1. An adjunction: $\Lambda \dashv \Gamma$

(Germs $\dashv$ Cross-sections)

The "bundle of germs" of P, $\Lambda P = \begin{pmatrix} \coprod_x P_x \\ \downarrow \\ X \end{pmatrix}$

The étalification of $Y \equiv \begin{pmatrix} Y \\ \downarrow \varphi \\ X \end{pmatrix}$

A presheaf $P : \mathcal{O}(X)^{op} \to \mathbf{Set}$, not necessarily a sheaf

$$\begin{tikzcd}
\Lambda\Gamma Y \arrow[d, "\varepsilon_Y"'] & \Lambda P \arrow[d] & P \arrow[l, mapsto, "\Lambda"'] \arrow[d] & P \arrow[d, "\eta_P"] \\
Y & Y \arrow[r, mapsto, "\Gamma"'] \arrow[ur, phantom, "\overset{\flat}{\longleftarrow}\ \underset{\#}{\longrightarrow}"] & \Gamma Y & \Gamma\Lambda P
\end{tikzcd}$$

A top. space $(Y, \mathcal{O}(Y))$ "displayed over X", i.e., $Y \equiv \begin{pmatrix} Y \\ \downarrow \varphi \\ X \end{pmatrix}$, φ continuous (not necessarily étale!)

The sheafification of P

$$\mathbf{Top}/X \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \mathbf{Set}^{\mathcal{O}(X)^{op}}$$

The "sheaf of cross-sections" of $\begin{pmatrix} Y \\ \downarrow \varphi \\ X \end{pmatrix}$, defined by:

$$(\Gamma Y)(U) := \left\{ s_U : U \to Y \text{ continuous such that this commutes: } \begin{tikzcd} & Y \arrow[d, "\varphi"] \\ U \arrow[ur, "s_U"] \arrow[r, hook] & X \end{tikzcd} \right\}$$

2. Order Topologies

Let $\mathbf{V}$ be this DAG:
$$\begin{tikzcd}[column sep=small] \alpha \arrow[dr] & & \beta \arrow[dl] \\ & \gamma & \end{tikzcd}$$

Let $\mathbf{V}^*$ be its reflexive-transitive closure:
$$\mathbf{V}^* = \begin{tikzcd}[column sep=small] \alpha \arrow[dr] \arrow[loop above] & & \beta \arrow[dl] \arrow[loop above] \\ & \gamma \arrow[loop below] & \end{tikzcd}$$

Let $\mathbb{V} \equiv (\mathbf{V}, \mathcal{O}(\mathbf{V}))$
$$= \left(\left\{\substack{\alpha,\ \beta,\\ \gamma}\right\}, \left\{\underset{O}{\underset{\parallel}{\substack{00\\0}}}, \underset{W}{\underset{\parallel}{\substack{00\\1}}}, \underset{V}{\underset{\parallel}{\substack{01\\1}}}, \underset{U}{\underset{\parallel}{\substack{10\\1}}}, \underset{X}{\underset{\parallel}{\substack{11\\1}}}\right\}\right)$$

Be $\mathbf{V}$ reinterpreted as a topological space with the order topology.

Then $\mathcal{O}(\mathbb{V})$ is this category, whose morphisms are the inclusions:

$$\begin{tikzcd}[column sep=tiny] & X \equiv \substack{11\\1} \equiv \left\{\substack{\alpha,\ \beta,\\ \gamma}\right\} & \\ U \equiv \substack{10\\1} \equiv \left\{\substack{\alpha,\ \cdot\\ \gamma}\right\} \arrow[ur] & & V \equiv \substack{01\\1} \equiv \left\{\substack{\cdot\ \ \beta,\\ \gamma}\right\} \arrow[ul] \\ & W \equiv \substack{00\\1} \equiv \left\{\substack{\cdot\ \ \cdot\\ \gamma}\right\} \arrow[ul] \arrow[ur] & \\ & O \equiv \substack{00\\0} \equiv \emptyset \arrow[u] & \end{tikzcd}$$

And $\mathcal{O}(\mathbb{V})^{op}$ is:

$$\begin{tikzcd}[column sep=small] & X \arrow[dl] \arrow[dr] & \\ U \arrow[dr] & & V \arrow[dl] \\ & W \arrow[d] & \\ & O & \end{tikzcd}$$

A presheaf $P : \mathcal{O}(\mathbb{V})^{op} \to \mathbf{Set}$ is given by 5 sets and 5 maps; diagrammatically,

$$\begin{tikzcd}[column sep=small] & PX \arrow[dl] \arrow[dr] & \\ PU \arrow[dr] & & PV \arrow[dl] \\ & PW \arrow[d] & \\ & PO & \end{tikzcd}$$

such that the $\Diamond$ commutes.

