematica* commands begin with a capital letter **(Dsolve),** whereas in *Maple* the same command begins with a lower case letter **(dsolve).** When discussing such common syntax, we compromise and write, for example, *dsolve.* See the *Student Resource and Solutions Manual* for the complete input commands used to solve a linear first-order DE.

EXERCISES 2.3

Answers to selected odd-numbered problems begin on page ANS-2.

In Problems 1–24 find the general solution of the given differential equation. Give the largest interval I over which the general solution is defined. Determine whether there are any transient terms in the general solution.

1. $\dfrac{dy}{dx} = 5y$

2. $\dfrac{dy}{dx} + 2y = 0$

3. $\dfrac{dy}{dx} + y = e^{3x}$

4. $3\dfrac{dy}{dx} + 12y = 4$

5. $y' + 3x^2y = x^2$

6. $y' + 2xy = x^3$

7. $x^2y' + xy = 1$

8. $y' = 2y + x^2 + 5$

9. $x\dfrac{dy}{dx} - y = x^2 \sin x$

10. $x\dfrac{dy}{dx} + 2y = 3$

11. $x\dfrac{dy}{dx} + 4y = x^3 - x$

12. $(1 + x)\dfrac{dy}{dx} - xy = x + x^2$

13. $x^2y' + x(x + 2)y = e^x$

14. $xy' + (1 + x)y = e^{-x}\sin 2x$

15. $y\,dx - 4(x + y^6)\,dy = 0$

16. $y\,dx = (ye^y - 2x)\,dy$

17. $\cos x\dfrac{dy}{dx} + (\sin x)y = 1$

18. $\cos^2 x\sin x\dfrac{dy}{dx} + (\cos^3 x)y = 1$

19. $(x + 1)\dfrac{dy}{dx} + (x + 2)y = 2xe^{-x}$

20. $(x + 2)^2\dfrac{dy}{dx} = 5 - 8y - 4xy$

21. $\dfrac{dr}{d\theta} + r\sec\theta = \cos\theta$

22. $\dfrac{dP}{dt} + 2tP = P + 4t - 2$

23. $x\dfrac{dy}{dx} + (3x + 1)y = e^{-3x}$

24. $(x^2 - 1)\dfrac{dy}{dx} + 2y = (x + 1)^2$

In Problems 25–30 solve the given initial-value problem. Give the largest interval I over which the solution is defined.

25. $xy' + y = e^x, \quad y(1) = 2$

26. $y\dfrac{dx}{dy} - x = 2y^2, \quad y(1) = 5$

27. $L\dfrac{di}{dt} + Ri = E, \quad i(0) = i_0,$
L, R, E, and i_0 constants

28. $\dfrac{dT}{dt} = k(T - T_m); \quad T(0) = T_0,$
k, T_m, and T_0 constants

29. $(x + 1)\dfrac{dy}{dx} + y = \ln x, \quad y(1) = 10$

30. $y' + (\tan x)y = \cos^2 x, \quad y(0) = -1$

In Problems 31–34 proceed as in Example 6 to solve the given initial-value problem. Use a graphing utility to graph the continuous function $y(x)$.

31. $\dfrac{dy}{dx} + 2y = f(x), y(0) = 0,$ where

$$f(x) = \begin{cases} 1, & 0 \le x \le 3 \\ 0, & x > 3 \end{cases}$$

32. $\dfrac{dy}{dx} + y = f(x), y(0) = 1,$ where

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ -1, & x > 1 \end{cases}$$

33. $\dfrac{dy}{dx} + 2xy = f(x), y(0) = 2,$ where

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases}$$

34. $(1 + x^2)\dfrac{dy}{dx} + 2xy = f(x), y(0) = 0,$ where

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ -x, & x \ge 1 \end{cases}$$

35. Proceed in a manner analogous to Example 6 to solve the initial-value problem $y' + P(x)y = 4x$, $y(0) = 3$, where

$$P(x) = \begin{cases} 2, & 0 \le x \le 1, \\ -2/x, & x > 1. \end{cases}$$

Use a graphing utility to graph the continuous function $y(x)$.

36. Consider the initial-value problem $y' + e^x y = f(x)$, $y(0) = 1$. Express the solution of the IVP for $x > 0$ as a nonelementary integral when $f(x) = 1$. What is the solution when $f(x) = 0$? When $f(x) = e^x$?

37. Express the solution of the initial-value problem $y' - 2xy = 1$, $y(1) = 1$, in terms of $\operatorname{erf}(x)$.

Discussion Problems

38. Reread the discussion following Example 2. Construct a linear first-order differential equation for which all nonconstant solutions approach the horizontal asymptote $y = 4$ as $x \to \infty$.

39. Reread Example 3 and then discuss, with reference to Theorem 1.2.1, the existence and uniqueness of a solution of the initial-value problem consisting of $xy' - 4y = x^6e^x$ and the given initial condition.

(a) $y(0) = 0$ **(b)** $y(0) = y_0, y_0 > 0$

(c) $y(x_0) = y_0, x_0 > 0, y_0 > 0$

40. Reread Example 4 and then find the general solution of the differential equation on the interval $(-3, 3)$.

41. Reread the discussion following Example 5. Construct a linear first-order differential equation for which all solutions are asymptotic to the line $y = 3x - 5$ as $x \to \infty$.

42. Reread Example 6 and then discuss why it is technically incorrect to say that the function in (13) is a "solution" of the IVP on the interval $[0, \infty)$.

43. **(a)** Construct a linear first-order differential equation of the form $xy' + a_0(x)y = g(x)$ for which $y_c = c/x^3$ and $y_p = x^3$. Give an interval on which $y = x^3 + c/x^3$ is the general solution of the DE.

(b) Give an initial condition $y(x_0) = y_0$ for the DE found in part (a) so that the solution of the IVP is $y = x^3 - 1/x^3$. Repeat if the solution is

$y = x^3 + 2/x^3$. Give an interval I of definition of each of these solutions. Graph the solution curves. Is there an initial-value problem whose solution is defined on $(-\infty, \infty)$?

(c) Is each IVP found in part (b) unique? That is, can there be more than one IVP for which, say, $y = x^3 - 1/x^3$, x in some interval I, is the solution?

44. In determining the integrating factor (5), we did not use a constant of integration in the evaluation of $\int P(x)\,dx$. Explain why using $\int P(x)\,dx + c$ has no effect on the solution of (2).

45. Suppose $P(x)$ is continuous on some interval I and a is a number in I. What can be said about the solution of the initial-value problem $y' + P(x)y = 0$, $y(a) = 0$?

Mathematical Models

46. Radioactive Decay Series The following system of differential equations is encountered in the study of the decay of a special type of radioactive series of elements:

$$\frac{dx}{dt} = -\lambda_1 x$$

$$\frac{dy}{dt} = \lambda_1 x - \lambda_2 y,$$

where λ_1 and λ_2 are constants. Discuss how to solve this system subject to $x(0) = x_0$, $y(0) = y_0$. Carry out your ideas.

47. Heart Pacemaker A heart pacemaker consists of a switch, a battery of constant voltage E_0, a capacitor with constant capacitance C, and the heart as a resistor with constant resistance R. When the switch is closed, the capacitor charges; when the switch is open, the capacitor discharges, sending an electrical stimulus to the heart. During the time the heart is being stimulated, the voltage E across the heart satisfies the linear differential equation

$$\frac{dE}{dt} = -\frac{1}{RC}E.$$

Solve the DE subject to $E(4) = E_0$.

Computer Lab Assignments

48. (a) Express the solution of the initial-value problem $y' - 2xy = -1$, $y(0) = \sqrt{\pi}/2$, in terms of erfc(x).

(b) Use tables or a CAS to find the value of $y(2)$. Use a CAS to graph the solution curve for the IVP on $(-\infty, \infty)$.

49. (a) The **sine integral function** is defined by $\text{Si}(x) = \int_0^x (\sin t/t)\,dt$, where the integrand is defined to be 1 at $t = 0$. Express the solution $y(x)$ of the initial-value problem $x^3y' + 2x^2y = 10\sin x$, $y(1) = 0$ in terms of Si(x).

(b) Use a CAS to graph the solution curve for the IVP for $x > 0$.

(c) Use a CAS to find the value of the absolute maximum of the solution $y(x)$ for $x > 0$.

50. (a) The **Fresnel sine integral** is defined by $S(x) = \int_0^x \sin(\pi t^2/2)\,dt$. Express the solution $y(x)$ of the initial-value problem $y' - (\sin x^2)y = 0$, $y(0) = 5$, in terms of $S(x)$.

(b) Use a CAS to graph the solution curve for the IVP on $(-\infty, \infty)$.

(c) It is known that $S(x) \to \frac{1}{2}$ as $x \to \infty$ and $S(x) \to -\frac{1}{2}$ as $x \to -\infty$. What does the solution $y(x)$ approach as $x \to \infty$? As $x \to -\infty$?

(d) Use a CAS to find the values of the absolute maximum and the absolute minimum of the solution $y(x)$.

2.4 EXACT EQUATIONS

REVIEW MATERIAL

- Multivariate calculus
- Partial differentiation and partial integration
- Differential of a function of two variables

INTRODUCTION Although the simple first-order equation

$$y\,dx + x\,dy = 0$$

is separable, we can solve the equation in an alternative manner by recognizing that the expression on the left-hand side of the equality is the differential of the function $f(x, y) = xy$; that is,

$$d(xy) = y\,dx + x\,dy.$$

In this section we examine first-order equations in differential form $M(x, y)\,dx + N(x, y)\,dy = 0$. By applying a simple test to M and N, we can determine whether $M(x, y)\,dx + N(x, y)\,dy$ is a differential of a function $f(x, y)$. If the answer is yes, we can construct f by partial integration.

DIFFERENTIAL OF A FUNCTION OF TWO VARIABLES If $z = f(x, y)$ is a function of two variables with continuous first partial derivatives in a region R of the xy-plane, then its differential is

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy. \tag{1}$$

In the special case when $f(x, y) = c$, where c is a constant, then (1) implies

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0. \tag{2}$$

In other words, given a one-parameter family of functions $f(x, y) = c$, we can generate a first-order differential equation by computing the differential of both sides of the equality. For example, if $x^2 - 5xy + y^3 = c$, then (2) gives the first-order DE

$$(2x - 5y)\,dx + (-5x + 3y^2)\,dy = 0. \tag{3}$$

A DEFINITION Of course, not every first-order DE written in differential form $M(x, y)\,dx + N(x, y)\,dy = 0$ corresponds to a differential of $f(x, y) = c$. So for our purposes it is more important to turn the foregoing example around; namely, if we are given a first-order DE such as (3), is there some way we can recognize that the differential expression $(2x - 5y)\,dx + (-5x + 3y^2)\,dy$ is the differential $d(x^2 - 5xy + y^3)$? If there is, then an implicit solution of (3) is $x^2 - 5xy + y^3 = c$. We answer this question after the next definition.

DEFINITION 2.4.1 Exact Equation

A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$ defined in R. A first-order differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an **exact equation** if the expression on the left-hand side is an exact differential.

For example, $x^2y^3\,dx + x^3y^2\,dy = 0$ is an exact equation, because its left-hand side is an exact differential:

$$d\left(\tfrac{1}{3}x^3y^3\right) = x^2y^3\,dx + x^3y^2\,dy.$$

Notice that if we make the identifications $M(x, y) = x^2y^3$ and $N(x, y) = x^3y^2$, then $\partial M/\partial y = 3x^2y^2 = \partial N/\partial x$. Theorem 2.4.1, given next, shows that the equality of the partial derivatives $\partial M/\partial y$ and $\partial N/\partial x$ is no coincidence.

THEOREM 2.4.1 Criterion for an Exact Differential

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b$, $c < y < d$. Then a necessary and sufficient condition that $M(x, y)\,dx + N(x, y)\,dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{4}$$

PROOF OF THE NECESSITY For simplicity let us assume that $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives for all (x, y). Now if the expression $M(x, y)\,dx + N(x, y)\,dy$ is exact, there exists some function f such that for all x in R,

$$M(x, y)\,dx + N(x, y)\,dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Therefore $$M(x, y) = \frac{\partial f}{\partial x}, \qquad N(x, y) = \frac{\partial f}{\partial y},$$

and $$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial N}{\partial x}.$$

The equality of the mixed partials is a consequence of the continuity of the first partial derivatives of $M(x, y)$ and $N(x, y)$. ■

The sufficiency part of Theorem 2.4.1 consists of showing that there exists a function f for which $\partial f/\partial x = M(x, y)$ and $\partial f/\partial y = N(x, y)$ whenever (4) holds. The construction of the function f actually reflects a basic procedure for solving exact equations.

METHOD OF SOLUTION Given an equation in the differential form $M(x, y)\,dx + N(x, y)\,dy = 0$, determine whether the equality in (4) holds. If it does, then there exists a function f for which

$$\frac{\partial f}{\partial x} = M(x, y).$$

We can find f by integrating $M(x, y)$ with respect to x while holding y constant:

$$f(x, y) = \int M(x, y)\,dx + g(y), \tag{5}$$

where the arbitrary function $g(y)$ is the "constant" of integration. Now differentiate (5) with respect to y and assume that $\partial f/\partial y = N(x, y)$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x, y)\,dx + g'(y) = N(x, y).$$

This gives $$g'(y) = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)\,dx. \tag{6}$$

Finally, integrate (6) with respect to y and substitute the result in (5). The implicit solution of the equation is $f(x, y) = c$.

Some observations are in order. First, it is important to realize that the expression $N(x, y) - (\partial/\partial y)\int M(x, y)\,dx$ in (6) is independent of x, because

$$\frac{\partial}{\partial x}\left[N(x, y) - \frac{\partial}{\partial y}\int M(x, y)\,dx\right] = \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\int M(x, y)\,dx\right) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

Second, we could just as well start the foregoing procedure with the assumption that $\partial f/\partial y = N(x, y)$. After integrating N with respect to y and then differentiating that result, we would find the analogues of (5) and (6) to be, respectively,

$$f(x, y) = \int N(x, y)\,dy + h(x) \qquad \text{and} \qquad h'(x) = M(x, y) - \frac{\partial}{\partial x}\int N(x, y)\,dy.$$

In either case *none of these formulas should be memorized.*

EXAMPLE 1 Solving an Exact DE

Solve $2xy\,dx + (x^2 - 1)\,dy = 0$.

SOLUTION With $M(x, y) = 2xy$ and $N(x, y) = x^2 - 1$ we have

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Thus the equation is exact, and so by Theorem 2.4.1 there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 - 1.$$

From the first of these equations we obtain, after integrating,

$$f(x, y) = x^2y + g(y).$$

Taking the partial derivative of the last expression with respect to y and setting the result equal to $N(x, y)$ gives

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1. \qquad \leftarrow N(x, y)$$

It follows that $g'(y) = -1$ and $g(y) = -y$. Hence $f(x, y) = x^2y - y$, so the solution of the equation in implicit form is $x^2y - y = c$. The explicit form of the solution is easily seen to be $y = c/(1 - x^2)$ and is defined on any interval not containing either $x = 1$ or $x = -1$. ■

NOTE The solution of the DE in Example 1 is *not* $f(x, y) = x^2y - y$. Rather, it is $f(x, y) = c$; if a constant is used in the integration of $g'(y)$, we can then write the solution as $f(x, y) = 0$. Note, too, that the equation could be solved by separation of variables.

EXAMPLE 2 Solving an Exact DE

Solve $(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0$.

SOLUTION The equation is exact because

$$\frac{\partial M}{\partial y} = 2e^{2y} + xy\sin xy - \cos xy = \frac{\partial N}{\partial x}.$$

Hence a function $f(x, y)$ exists for which

$$M(x, y) = \frac{\partial f}{\partial x} \quad \text{and} \quad N(x, y) = \frac{\partial f}{\partial y}.$$

Now for variety we shall start with the assumption that $\partial f/\partial y = N(x, y)$; that is,

$$\frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$$

$$f(x, y) = 2x\int e^{2y}\,dy - x\int \cos xy\,dy + 2\int y\,dy.$$

Remember, the reason x can come out in front of the symbol $\int$ is that in the integration with respect to y, x is treated as an ordinary constant. It follows that

$$f(x, y) = xe^{2y} - \sin xy + y^2 + h(x)$$

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy + h'(x) = e^{2y} - y\cos xy, \quad \leftarrow M(x, y)$$

and so $h'(x) = 0$ or $h(x) = c$. Hence a family of solutions is

$$xe^{2y} - \sin xy + y^2 + c = 0.$$

■

EXAMPLE 3 An Initial-Value Problem

Solve $\dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y(1 - x^2)}, \quad y(0) = 2.$

SOLUTION By writing the differential equation in the form

$$(\cos x \sin x - xy^2)\,dx + y(1 - x^2)\,dy = 0,$$

we recognize that the equation is exact because

$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}.$$

Now
$$\frac{\partial f}{\partial y} = y(1 - x^2)$$

$$f(x, y) = \frac{y^2}{2}(1 - x^2) + h(x)$$

$$\frac{\partial f}{\partial x} = -xy^2 + h'(x) = \cos x \sin x - xy^2.$$

The last equation implies that $h'(x) = \cos x \sin x$. Integrating gives

$$h(x) = -\int (\cos x)(-\sin x\,dx) = -\frac{1}{2}\cos^2 x.$$

Thus
$$\frac{y^2}{2}(1 - x^2) - \frac{1}{2}\cos^2 x = c_1 \quad \text{or} \quad y^2(1 - x^2) - \cos^2 x = c, \tag{7}$$

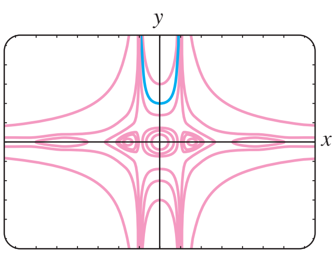


FIGURE 2.4.1 Some graphs of members of the family $y^2(1 - x^2) - \cos^2 x = c$

where $2c_1$ has been replaced by c. The initial condition $y = 2$ when $x = 0$ demands that $4(1) - \cos^2(0) = c$, and so $c = 3$. An implicit solution of the problem is then $y^2(1 - x^2) - \cos^2 x = 3$.