The construction of the "bundle of germs" $\Lambda P := \begin{pmatrix} \coprod_x P_x \\ \downarrow \\ X \end{pmatrix}$ involves a top. space

$$\coprod_x P_x \equiv \left(\coprod_x P_x,\ \mathcal{O}\left(\coprod_x P_x\right)\right),$$

where the set $\coprod_x P_x$ is a disjoint union of spaces of germs:

$$\coprod_x P_x := \{(x, p_x) \mid x \in X,\ p_x \in P_x\};$$

for each point $x \in X$ the space of germs P_x is the quotient

$$P_x := \left(\coprod_{U \ni x} PU\right) \Big/ \underset{x}{\sim},$$

where "$/\underset{x}{\sim}$" identifies a $p_U \in PU$ and a $p_V \in PV$ exactly when there is an open set $W \ni x$ such that $p_U|_{U \cap W} = p_V|_{V \cap W}$.

In a presheaf $P : \mathcal{O}(\mathbb{V})^{op} \to \mathbf{Set}$ these spaces of germs are easy to calculate.

For example, take $x := \gamma$. The open sets containing γ are
$$\begin{tikzcd}[column sep=tiny, row sep=tiny] & X & \\ U \arrow[ur] & & V \arrow[ul] \\ & W \arrow[ul] \arrow[ur] & \end{tikzcd},$$
and the quotient

$$P_\gamma := \left(\coprod_{U \ni \gamma} PU\right) \Big/ \underset{\gamma}{\sim}$$
$$= (PX \amalg PU \amalg PV \amalg PW) \big/ \underset{\gamma}{\sim}$$

is the colimit of this diagram:

$$\begin{tikzcd}[column sep=small] & PX \arrow[dl] \arrow[dr] & \\ PU \arrow[dr] & & PV \arrow[dl] \\ & PW & \end{tikzcd}$$

Each equivalence class $[p_U]_\gamma \in P_\gamma$ has exactly one element in the lower object above, PW, which is P applied to the smallest open set containing γ; so

$$P_\gamma = \{[p_W]_\gamma \mid p_W \in PW\}$$
$$\cong PW,$$

and

$$P_\alpha \cong PU, \quad P_\beta \cong PV,$$
$$P_\gamma \cong PW.$$

3. THE EVIL PRESHEAF

SUPPOSE FOR A MOMENT THAT:

$$\begin{tikzcd}[column sep=small]
 & X=(-\infty,+\infty) & \\
U=(-\infty,1) \arrow[ur] & & V=(0,+\infty) \arrow[ul] \\
 & W=(0,1) \arrow[ul] \arrow[ur] & \\
 & O=\emptyset \arrow[u] &
\end{tikzcd}$$

I.E., WE ARE (TEMPORARILY) REGARDING $\mathcal{O}(\mathbb{V})$ AS A SUBTOPOLOGY OF $\mathcal{O}(\mathbb{R})$.

THEN IT MAKES SENSE TO SPEAK OF THE PRESHEAF $C^\infty : \mathcal{O}(\mathbb{V})^{op} \to \mathbf{Set}$,

$$\begin{tikzcd}[column sep=small]
 & C^\infty(X) \arrow[dl] \arrow[dr] & \\
C^\infty(U) \arrow[dr] & & C^\infty(V) \arrow[dl] \\
 & C^\infty(W) \arrow[d] & \\
 & C^\infty(O) &
\end{tikzcd}$$

AND IT IS A <u>SHEAF</u>, IN THE FOLLOWING SENSE:

<u>DEF:</u> A PRESHEAF P IS A <u>SHEAF</u> IFF EACH COMPATIBLE FAMILY IN IT HAS EXACTLY ONE GLUEING.