The solution curve of the IVP is the curve drawn in dark blue in Figure 2.4.1; it is part of an interesting family of curves. The graphs of the members of the one-parameter family of solutions given in (7) can be obtained in several ways, two of which are using software to graph level curves (as discussed in Section 2.2) and using a graphing utility to carefully graph the explicit functions obtained for various values of c by solving $y^2 = (c + \cos^2 x)/(1 - x^2)$ for y. ■

INTEGRATING FACTORS Recall from Section 2.3 that the left-hand side of the linear equation $y' + P(x)y = f(x)$ can be transformed into a derivative when we multiply the equation by an integrating factor. The same basic idea sometimes works for a nonexact differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$. That is, it is sometimes possible to find an **integrating factor** $\mu(x, y)$ so that after multiplying, the left-hand side of

$$\mu(x, y)M(x, y)\,dx + \mu(x, y)N(x, y)\,dy = 0 \tag{8}$$

is an exact differential. In an attempt to find μ, we turn to the criterion (4) for exactness. Equation (8) is exact if and only if $(\mu M)_y = (\mu N)_x$, where the subscripts denote partial derivatives. By the Product Rule of differentiation the last equation is the same as $\mu M_y + \mu_y M = \mu N_x + \mu_x N$ or

$$\mu_x N - \mu_y M = (M_y - N_x)\mu. \tag{9}$$

Although M, N, M_y, and N_x are known functions of x and y, the difficulty here in determining the unknown $\mu(x, y)$ from (9) is that we must solve a partial differential equation. Since we are not prepared to do that, we make a simplifying assumption. Suppose μ is a function of one variable; for example, say that μ depends only on x. In this case, $\mu_x = d\mu/dx$ and $\mu_y = 0$, so (9) can be written as

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu. \tag{10}$$

We are still at an impasse if the quotient $(M_y - N_x)/N$ depends on both x and y. However, if after all obvious algebraic simplifications are made, the quotient $(M_y - N_x)/N$ turns out to depend solely on the variable x, then (10) is a first-order ordinary differential equation. We can finally determine μ because (10) is *separable* as well as *linear*. It follows from either Section 2.2 or Section 2.3 that $\mu(x) = e^{\int((M_y - N_x)/N)\,dx}$. In like manner, it follows from (9) that if μ depends only on the variable y, then

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu. \tag{11}$$

In this case, if $(N_x - M_y)/M$ is a function of y only, then we can solve (11) for μ.

We summarize the results for the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0. \tag{12}$$

- If $(M_y - N_x)/N$ is a function of x alone, then an integrating factor for (12) is

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}dx}. \tag{13}$$

- If $(N_x - M_y)/M$ is a function of y alone, then an integrating factor for (12) is

$$\mu(y) = e^{\int \frac{N_x - M_y}{M}dy}. \tag{14}$$

EXAMPLE 4 A Nonexact DE Made Exact

The nonlinear first-order differential equation

$$xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$$

is not exact. With the identifications $M = xy$, $N = 2x^2 + 3y^2 - 20$, we find the partial derivatives $M_y = x$ and $N_x = 4x$. The first quotient from (13) gets us nowhere, since

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20}$$

depends on x and y. However, (14) yields a quotient that depends only on y:

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y}.$$

The integrating factor is then $e^{\int 3dy/y} = e^{3\ln y} = e^{\ln y^3} = y^3$. After we multiply the given DE by $\mu(y) = y^3$, the resulting equation is

$$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0.$$

You should verify that the last equation is now exact as well as show, using the method of this section, that a family of solutions is $\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c$. ■

REMARKS

(*i*) When testing an equation for exactness, make sure it is of the precise form $M(x, y)\,dx + N(x, y)\,dy = 0$. Sometimes a differential equation is written $G(x, y)\,dx = H(x, y)\,dy$. In this case, first rewrite it as $G(x, y)\,dx - H(x, y)\,dy = 0$ and then identify $M(x, y) = G(x, y)$ and $N(x, y) = -H(x, y)$ before using (4).

(*ii*) In some texts on differential equations the study of exact equations precedes that of linear DEs. Then the method for finding integrating factors just discussed can be used to derive an integrating factor for $y' + P(x)y = f(x)$. By rewriting the last equation in the differential form $(P(x)y - f(x))\,dx + dy = 0$, we see that

$$\frac{M_y - N_x}{N} = P(x).$$

From (13) we arrive at the already familiar integrating factor $e^{\int P(x)dx}$, used in Section 2.3.

EXERCISES 2.4

Answers to selected odd-numbered problems begin on page ANS-2.

In Problems 1–20 determine whether the given differential equation is exact. If it is exact, solve it.

1. $(2x - 1)\,dx + (3y + 7)\,dy = 0$

2. $(2x + y)\,dx - (x + 6y)\,dy = 0$

3. $(5x + 4y)\,dx + (4x - 8y^3)\,dy = 0$

4. $(\sin y - y\sin x)\,dx + (\cos x + x\cos y - y)\,dy = 0$

5. $(2xy^2 - 3)\,dx + (2x^2y + 4)\,dy = 0$

6. $\left(2y - \dfrac{1}{x} + \cos 3x\right)\dfrac{dy}{dx} + \dfrac{y}{x^2} - 4x^3 + 3y\sin 3x = 0$

7. $(x^2 - y^2)\,dx + (x^2 - 2xy)\,dy = 0$

8. $\left(1 + \ln x + \dfrac{y}{x}\right)dx = (1 - \ln x)\,dy$

9. $(x - y^3 + y^2\sin x)\,dx = (3xy^2 + 2y\cos x)\,dy$

10. $(x^3 + y^3)\,dx + 3xy^2\,dy = 0$

11. $(y\ln y - e^{-xy})\,dx + \left(\dfrac{1}{y} + x\ln y\right)dy = 0$

12. $(3x^2y + e^y)\,dx + (x^3 + xe^y - 2y)\,dy = 0$

13. $x\dfrac{dy}{dx} = 2xe^x - y + 6x^2$

14. $\left(1 - \dfrac{3}{y} + x\right)\dfrac{dy}{dx} + y = \dfrac{3}{x} - 1$

15. $\left(x^2y^3 - \dfrac{1}{1 + 9x^2}\right)\dfrac{dx}{dy} + x^3y^2 = 0$

16. $(5y - 2x)y' - 2y = 0$

17. $(\tan x - \sin x\sin y)\,dx + \cos x\cos y\,dy = 0$

18. $(2y\sin x\cos x - y + 2y^2e^{xy^2})\,dx = (x - \sin^2 x - 4xye^{xy^2})\,dy$

19. $(4t^3y - 15t^2 - y)\,dt + (t^4 + 3y^2 - t)\,dy = 0$

20. $\left(\dfrac{1}{t} + \dfrac{1}{t^2} - \dfrac{y}{t^2 + y^2}\right)dt + \left(ye^y + \dfrac{t}{t^2 + y^2}\right)dy = 0$

In Problems 21–26 solve the given initial-value problem.

21. $(x + y)^2\,dx + (2xy + x^2 - 1)\,dy = 0, \quad y(1) = 1$

22. $(e^x + y)\,dx + (2 + x + ye^y)\,dy = 0, \quad y(0) = 1$

23. $(4y + 2t - 5)\,dt + (6y + 4t - 1)\,dy = 0, \quad y(-1) = 2$

24. $\left(\dfrac{3y^2 - t^2}{y^5}\right)\dfrac{dy}{dt} + \dfrac{t}{2y^4} = 0, \quad y(1) = 1$

25. $(y^2 \cos x - 3x^2y - 2x)\,dx$
$+ (2y \sin x - x^3 + \ln y)\,dy = 0, \quad y(0) = e$

26. $\left(\dfrac{1}{1 + y^2} + \cos x - 2xy\right)\dfrac{dy}{dx} = y(y + \sin x), \; y(0) = 1$

In Problems 27 and 28 find the value of k so that the given differential equation is exact.

27. $(y^3 + kxy^4 - 2x)\,dx + (3xy^2 + 20x^2y^3)\,dy = 0$

28. $(6xy^3 + \cos y)\,dx + (2kx^2y^2 - x \sin y)\,dy = 0$

In Problems 29 and 30 verify that the given differential equation is not exact. Multiply the given differential equation by the indicated integrating factor $\mu(x, y)$ and verify that the new equation is exact. Solve.

29. $(-xy \sin x + 2y \cos x)\,dx + 2x \cos x\,dy = 0;$
$\mu(x, y) = xy$

30. $(x^2 + 2xy - y^2)\,dx + (y^2 + 2xy - x^2)\,dy = 0;$
$\mu(x, y) = (x + y)^{-2}$

In Problems 31–36 solve the given differential equation by finding, as in Example 4, an appropriate integrating factor.

31. $(2y^2 + 3x)\,dx + 2xy\,dy = 0$

32. $y(x + y + 1)\,dx + (x + 2y)\,dy = 0$

33. $6xy\,dx + (4y + 9x^2)\,dy = 0$

34. $\cos x\,dx + \left(1 + \dfrac{2}{y}\right)\sin x\,dy = 0$

35. $(10 - 6y + e^{-3x})\,dx - 2\,dy = 0$

36. $(y^2 + xy^3)\,dx + (5y^2 - xy + y^3 \sin y)\,dy = 0$

In Problems 37 and 38 solve the given initial-value problem by finding, as in Example 4, an appropriate integrating factor.

37. $x\,dx + (x^2y + 4y)\,dy = 0, \quad y(4) = 0$

38. $(x^2 + y^2 - 5)\,dx = (y + xy)\,dy, \quad y(0) = 1$

39. **(a)** Show that a one-parameter family of solutions of the equation

$$(4xy + 3x^2)\,dx + (2y + 2x^2)\,dy = 0$$

is $x^3 + 2x^2y + y^2 = c$.

(b) Show that the initial conditions $y(0) = -2$ and $y(1) = 1$ determine the same implicit solution.

(c) Find explicit solutions $y_1(x)$ and $y_2(x)$ of the differential equation in part (a) such that $y_1(0) = -2$ and $y_2(1) = 1$. Use a graphing utility to graph $y_1(x)$ and $y_2(x)$.

Discussion Problems

40. Consider the concept of an integrating factor used in Problems 29–38. Are the two equations $M\,dx + N\,dy = 0$ and $\mu M\,dx + \mu N\,dy = 0$ necessarily equivalent in the sense that a solution of one is also a solution of the other? Discuss.

41. Reread Example 3 and then discuss why we can conclude that the interval of definition of the explicit solution of the IVP (the blue curve in Figure 2.4.1) is $(-1, 1)$.

42. Discuss how the functions $M(x, y)$ and $N(x, y)$ can be found so that each differential equation is exact. Carry out your ideas.

(a) $M(x, y)\,dx + \left(xe^{xy} + 2xy + \dfrac{1}{x}\right)dy = 0$

(b) $\left(x^{-1/2}y^{1/2} + \dfrac{x}{x^2 + y}\right)dx + N(x, y)\,dy = 0$

43. Differential equations are sometimes solved by having a clever idea. Here is a little exercise in cleverness: Although the differential equation $(x - \sqrt{x^2 + y^2})\,dx + y\,dy = 0$ is not exact, show how the rearrangement $(x\,dx + y\,dy)/\sqrt{x^2 + y^2} = dx$ and the observation $\frac{1}{2}d(x^2 + y^2) = x\,dx + y\,dy$ can lead to a solution.

44. True or False: Every separable first-order equation $dy/dx = g(x)h(y)$ is exact.

Mathematical Model

45. **Falling Chain** A portion of a uniform chain of length 8 ft is loosely coiled around a peg at the edge of a high horizontal platform, and the remaining portion of the chain hangs at rest over the edge of the platform. See Figure 2.4.2. Suppose that the length of the overhanging chain is 3 ft, that the chain weighs 2 lb/ft, and that the positive direction is downward. Starting at $t = 0$ seconds, the weight of the overhanging portion causes the chain on the table to uncoil smoothly and to fall to the floor. If $x(t)$ denotes the length of the chain overhanging the table at time $t > 0$, then $v = dx/dt$ is its velocity. When all resistive forces are ignored, it can be shown that a mathematical model relating v to x is given by

$$xv\frac{dv}{dx} + v^2 = 32x.$$

(a) Rewrite this model in differential form. Proceed as in Problems 31–36 and solve the DE for v in terms of x by finding an appropriate integrating factor. Find an explicit solution $v(x)$.

(b) Determine the velocity with which the chain leaves the platform.

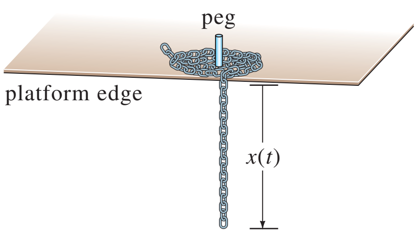


FIGURE 2.4.2 Uncoiling chain in Problem 45

Computer Lab Assignments

46. Streamlines

(a) The solution of the differential equation

$$\frac{2xy}{(x^2 + y^2)^2}\,dx + \left[1 + \frac{y^2 - x^2}{(x^2 + y^2)^2}\right]dy = 0$$

is a family of curves that can be interpreted as streamlines of a fluid flow around a circular object whose boundary is described by the equation $x^2 + y^2 = 1$. Solve this DE and note the solution $f(x, y) = c$ for $c = 0$.

(b) Use a CAS to plot the streamlines for $c = 0,\ \pm0.2,\ \pm0.4,\ \pm0.6$, and ±0.8 in three different ways. First, use the *contourplot* of a CAS. Second, solve for x in terms of the variable y. Plot the resulting two functions of y for the given values of c, and then combine the graphs. Third, use the CAS to solve a cubic equation for y in terms of x.

2.5 SOLUTIONS BY SUBSTITUTIONS

REVIEW MATERIAL

- Techniques of integration
- Separation of variables
- Solution of linear DEs

INTRODUCTION We usually solve a differential equation by recognizing it as a certain kind of equation (say, separable, linear, or exact) and then carrying out a procedure, consisting of *equation-specific mathematical steps,* that yields a solution of the equation. But it is not uncommon to be stumped by a differential equation because it does not fall into one of the classes of equations that we know how to solve. The procedures that are discussed in this section may be helpful in this situation.

SUBSTITUTIONS Often the first step in solving a differential equation consists of transforming it into another differential equation by means of a **substitution.** For example, suppose we wish to transform the first-order differential equation $dy/dx = f(x, y)$ by the substitution $y = g(x, u)$, where u is regarded as a function of the variable x. If g possesses first-partial derivatives, then the Chain Rule

$$\frac{dy}{dx} = \frac{\partial g}{\partial x}\frac{dx}{dx} + \frac{\partial g}{\partial u}\frac{du}{dx} \qquad \text{gives} \qquad \frac{dy}{dx} = g_x(x, u) + g_u(x, u)\frac{du}{dx}.$$

If we replace dy/dx by the foregoing derivative and replace y in $f(x, y)$ by $g(x, u)$, then the DE $dy/dx = f(x, y)$ becomes $g_x(x, u) + g_u(x, u)\dfrac{du}{dx} = f(x, g(x, u))$, which, solved for du/dx, has the form $\dfrac{du}{dx} = F(x, u)$. If we can determine a solution $u = \phi(x)$ of this last equation, then a solution of the original differential equation is $y = g(x, \phi(x))$.

In the discussion that follows we examine three different kinds of first-order differential equations that are solvable by means of a substitution.

HOMOGENEOUS EQUATIONS If a function f possesses the property $f(tx, ty) = t^{\alpha} f(x, y)$ for some real number α, then f is said to be a **homogeneous function** of degree α. For example, $f(x, y) = x^3 + y^3$ is a homogeneous function of degree 3, since

$$f(tx, ty) = (tx)^3 + (ty)^3 = t^3(x^3 + y^3) = t^3 f(x, y),$$

whereas $f(x, y) = x^3 + y^3 + 1$ is not homogeneous. A first-order DE in differential form

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{1}$$

is said to be **homogeneous*** if both coefficient functions M and N are homogeneous equations of the *same* degree. In other words, (1) is homogeneous if

$$M(tx, ty) = t^{\alpha} M(x, y) \qquad \text{and} \qquad N(tx, ty) = t^{\alpha} N(x, y).$$

In addition, if M and N are homogeneous functions of degree α, we can also write

$$M(x, y) = x^{\alpha} M(1, u) \qquad \text{and} \qquad N(x, y) = x^{\alpha} N(1, u), \quad \text{where } u = y/x, \tag{2}$$

and

$$M(x, y) = y^{\alpha} M(v, 1) \qquad \text{and} \qquad N(x, y) = y^{\alpha} N(v, 1), \quad \text{where } v = x/y. \tag{3}$$

See Problem 31 in Exercises 2.5. Properties (2) and (3) suggest the substitutions that can be used to solve a homogeneous differential equation. Specifically, *either* of the substitutions $y = ux$ or $x = vy$, where u and v are new dependent variables, will reduce a homogeneous equation to a *separable* first-order differential equation. To show this, observe that as a consequence of (2) a homogeneous equation $M(x, y)\, dx + N(x, y)\, dy = 0$ can be rewritten as

$$x^{\alpha} M(1, u)\, dx + x^{\alpha} N(1, u)\, dy = 0 \qquad \text{or} \qquad M(1, u)\, dx + N(1, u)\, dy = 0,$$

where $u = y/x$ or $y = ux$. By substituting the differential $dy = u\, dx + x\, du$ into the last equation and gathering terms, we obtain a separable DE in the variables u and x:

$$M(1, u)\, dx + N(1, u)[u\, dx + x\, du] = 0$$

$$[M(1, u) + uN(1, u)]\, dx + xN(1, u)\, du = 0$$

or

$$\frac{dx}{x} + \frac{N(1, u)\, du}{M(1, u) + uN(1, u)} = 0.$$

At this point we offer the same advice as in the preceding sections: Do not memorize anything here (especially the last formula); rather, *work through the procedure each time.* The proof that the substitutions $x = vy$ and $dx = v\, dy + y\, dv$ also lead to a separable equation follows in an analogous manner from (3).

EXAMPLE 1 Solving a Homogeneous DE

Solve $(x^2 + y^2)\, dx + (x^2 - xy)\, dy = 0$.

SOLUTION Inspection of $M(x, y) = x^2 + y^2$ and $N(x, y) = x^2 - xy$ shows that these coefficients are homogeneous functions of degree 2. If we let $y = ux$, then

*Here the word *homogeneous* does not mean the same as it did in Section 2.3. Recall that a linear first-order equation $a_1(x)y' + a_0(x)y = g(x)$ is homogeneous when $g(x) = 0$.