IN C^∞, IF f_U AND f_V ARE COMPATIBLE, I.E., IF

$f_U : U \to \mathbb{R}$

AND $f_V : V \to \mathbb{R}$

ARE C^∞ AND THE RESTRICTIONS

$f_U|_{U\cap V} = f_U|_W$

AND $f_V|_{U\cap V} = f_V|_W$

COINCIDE, THEN THERE IS EXACTLY ONE "GLUEING"

$f_X : X \to \mathbb{R}$

IN $C^\infty(X)$ SUCH THAT

$f_X|_U = f_U$

AND $f_X|_V = f_V$.

NOTE THAT:

- EACH COMPATIBLE FAMILY HAS AT MOST ONE ELEMENT IN EACH PU,
- COMPATIBILITY MEANS THAT "ALL RESTRICTIONS ARE WELL-DEFINED"
- GLUEING $\{f_U, f_W\}$ CORRESPONDS TO:

$$\begin{pmatrix} f_U\ f_V \\ \cdot \\ \cdot \end{pmatrix} \to \begin{pmatrix} f_U\ f_V \\ f_W \\ f_O \end{pmatrix} \to \begin{pmatrix} f_X \\ f_U\ f_V \\ f_W \\ f_O \end{pmatrix} \to \begin{pmatrix} f_W \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

SATURATE DOWNWARDS

SATURATE UPWARDS (ADD UNIONS)

THE <u>EVIL PRESHEAF</u>, $E : \mathcal{O}(\mathbb{V})^{op} \to \mathbf{Set}$, DEFINED AS:

$$\begin{tikzcd}[column sep=small]
 & \{e_X, e'_X\} \arrow[dl] \arrow[dr] & \\
\{e_U\} \arrow[dr] & & \{e_V, e'_V\} \arrow[dl] \\
 & \{e_W\} \arrow[d] & \\
 & \{e_O\} &
\end{tikzcd}$$

FAILS THE TWO CONDITIONS THAT A PRESHEAF MUST OBEY TO BE A SHEAF: IT IS NOT <u>SEPARATED</u> BECAUSE THE COMPATIBLE FAMILY $\{e_U, e_V\}$ HAS TWO DIFFERENT GLUEINGS, e_X AND e'_X, AND IT IS NOT <u>COLLATED</u> BECAUSE THE COMPATIBLE FAMILY $\{e_U, e'_V\}$ DOESN'T HAVE A GLUEING.

NOTE THAT WE CAN LOOK AT THESE CONDITIONS "FROM THE OTHER DIRECTION": EACH SET OF OPEN SETS $\mathcal{U} \subseteq \mathcal{O}(X)$, E.G., $\{U,V\}$, INDUCES A MAP FROM "GLUEINGS" TO "COMPATIBLE FAMILIES",

WHICH IN THE CASE OF THE "COVER" $\{U,V\}$ IS:

$$\begin{tikzcd}
\boxed{\begin{matrix}(e_U,e_V)\\(e_U,e'_V)\end{matrix}} & \boxed{\begin{matrix}e_X\\e'_X\end{matrix}} \arrow[l]
\end{tikzcd}$$

$$PU \times PV \leftarrow PX$$

THIS FAILS TO BE INJECTIVE ("SEPARATEDNESS FAILS") AND FAILS TO BE SURJECTIVE ("COLLATEDNESS FAILS").

IN THE GENERAL CASE, FOR A PRESHEAF [redacted] $P : \mathcal{O}(X)^{op} \to \mathbf{Set}$, WE WOULD HAVE TO CONSIDER, FOR EACH "COVER" $\mathcal{U} \subseteq \mathcal{O}(X)$ OF $\bigcup_{U\in\mathcal{U}} U$, THE ASSOCIATED MAP FROM $P(\bigcup_{U\in\mathcal{U}} U)$ TO THE [redacted] SET OF COMPATIBLE FAMILIES WITH SUPPORT $\mathcal{U}$; THEN "P IS SEPARATED" MEANS THAT ALL THESE MAPS ARE INJECTIVE, "P IS COLLATED" MEANS THAT THEY ARE ALL SURJECTIVE.

4. Sheafifying the evil presheaf

$$
\left(
\begin{tikzcd}
{[e_U]_\alpha} \arrow[dr] & & {[e_V]_\beta, [e'_V]_\beta} \arrow[dl] \arrow[dl, bend left] \\
& {[e_W]_\gamma} \arrow[d] & \\
& \left(\begin{tikzcd}[sep=small] \alpha \arrow[dr] & & \beta \arrow[dl] \\ & \gamma & \end{tikzcd} \right) &
\end{tikzcd}
\right)
= \Lambda E
$$

$$
\begin{tikzcd}
\Lambda E \arrow[d, "\mathrm{id}_{\Lambda E}"'] & & E \arrow[ll, mapsto] \arrow[d, "{\eta_E := (\mathrm{id}_{\Lambda E})^\#}"] \\
\Lambda E \arrow[rr, mapsto] & \arrow[r, phantom, "\longmapsto"] & \Gamma\Lambda E
\end{tikzcd}
$$

$$
E = \left(
\begin{tikzcd}[sep=small]
& \{e_X, e'_X\} \arrow[dl] \arrow[dr] & \\
\{e_U\} \arrow[dr] & & \{e_V, e'_V\} \arrow[dl] \\
& \{e_W\} \arrow[d] & \\
& \{e_O\} &
\end{tikzcd}
\right)
\qquad
\Gamma\Lambda E = \left(
\begin{tikzcd}[sep=small]
& \{\underline{e}_X, \underline{e}'_X\} \arrow[dl] \arrow[dr] & \\
\{\underline{e}_U\} \arrow[dr] & & \{\underline{e}_V, \underline{e}'_V\} \arrow[dl] \\
& \{\underline{e}_W\} \arrow[d] & \\
& \{\underline{e}_O\} &
\end{tikzcd}
\right)
$$

$$
\begin{tikzcd}[sep=small]
\{e_X, e'_X\} \arrow[r] \arrow[d] \arrow[dr] & \{e_V, e'_V\} \arrow[d] \\
\{\underline{e}_X, \underline{e}'_X\} \arrow[r] & \{\underline{e}_V, \underline{e}'_V\}
\end{tikzcd}
\Longrightarrow \eta_E
$$

$$
\mathrm{Top}/\mathbb{V} \underset{\Gamma}{\overset{\Lambda}{\leftrightarrows}} \mathrm{Set}^{\mathcal{O}(\mathbb{V})^{op}}
$$

5. The topology on ΛP

Look again at the action of the functor Λ on objects:

$$\begin{pmatrix} \coprod_x P_x \\ \downarrow \\ X \end{pmatrix} = \Lambda P \longleftarrow P$$

$$\mathbf{Top}/X \xleftarrow{\Lambda} \mathbf{Set}^{\mathcal{O}(X)^{op}}$$

We saw how to construct the set of germs, $\coprod_x P_x$, for a given presheaf $P : \mathcal{O}(X)^{op} \to \mathbf{Set}$, but we don't know yet the topology $\mathcal{O}(\coprod_x P_x)$.

We need the notion of "étale map".

Fix a displayed space $\begin{pmatrix} Y \\ \downarrow p \\ X \end{pmatrix}$ (over X).

We will use letters with primes for points and subsets of Y and the same letters without primes for their images in X – for example, if $\alpha' \in U' \in \mathcal{O}(Y)$ then $\alpha := p(\alpha')$ and $U := p(U')$.

We will say that an open set $U' \in \mathcal{O}(Y)$ is "similar to its image" when:

- $U \in \mathcal{O}(X)$,
- $(p|_{U'}) : U' \to U$ is a bijection,
- $(p|_{U'})$ is continuous,
- $(p|_{U'})^{-1}$ is continuous.

Notations:

$\mathcal{L}_p(Y) := \{ U' \in \mathcal{O}(Y) \mid U'$ is similar to its image$\}$,

$\mathcal{L}_p(\alpha') := \{ U' \in \mathcal{O}(Y) \mid \alpha' \in U'$ and U' is similar to its image$\}$.