$dy = u\,dx + x\,du$, so after substituting, the given equation becomes

$$
\begin{aligned}
(x^2 + u^2x^2)\,dx + (x^2 - ux^2)[u\,dx + x\,du] &= 0 \\
x^2(1 + u)\,dx + x^3(1 - u)\,du &= 0 \\
\frac{1-u}{1+u}\,du + \frac{dx}{x} &= 0 \\
\left[-1 + \frac{2}{1+u}\right]du + \frac{dx}{x} &= 0. \quad \leftarrow \text{long division}
\end{aligned}
$$

After integration the last line gives

$$
\begin{aligned}
-u + 2\ln|1+u| + \ln|x| &= \ln|c| \\
-\frac{y}{x} + 2\ln\left|1 + \frac{y}{x}\right| + \ln|x| &= \ln|c|. \quad \leftarrow \text{resubstituting } u = y/x
\end{aligned}
$$

Using the properties of logarithms, we can write the preceding solution as

$$\ln\left|\frac{(x+y)^2}{cx}\right| = \frac{y}{x} \qquad \text{or} \qquad (x+y)^2 = cxe^{y/x}.$$ ■

Although either of the indicated substitutions can be used for every homogeneous differential equation, in practice we try $x = vy$ whenever the function $M(x, y)$ is simpler than $N(x, y)$. Also it could happen that after using one substitution, we may encounter integrals that are difficult or impossible to evaluate in closed form; switching substitutions may result in an easier problem.

BERNOULLI'S EQUATION The differential equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n, \tag{4}$$

where n is any real number, is called **Bernoulli's equation.** Note that for $n = 0$ and $n = 1$, equation (4) is linear. For $n \neq 0$ and $n \neq 1$ the substitution $u = y^{1-n}$ reduces any equation of form (4) to a linear equation.

EXAMPLE 2 Solving a Bernoulli DE

Solve $x\dfrac{dy}{dx} + y = x^2y^2$.

SOLUTION We first rewrite the equation as

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2$$

by dividing by x. With $n = 2$ we have $u = y^{-1}$ or $y = u^{-1}$. We then substitute

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -u^{-2}\frac{du}{dx} \qquad \leftarrow \text{Chain Rule}$$

into the given equation and simplify. The result is

$$\frac{du}{dx} - \frac{1}{x}u = -x.$$

The integrating factor for this linear equation on, say, $(0, \infty)$ is

$$e^{-\int dx/x} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}.$$

Integrating
$$\frac{d}{dx}[x^{-1}u] = -1$$

gives $x^{-1}u = -x + c$ or $u = -x^2 + cx$. Since $u = y^{-1}$, we have $y = 1/u$, so a solution of the given equation is $y = 1/(-x^2 + cx)$. ■

Note that we have not obtained the general solution of the original nonlinear differential equation in Example 2, since $y = 0$ is a singular solution of the equation.

REDUCTION TO SEPARATION OF VARIABLES A differential equation of the form

$$\frac{dy}{dx} = f(Ax + By + C) \tag{5}$$

can always be reduced to an equation with separable variables by means of the substitution $u = Ax + By + C$, $B \neq 0$. Example 3 illustrates the technique.

EXAMPLE 3 An Initial-Value Problem

Solve $\dfrac{dy}{dx} = (-2x + y)^2 - 7, \quad y(0) = 0.$

SOLUTION If we let $u = -2x + y$, then $du/dx = -2 + dy/dx$, so the differential equation is transformed into

$$\frac{du}{dx} + 2 = u^2 - 7 \qquad \text{or} \qquad \frac{du}{dx} = u^2 - 9.$$

The last equation is separable. Using partial fractions

$$\frac{du}{(u-3)(u+3)} = dx \qquad \text{or} \qquad \frac{1}{6}\left[\frac{1}{u-3} - \frac{1}{u+3}\right]du = dx$$

and then integrating yields

$$\frac{1}{6}\ln\left|\frac{u-3}{u+3}\right| = x + c_1 \qquad \text{or} \qquad \frac{u-3}{u+3} = e^{6x+6c_1} = ce^{6x}. \qquad \leftarrow \text{replace } e^{6c_1} \text{ by } c$$

Solving the last equation for u and then resubstituting gives the solution

$$u = \frac{3(1 + ce^{6x})}{1 - ce^{6x}} \qquad \text{or} \qquad y = 2x + \frac{3(1 + ce^{6x})}{1 - ce^{6x}}. \tag{6}$$

Finally, applying the initial condition $y(0) = 0$ to the last equation in (6) gives $c = -1$. Figure 2.5.1, obtained with the aid of a graphing utility, shows the graph of the particular solution $y = 2x + \dfrac{3(1 - e^{6x})}{1 + e^{6x}}$ in dark blue, along with the graphs of some other members of the family of solutions (6). ■

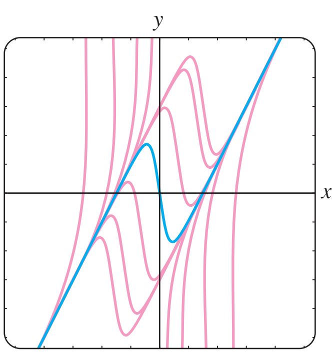


FIGURE 2.5.1 Some solutions of $y' = (-2x + y)^2 - 7$

EXERCISES 2.5

Answers to selected odd-numbered problems begin on page ANS-2.

Each DE in Problems 1–14 is homogeneous.

In Problems 1–10 solve the given differential equation by using an appropriate substitution.

1. $(x - y)\,dx + x\,dy = 0$

2. $(x + y)\,dx + x\,dy = 0$

3. $x\,dx + (y - 2x)\,dy = 0$

4. $y\,dx = 2(x + y)\,dy$

5. $(y^2 + yx)\,dx - x^2\,dy = 0$

6. $(y^2 + yx)\,dx + x^2\,dy = 0$

7. $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$

8. $\dfrac{dy}{dx} = \dfrac{x + 3y}{3x + y}$

9. $-y\,dx + \left(x + \sqrt{xy}\right)dy = 0$

10. $x\dfrac{dy}{dx} = y + \sqrt{x^2 - y^2}, \quad x > 0$

In Problems 11–14 solve the given initial-value problem.

11. $xy^2\dfrac{dy}{dx} = y^3 - x^3, \quad y(1) = 2$

12. $(x^2 + 2y^2)\dfrac{dx}{dy} = xy, \quad y(-1) = 1$

13. $(x + ye^{y/x})\,dx - xe^{y/x}\,dy = 0, \quad y(1) = 0$

14. $y\,dx + x(\ln x - \ln y - 1)\,dy = 0, \quad y(1) = e$

Each DE in Problems 15–22 is a Bernoulli equation.

In Problems 15–20 solve the given differential equation by using an appropriate substitution.

15. $x\dfrac{dy}{dx} + y = \dfrac{1}{y^2}$

16. $\dfrac{dy}{dx} - y = e^x y^2$

17. $\dfrac{dy}{dx} = y(xy^3 - 1)$

18. $x\dfrac{dy}{dx} - (1 + x)y = xy^2$

19. $t^2\dfrac{dy}{dt} + y^2 = ty$

20. $3(1 + t^2)\dfrac{dy}{dt} = 2ty(y^3 - 1)$

In Problems 21 and 22 solve the given initial-value problem.

21. $x^2\dfrac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \frac{1}{2}$

22. $y^{1/2}\dfrac{dy}{dx} + y^{3/2} = 1, \quad y(0) = 4$

Each DE in Problems 23–30 is of the form given in (5).

In Problems 23–28 solve the given differential equation by using an appropriate substitution.

23. $\dfrac{dy}{dx} = (x + y + 1)^2$

24. $\dfrac{dy}{dx} = \dfrac{1 - x - y}{x + y}$

25. $\dfrac{dy}{dx} = \tan^2(x + y)$

26. $\dfrac{dy}{dx} = \sin(x + y)$

27. $\dfrac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$

28. $\dfrac{dy}{dx} = 1 + e^{y - x + 5}$

In Problems 29 and 30 solve the given initial-value problem.

29. $\dfrac{dy}{dx} = \cos(x + y), \quad y(0) = \pi/4$

30. $\dfrac{dy}{dx} = \dfrac{3x + 2y}{3x + 2y + 2}, \quad y(-1) = -1$

Discussion Problems

31. Explain why it is always possible to express any homogeneous differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right).$$

You might start by proving that

$$M(x, y) = x^\alpha M(1, y/x) \quad \text{and} \quad N(x, y) = x^\alpha N(1, y/x).$$

32. Put the homogeneous differential equation

$$(5x^2 - 2y^2)\,dx - xy\,dy = 0$$

into the form given in Problem 31.

33. **(a)** Determine two singular solutions of the DE in Problem 10.

(b) If the initial condition $y(5) = 0$ is as prescribed in Problem 10, then what is the largest interval I over which the solution is defined? Use a graphing utility to graph the solution curve for the IVP.

34. In Example 3 the solution $y(x)$ becomes unbounded as $x \to \pm\infty$. Nevertheless, $y(x)$ is asymptotic to a curve as $x \to -\infty$ and to a different curve as $x \to \infty$. What are the equations of these curves?

35. The differential equation $dy/dx = P(x) + Q(x)y + R(x)y^2$ is known as **Riccati's equation.**

(a) A Riccati equation can be solved by a succession of two substitutions *provided* that we know a particular solution y_1 of the equation. Show that the substitution $y = y_1 + u$ reduces Riccati's equation to a Bernoulli equation (4) with $n = 2$. The Bernoulli equation can then be reduced to a linear equation by the substitution $w = u^{-1}$.

(b) Find a one-parameter family of solutions for the differential equation

$$\frac{dy}{dx} = -\frac{4}{x^2} - \frac{1}{x}y + y^2$$

where $y_1 = 2/x$ is a known solution of the equation.

36. Determine an appropriate substitution to solve

$$xy' = y\ln(xy).$$

Mathematical Models

37. Falling Chain In Problem 45 in Exercises 2.4 we saw that a mathematical model for the velocity v of a chain slipping off the edge of a high horizontal platform is

$$xv\frac{dv}{dx} + v^2 = 32x.$$

In that problem you were asked to solve the DE by converting it into an exact equation using an integrating factor. This time solve the DE using the fact that it is a Bernoulli equation.