<u>Definition:</u> A displayed space $\begin{pmatrix} Y \\ \downarrow p \\ X \end{pmatrix}$ is <u>étale</u> when $\mathcal{L}_p(Y)$ covers Y, or, equivalently, when:

$$\forall \alpha' \in Y.\ \mathcal{L}_p(\alpha') \neq \emptyset.$$

Now for each $U \in \mathcal{O}(X)$ and for each $p_U \in PU$ we will define the "natural section" $\dot{p}_U$ as this:

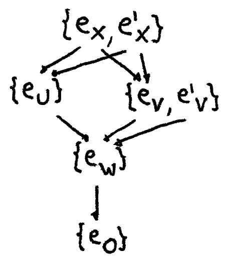
$$\begin{aligned} \dot{p}_U : U &\to \textstyle\coprod_x P_x \\ \alpha &\mapsto [p_U]_\alpha \end{aligned}$$

Note that each $\dot{p}_U$ makes the diagram

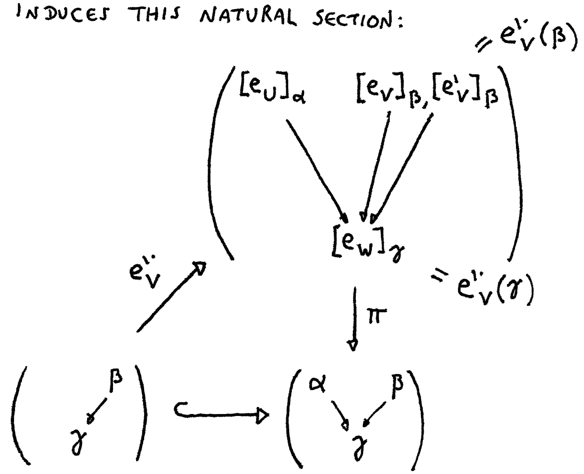
$$\begin{tikzcd} & \coprod_x P_x \arrow[d, "\pi"] \\ U \arrow[ur, "\dot{p}_U"] \arrow[r, hook] & X \end{tikzcd}$$

commute, where $U \hookrightarrow X$ is the inclusion.

For example, in the evil presheaf $E : \mathcal{O}(V)^{op} \to \mathbf{Set}$,



The element $e'_V \in PV$ induces this natural section:



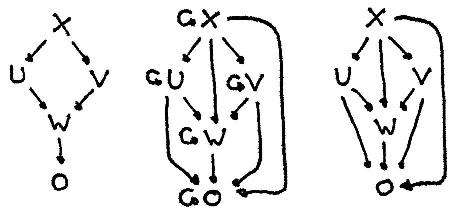
The usual definition for the topology $\mathcal{O}(\coprod_x P_x)$ is that it is the weakest topology on $\coprod_x P_x$ that has all the sets of the form $\dot{p}_U(U)$ as open sets (and as sets "similar to their images")... But we will see that when X and P are finite (and X is T_0) then $\mathcal{O}(\coprod_x P_x)$ is trivial to calculate.

6. More on finite DAGs

Let $R \subseteq A \times A$ be a relation on a set A of vertices. (A is fixed).

Let's denote by R^* its transitive-reflexive closure and by R^- "R minus its identity arrows".

Example:



$R \qquad R^* \qquad R^{*-}$

We will say that R and R' are <u>equivalent</u> when $R^* = R'^*$.

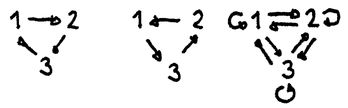
Each equivalence class

$$[R] := \{R' \mid R'^* = R^*\}$$

has a biggest element, R^*, but R^* is clumsy to draw.

In some cases $[R]$ will also have a smallest element, R^{ess}, that can be calculated by dropping all the "non-essential arrows" from R.

Note that in this case



$T \qquad T' \qquad T^* = T'^*$

We have $T^* = T'^*$, but $T \neq T'$ and both T and T' are minimal, in the sense that it is not possible to drop arrows from T or T' and still get equivalent relations... so $[T]$ does not have a smallest element. Also, if $\leq \subseteq \mathbb{R} \times \mathbb{R}$ is

$$\leq := \{(a,b) \in \mathbb{R} \times \mathbb{R} \mid a \leq b\}$$

then we can drop any finite number of arrows from $\leq$ and get a smaller relation, $\leq'$, such that $[\leq] = [\leq']$.

So $[\leq]$ also does not have a smallest element.

<u>Definition:</u> We say that R is <u>weakly acyclical</u> when R^- is acyclical in the usual sense.

<u>Definition:</u> We say that an arrow $(\alpha, \beta) \in R$ is <u>essential</u> when $(R \backslash \{(\alpha,\beta)\})^* \neq R^*$. We write the set of essential arrows of R as R^{ess}.