38. Population Growth In the study of population dynamics one of the most famous models for a growing but bounded population is the **logistic equation**

$$\frac{dP}{dt} = P(a - bP),$$

where a and b are positive constants. Although we will come back to this equation and solve it by an alternative method in Section 3.2, solve the DE this first time using the fact that it is a Bernoulli equation.

2.6 A NUMERICAL METHOD

INTRODUCTION A first-order differential equation $dy/dx = f(x, y)$ is a source of information. We started this chapter by observing that we could garner *qualitative* information from a first-order DE about its solutions even before we attempted to solve the equation. Then in Sections 2.2–2.5 we examined first-order DEs *analytically*—that is, we developed some procedures for obtaining explicit and implicit solutions. But a differential equation can a possess a solution yet we may not be able to obtain it analytically. So to round out the picture of the different types of analyses of differential equations, we conclude this chapter with a method by which we can "solve" the differential equation *numerically*—this means that the DE is used as the cornerstone of an algorithm for approximating the unknown solution.

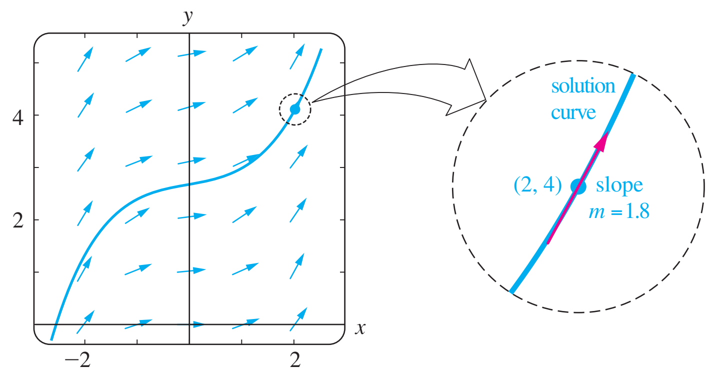
In this section we are going to develop only the simplest of numerical methods—a method that utilizes the idea that a tangent line can be used to approximate the values of a function in a small neighborhood of the point of tangency. A more extensive treatment of numerical methods for ordinary differential equations is given in Chapter 9.

USING THE TANGENT LINE Let us assume that the first-order initial-value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{1}$$

possesses a solution. One way of approximating this solution is to use tangent lines. For example, let $y(x)$ denote the unknown solution of the first-order initial-value problem $y' = 0.1\sqrt{y} + 0.4x^2$, $y(2) = 4$. The nonlinear differential equation in this IVP cannot be solved directly by any of the methods considered in Sections 2.2, 2.4, and 2.5; nevertheless, we can still find approximate numerical values of the unknown $y(x)$. Specifically, suppose we wish to know the value of $y(2.5)$. The IVP has a solution, and as the flow of the direction field of the DE in Figure 2.6.1(a) suggests, a solution curve must have a shape similar to the curve shown in blue.

The direction field in Figure 2.6.1(a) was generated with lineal elements passing through points in a grid with integer coordinates. As the solution curve passes through the initial point (2, 4), the lineal element at this point is a tangent line with slope given by $f(2, 4) = 0.1\sqrt{4} + 0.4(2)^2 = 1.8$. As is apparent in Figure 2.6.1(a) and the "zoom in" in Figure 2.6.1(b), when x is close to 2, the points on the solution curve are close to the points on the tangent line (the lineal element). Using the point (2, 4), the slope $f(2, 4) = 1.8$, and the point-slope form of a line, we find that an equation of the tangent line is $y = L(x)$, where $L(x) = 1.8x + 0.4$. This last equation, called a **linearization** of $y(x)$ at $x = 2$, can be used to approximate values of $y(x)$ within a small neighborhood of $x = 2$. If $y_1 = L(x_1)$ denotes the y-coordinate on the tangent line and $y(x_1)$ is the y-coordinate on the solution curve corresponding to an x-coordinate x_1 that is close to $x = 2$, then $y(x_1) \approx y_1$. If we choose, say, $x_1 = 2.1$, then $y_1 = L(2.1) = 1.8(2.1) + 0.4 = 4.18$, so $y(2.1) \approx 4.18$.



(a) direction field for $y \geq 0$ **(b)** lineal element at (2, 4)

FIGURE 2.6.1 Magnification of a neighborhood about the point (2, 4)

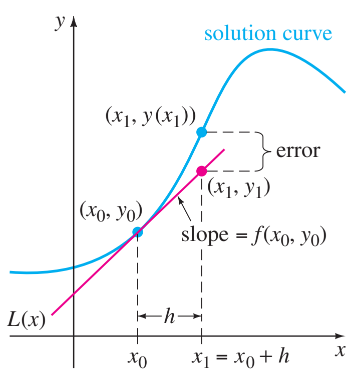


FIGURE 2.6.2 Approximating $y(x_1)$ using a tangent line

EULER'S METHOD To generalize the procedure just illustrated, we use the linearization of the unknown solution $y(x)$ of (1) at $x = x_0$:

$$L(x) = y_0 + f(x_0, y_0)(x - x_0). \tag{2}$$

The graph of this linearization is a straight line tangent to the graph of $y = y(x)$ at the point (x_0, y_0). We now let h be a positive increment of the x-axis, as shown in Figure 2.6.2. Then by replacing x by $x_1 = x_0 + h$ in (2), we get

$$L(x_1) = y_0 + f(x_0, y_0)(x_0 + h - x_0) \quad \text{or} \quad y_1 = y_0 + hf(x_1, y_1),$$

where $y_1 = L(x_1)$. The point (x_1, y_1) on the tangent line is an approximation to the point $(x_1, y(x_1))$ on the solution curve. Of course, the accuracy of the approximation $L(x_1) \approx y(x_1)$ or $y_1 \approx y(x_1)$ depends heavily on the size of the increment h. Usually, we must choose this **step size** to be "reasonably small." We now repeat the process using a second "tangent line" at (x_1, y_1).* By identifying the new starting point as (x_1, y_1) with (x_0, y_0) in the above discussion, we obtain an approximation $y_2 \approx y(x_2)$ corresponding to two steps of length h from x_0, that is, $x_2 = x_1 + h = x_0 + 2h$, and

$$y(x_2) = y(x_0 + 2h) = y(x_1 + h) \approx y_2 = y_1 + hf(x_1, y_1).$$

Continuing in this manner, we see that $y_1, y_2, y_3, \ldots,$ can be defined recursively by the general formula

$$y_{n+1} = y_n + hf(x_n, y_n), \tag{3}$$

where $x_n = x_0 + nh$, $n = 0, 1, 2, \ldots.$ This procedure of using successive "tangent lines" is called **Euler's method.**

*This is not an actual tangent line, since (x_1, y_1) lies on the first tangent and not on the solution curve.

EXAMPLE 1 Euler's Method

Consider the initial-value problem $y' = 0.1\sqrt{y} + 0.4x^2$, $y(2) = 4$. Use Euler's method to obtain an approximation of $y(2.5)$ using first $h = 0.1$ and then $h = 0.05$.

SOLUTION With the identification $f(x, y) = 0.1\sqrt{y} + 0.4x^2$, (3) becomes

$$y_{n+1} = y_n + h\left(0.1\sqrt{y_n} + 0.4x_n^2\right).$$

Then for $h = 0.1$, $x_0 = 2$, $y_0 = 4$, and $n = 0$ we find

$$y_1 = y_0 + h\left(0.1\sqrt{y_0} + 0.4x_0^2\right) = 4 + 0.1\left(0.1\sqrt{4} + 0.4(2)^2\right) = 4.18,$$

which, as we have already seen, is an estimate to the value of $y(2.1)$. However, if we use the smaller step size $h = 0.05$, it takes two steps to reach $x = 2.1$. From

$$y_1 = 4 + 0.05\left(0.1\sqrt{4} + 0.4(2)^2\right) = 4.09$$

$$y_2 = 4.09 + 0.05\left(0.1\sqrt{4.09} + 0.4(2.05)^2\right) = 4.18416187$$

we have $y_1 \approx y(2.05)$ and $y_2 \approx y(2.1)$. The remainder of the calculations were carried out by using software. The results are summarized in Tables 2.1 and 2.2, where each entry has been rounded to four decimal places. We see in Tables 2.1 and 2.2 that it takes five steps with $h = 0.1$ and 10 steps with $h = 0.05$, respectively, to get to $x = 2.5$. Intuitively, we would expect that $y_{10} = 5.0997$ corresponding to $h = 0.05$ is the better approximation of $y(2.5)$ than the value $y_5 = 5.0768$ corresponding to $h = 0.1$. ■

TABLE 2.1 $h = 0.1$

x_n	y_n
2.00	4.0000
2.10	4.1800
2.20	4.3768
2.30	4.5914
2.40	4.8244
2.50	5.0768

TABLE 2.2 $h = 0.05$

x_n	y_n
2.00	4.0000
2.05	4.0900
2.10	4.1842
2.15	4.2826
2.20	4.3854
2.25	4.4927
2.30	4.6045
2.35	4.7210
2.40	4.8423
2.45	4.9686
2.50	5.0997

In Example 2 we apply Euler's method to a differential equation for which we have already found a solution. We do this to compare the values of the approximations y_n at each step with the true or actual values of the solution $y(x_n)$ of the initial-value problem.

EXAMPLE 2 Comparison of Approximate and Actual Values

Consider the initial-value problem $y' = 0.2xy$, $y(1) = 1$. Use Euler's method to obtain an approximation of $y(1.5)$ using first $h = 0.1$ and then $h = 0.05$.

SOLUTION With the identification $f(x, y) = 0.2xy$, (3) becomes

$$y_{n+1} = y_n + h(0.2x_n y_n)$$

where $x_0 = 1$ and $y_0 = 1$. Again with the aid of computer software we obtain the values in Tables 2.3 and 2.4.

TABLE 2.3 $h = 0.1$

x_n	y_n	Actual value	Abs. error	% Rel. error
1.00	1.0000	1.0000	0.0000	0.00
1.10	1.0200	1.0212	0.0012	0.12
1.20	1.0424	1.0450	0.0025	0.24
1.30	1.0675	1.0714	0.0040	0.37
1.40	1.0952	1.1008	0.0055	0.50
1.50	1.1259	1.1331	0.0073	0.64

TABLE 2.4 $h = 0.05$

x_n	y_n	Actual value	Abs. error	% Rel. error
1.00	1.0000	1.0000	0.0000	0.00
1.05	1.0100	1.0103	0.0003	0.03
1.10	1.0206	1.0212	0.0006	0.06
1.15	1.0318	1.0328	0.0009	0.09
1.20	1.0437	1.0450	0.0013	0.12
1.25	1.0562	1.0579	0.0016	0.16
1.30	1.0694	1.0714	0.0020	0.19
1.35	1.0833	1.0857	0.0024	0.22
1.40	1.0980	1.1008	0.0028	0.25
1.45	1.1133	1.1166	0.0032	0.29
1.50	1.1295	1.1331	0.0037	0.32

■

In Example 1 the true or actual values were calculated from the known solution $y = e^{0.1(x^2-1)}$. (Verify.) The **absolute error** is defined to be

$$|\textit{actual value} - \textit{approximation}|.$$

The **relative error** and **percentage relative error** are, in turn,

$$\frac{\textit{absolute error}}{|\textit{actual value}|} \quad \text{and} \quad \frac{\textit{absolute error}}{|\textit{actual value}|} \times 100.$$

It is apparent from Tables 2.3 and 2.4 that the accuracy of the approximations improves as the step size h decreases. Also, we see that even though the percentage relative error is growing with each step, it does not appear to be that bad. But you should not be deceived by one example. If we simply change the coefficient of the right side of the DE in Example 2 from 0.2 to 2, then at $x_n = 1.5$ the percentage relative errors increase dramatically. See Problem 4 in Exercises 2.6.

A CAVEAT Euler's method is just one of many different ways in which a solution of a differential equation can be approximated. Although attractive for its simplicity, *Euler's method is seldom used in serious calculations.* It was introduced here simply to give you a first taste of numerical methods. We will go into greater detail in discussing numerical methods that give significantly greater accuracy, notably the **fourth order Runge-Kutta method,** referred to as the **RK4 method,** in Chapter 9.

NUMERICAL SOLVERS Regardless of whether we can actually find an explicit or implicit solution, if a solution of a differential equation exists, it represents a smooth curve in the Cartesian plane. The basic idea behind *any* numerical method for first-order ordinary differential equations is to somehow approximate the y-values of a solution for preselected values of x. We start at a specified initial point (x_0, y_0) on a solution curve and proceed to calculate in a step-by-step fashion a sequence of points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ whose y-coordinates y_i approximate the y-coordinates $y(x_i)$ of points $(x_1, y(x_1)), (x_2, y(x_2)), \ldots, (x_n, y(x_n))$ that lie on the graph of the usually unknown solution $y(x)$. By taking the x-coordinates close together (that is, for small values of h) and by joining the points (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$ with short line segments, we obtain a polygonal curve whose qualitative characteristics we hope are close to those of an actual solution curve. Drawing curves is something that is well suited to a computer. A computer program written to either implement a numerical method or render a visual representation of an approximate solution curve fitting the numerical data produced by this method is referred to as a **numerical solver.** Many different numerical solvers are commercially available, either embedded in a larger software package, such as a computer algebra system, or provided as a stand-alone package. Some software packages simply plot the generated numerical approximations, whereas others generate hard numerical data as well as the corresponding approximate or **numerical solution curves.** By way of illustration of the connect-the-dots nature of the graphs produced by a numerical solver, the two colored polygonal graphs in Figure 2.6.3 are the numerical solution curves for the initial-value problem $y' = 0.2xy$, $y(0) = 1$ on the interval [0, 4] obtained from Euler's method and the RK4 method using the step size $h = 1$. The blue smooth curve is the graph of the exact solution $y = e^{0.1x^2}$ of the IVP. Notice in Figure 2.6.3 that, even with the ridiculously large step size of $h = 1$, the RK4 method produces the more believable "solution curve." The numerical solution curve obtained from the RK4 method is indistinguishable from the actual solution curve on the interval [0, 4] when a more typical step size of $h = 0.1$ is used.

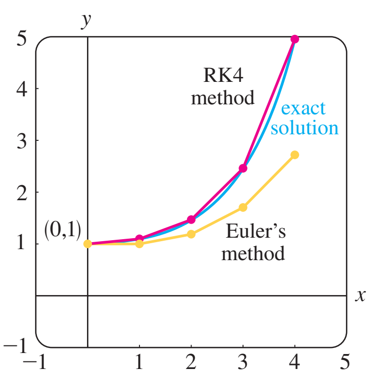


FIGURE 2.6.3 Comparison of the Runge-Kutta (RK4) and Euler methods

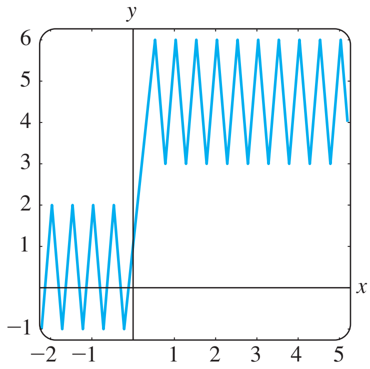


FIGURE 2.6.4 A not very helpful numerical solution curve

USING A NUMERICAL SOLVER Knowledge of the various numerical methods is not necessary in order to use a numerical solver. A solver usually requires that the differential equation be expressed in normal form $dy/dx = f(x, y)$. Numerical solvers that generate only curves usually require that you supply $f(x, y)$ and the initial data x_0 and y_0 and specify the desired numerical method. If the idea is to approximate the numerical value of $y(a)$, then a solver may additionally require that you state a value for h or, equivalently, give the number of steps that you want to take to get from $x = x_0$ to $x = a$. For example, if we wanted to approximate $y(4)$ for the IVP illustrated in Figure 2.6.3, then, starting at $x = 0$ it would take four steps to reach $x = 4$ with a step size of $h = 1$; 40 steps is equivalent to a step size of $h = 0.1$. Although we will not delve here into the many problems that one can encounter when attempting to approximate mathematical quantities, you should at least be aware of the fact that a numerical solver may break down near certain points or give an incomplete or misleading picture when applied to some first-order differential equations in the normal form. Figure 2.6.4 illustrates the graph obtained by applying Euler's method to a certain first-order initial-value problem $dy/dx = f(x, y)$, $y(0) = 1$. Equivalent results were obtained using three different commercial numerical solvers, yet the graph is hardly a plausible solution curve. (Why?) There are several avenues of recourse when a numerical solver has difficulties; three of the more obvious are decrease the step size, use another numerical method, and try a different numerical solver.

EXERCISES 2.6

Answers to selected odd-numbered problems begin on page ANS-2.

In Problems 1 and 2 use Euler's method to obtain a four-decimal approximation of the indicated value. Carry out the recursion of (3) by hand, first using $h = 0.1$ and then using $h = 0.05$.

1. $y' = 2x - 3y + 1$, $y(1) = 5$; $y(1.2)$

2. $y' = x + y^2$, $y(0) = 0$; $y(0.2)$

In Problems 3 and 4 use Euler's method to obtain a four-decimal approximation of the indicated value. First use $h = 0.1$ and then use $h = 0.05$. Find an explicit solution for each initial-value problem and then construct tables similar to Tables 2.3 and 2.4.

3. $y' = y$, $y(0) = 1$; $y(1.0)$

4. $y' = 2xy$, $y(1) = 1$; $y(1.5)$

In Problems 5–10 use a numerical solver and Euler's method to obtain a four-decimal approximation of the indicated value. First use $h = 0.1$ and then use $h = 0.05$.

5. $y' = e^{-y}$, $y(0) = 0$; $y(0.5)$

6. $y' = x^2 + y^2$, $y(0) = 1$; $y(0.5)$

7. $y' = (x - y)^2$, $y(0) = 0.5$; $y(0.5)$

8. $y' = xy + \sqrt{y}$, $y(0) = 1$; $y(0.5)$

9. $y' = xy^2 - \dfrac{y}{x}$, $y(1) = 1$; $y(1.5)$

10. $y' = y - y^2$, $y(0) = 0.5$; $y(0.5)$

In Problems 11 and 12 use a numerical solver to obtain a numerical solution curve for the given initial-value problem. First use Euler's method and then the RK4 method. Use $h = 0.25$ in each case. Superimpose both solution curves on the same coordinate axes. If possible, use a different color for each curve. Repeat, using $h = 0.1$ and $h = 0.05$.

11. $y' = 2(\cos x)y$, $y(0) = 1$

12. $y' = y(10 - 2y)$, $y(0) = 1$

Discussion Problems

13. Use a numerical solver and Euler's method to approximate $y(1.0)$, where $y(x)$ is the solution to $y' = 2xy^2$, $y(0) = 1$. First use $h = 0.1$ and then use $h = 0.05$. Repeat, using the RK4 method. Discuss what might cause the approximations to $y(1.0)$ to differ so greatly.

Computer Lab Assignments

14. **(a)** Use a numerical solver and the RK4 method to graph the solution of the initial-value problem $y' = -2xy + 1$, $y(0) = 0$.

(b) Solve the initial-value problem by one of the analytic procedures developed earlier in this chapter.

(c) Use the analytic solution $y(x)$ found in part (b) and a CAS to find the coordinates of all relative extrema.

CHAPTER 2 IN REVIEW

Answers to selected odd-numbered problems begin on page ANS-3.

Answer Problems 1–4 without referring back to the text. Fill in the blanks or answer true or false.

1. The linear DE, $y' - ky = A$, where k and A are constants, is autonomous. The critical point __________ of the equation is a(n) __________ (attractor or repeller) for $k > 0$ and a(n) __________ (attractor or repeller) for $k < 0$.

2. The initial-value problem $x\dfrac{dy}{dx} - 4y = 0$, $y(0) = k$, has an infinite number of solutions for $k =$ __________ and no solution for $k =$ __________.

3. The linear DE, $y' + k_1 y = k_2$, where k_1 and k_2 are nonzero constants, always possesses a constant solution. __________

4. The linear DE, $a_1(x)y' + a_2(x)y = 0$ is also separable. __________

In Problems 5 and 6 construct an autonomous first-order differential equation $dy/dx = f(y)$ whose phase portrait is consistent with the given figure.

5.

FIGURE 2.R.1 Graph for Problem 5

6.

FIGURE 2.R.2 Graph for Problem 6

7. The number 0 is a critical point of the autonomous differential equation $dx/dt = x^n$, where n is a positive integer. For what values of n is 0 asymptotically stable? Semi-stable? Unstable? Repeat for the differential equation $dx/dt = -x^n$.

8. Consider the differential equation $dP/dt = f(P)$, where

$$f(P) = -0.5P^3 - 1.7P + 3.4.$$

The function $f(P)$ has one real zero, as shown in Figure 2.R.3. Without attempting to solve the differential equation, estimate the value of $\lim_{t\to\infty} P(t)$.

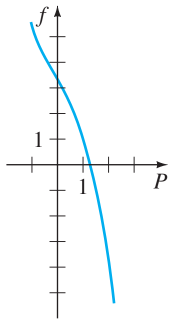


FIGURE 2.R.3 Graph for Problem 8

9. Figure 2.R.4 is a portion of a direction field of a differential equation $dy/dx = f(x, y)$. By hand, sketch two different solution curves—one that is tangent to the lineal element shown in black and one that is tangent to the lineal element shown in color.

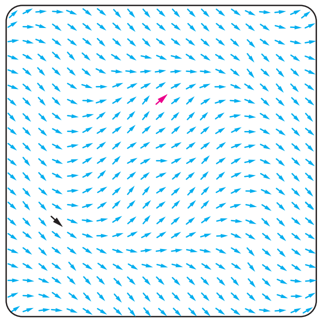


FIGURE 2.R.4 Portion of a direction field for Problem 9

10. Classify each differential equation as separable, exact, linear, homogeneous, or Bernoulli. Some equations may be more than one kind. Do not solve.

(a) $\dfrac{dy}{dx} = \dfrac{x - y}{x}$ **(b)** $\dfrac{dy}{dx} = \dfrac{1}{y - x}$

(c) $(x + 1)\dfrac{dy}{dx} = -y + 10$ **(d)** $\dfrac{dy}{dx} = \dfrac{1}{x(x - y)}$

(e) $\dfrac{dy}{dx} = \dfrac{y^2 + y}{x^2 + x}$ **(f)** $\dfrac{dy}{dx} = 5y + y^2$

(g) $y\,dx = (y - xy^2)\,dy$ **(h)** $x\dfrac{dy}{dx} = ye^{x/y} - x$

(i) $xy\,y' + y^2 = 2x$ **(j)** $2xy\,y' + y^2 = 2x^2$

(k) $y\,dx + x\,dy = 0$

(l) $\left(x^2 + \dfrac{2y}{x}\right)dx = (3 - \ln x^2)\,dy$

(m) $\dfrac{dy}{dx} = \dfrac{x}{y} + \dfrac{y}{x} + 1$ **(n)** $\dfrac{y}{x^2}\dfrac{dy}{dx} + e^{2x^3 + y^2} = 0$

In Problems 11–18 solve the given differential equation.

11. $(y^2 + 1)\,dx = y \sec^2 x\,dy$

12. $y(\ln x - \ln y)\,dx = (x \ln x - x \ln y - y)\,dy$

13. $(6x + 1)y^2 \dfrac{dy}{dx} + 3x^2 + 2y^3 = 0$

14. $\dfrac{dx}{dy} = -\dfrac{4y^2 + 6xy}{3y^2 + 2x}$

15. $t\dfrac{dQ}{dt} + Q = t^4 \ln t$

16. $(2x + y + 1)y' = 1$

17. $(x^2 + 4)\,dy = (2x - 8xy)\,dx$

18. $(2r^2 \cos\theta \sin\theta + r\cos\theta)\,d\theta$
$+ (4r + \sin\theta - 2r\cos^2\theta)\,dr = 0$

In Problems 19 and 20 solve the given initial-value problem and give the largest interval *I* on which the solution is defined.

19. $\sin x \dfrac{dy}{dx} + (\cos x)y = 0, \quad y\left(\dfrac{7\pi}{6}\right) = -2$

20. $\dfrac{dy}{dt} + 2(t + 1)y^2 = 0, \quad y(0) = -\frac{1}{8}$

21. **(a)** Without solving, explain why the initial-value problem

$$\frac{dy}{dx} = \sqrt{y}, \quad y(x_0) = y_0$$

has no solution for $y_0 < 0$.

(b) Solve the initial-value problem in part (a) for $y_0 > 0$ and find the largest interval *I* on which the solution is defined.

22. **(a)** Find an implicit solution of the initial-value problem

$$\frac{dy}{dx} = \frac{y^2 - x^2}{xy}, \quad y(1) = -\sqrt{2}.$$

(b) Find an explicit solution of the problem in part (a) and give the largest interval *I* over which the solution is defined. A graphing utility may be helpful here.

23. Graphs of some members of a family of solutions for a first-order differential equation $dy/dx = f(x, y)$ are shown in Figure 2.R.5. The graphs of two implicit solutions, one that passes through the point $(1, -1)$ and one that passes through $(-1, 3)$, are shown in red. Reproduce the figure on a piece of paper. With colored pencils trace out the solution curves for the solutions $y = y_1(x)$ and $y = y_2(x)$ defined by the implicit solutions such that $y_1(1) = -1$ and $y_2(-1) = 3$, respectively. Estimate the intervals on which the solutions $y = y_1(x)$ and $y = y_2(x)$ are defined.

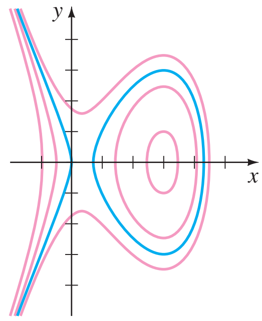


FIGURE 2.R.5 Graph for Problem 23

24. Use Euler's method with step size $h = 0.1$ to approximate $y(1.2)$, where $y(x)$ is a solution of the initial-value problem $y' = 1 + x\sqrt{y}$, $y(1) = 9$.

In Problems 25 and 26 each figure represents a portion of a direction field of an autonomous first-order differential equation $dy/dx = f(y)$. Reproduce the figure on a separate piece of paper and then complete the direction field over the grid. The points of the grid are (mh, nh), where $h = \frac{1}{2}$, m and n integers, $-7 \le m \le 7$, $-7 \le n \le 7$. In each direction field, sketch by hand an approximate solution curve that passes through each of the solid points shown in red. Discuss: Does it appear that the DE possesses critical points in the interval $-3.5 \le y \le 3.5$? If so, classify the critical points as asymptotically stable, unstable, or semi-stable.

25.

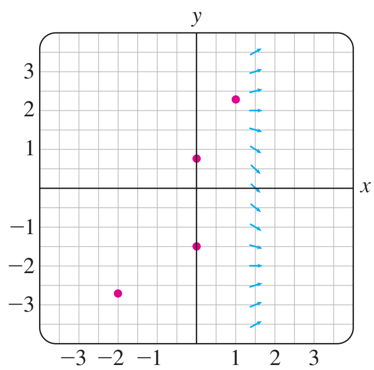


FIGURE 2.R.6 Portion of a direction field for Problem 25

26.

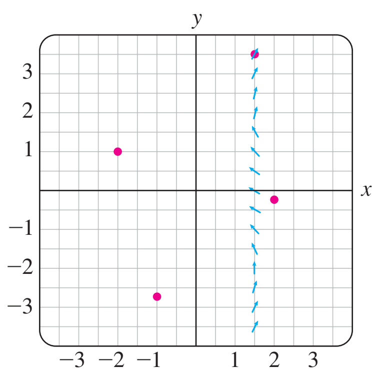


FIGURE 2.R.7 Portion of a direction field for Problem 26