In the examples that we have just described we have:

$$R = R^{ess} = R^{*ess} = R^{*-ess}$$
$$T = T^{ess}$$
$$T' = T'^{ess}$$
$$T^{*ess} = T'^{*ess} = \emptyset$$
$$\leq^{ess} = \emptyset$$

<u>Theorem:</u> If A is finite and $R \subseteq A \times A$ is weakly acyclical, then $[R]$ has a minimal element, and it is R^{ess}.

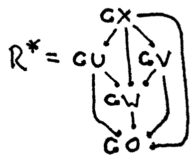
<u>Corollary:</u> If A is finite and $R, R' \subseteq A \times A$ are weakly acyclical, then:

$$R \sim R' \overset{\text{def}}{\Leftrightarrow} R^* = R'^*$$
$$\Leftrightarrow [R] = [R']$$
$$\Leftrightarrow R^{ess} = R'^{ess}.$$

<u>Sketch of the proof.</u> If we treat the vertices as characters we can represent paths as strings and sets of paths as sets of strings.

We can define precisely what is the set of all paths on R, the set of all paths on R from α to β, and functions that calculate the endpoints of a path and that replace all occurrences of a certain arrow in a path – or in a set of paths – by a path whose endpoints are that arrow.

In the example



the set Paths_{R^*} is infinite, because it contains paths with an arbitrary number of "identity arrows" – for example,

"XXXUWWOOO" $\in \mathrm{Paths}_{R^*}$

But we can apply to Paths_{R^*} the series of replacements

Repl("XX", "X")
Repl("UU", "U")
Repl("VV", "V")
Repl("WW", "W")
Repl("OO", "O")

which will take all paths in Paths_{R^*} to paths in $\mathrm{Paths}_{R^{*-}}$, and then

Repl("XO", "XWO")
Repl("XW", "XVW")
Repl("UO", "UWO")
Repl("VO", "VWO")

to map $\mathrm{Paths}_{R^{*-}}$ to Paths_R ...

7. ALEXANDROFF TOPOLOGIES

WE HAVE BEEN REPRESENTING (CERTAIN) TOPOLOGIES AS DAGs, BUT HAVEN'T SEEN PRECISELY YET HOW – AND WHEN – THIS CAN BE DONE...

A TOPOLOGY $\mathcal{O}(X)$ IS SAID TO BE <u>ALEXANDROFF</u> IF ALL INTERSECTIONS OF OPEN SETS IN IT YIELD OPEN SETS. FOR EXAMPLE, $\mathcal{O}(\mathbb{R})$ IS NOT ALEXANDROFF, BUT EVERY TOPOLOGY ON A <u>FINITE</u> SET X IS ALEXANDROFF.

IF $\mathcal{O}(X)$ IS ALEXANDROFF AND $\alpha \in X$ THEN THE CONSTRUCTION

$$\downarrow\alpha := \bigcap \{ U \in \mathcal{O}(X) \mid \alpha \in U \}$$

YIELDS AN OPEN SET – THE SMALLEST OPEN SET CONTAINING α – AND THE SETS OF THE FORM $\downarrow\alpha$ FORM A BASIS FOR $\mathcal{O}(X)$: FOR ANY OPEN SET $U \in \mathcal{O}(X)$ WE HAVE

$$U = \bigcup_{\alpha \in U} \downarrow\alpha .$$

THE CONSTRUCTION $\bigcup_{\alpha \in S} \downarrow\alpha$ MAKES SENSE FOR ARBITRARY [redacted]SUBSETS $S \subseteq X$, AND – IN ALEXANDROFF SPACES – IT YIELDS THE SMALLEST OPEN SET CONTAINING S. I.E., WE HAVE

$$\bigcup_{\alpha \in S} \downarrow\alpha = \bigcap \{ U \in \mathcal{O}(X) \mid U \supseteq S \}.$$

THIS IS DUAL – IN A SENSE THAT WE WILL MAKE PRECISE LATER – TO THE CONSTRUCTION OF THE INTERIOR OF A SET AS THE UNION OF ALL OPEN SETS CONTAINED IN IT; WE CALL THIS NEW OPERATION THE "COINTERIOR". FORMALLY,

$$S^{\mathrm{coint}} := \bigcup_{\alpha \in S} \downarrow\alpha = \bigcap \{ U \in \mathcal{O}(X) \mid U \supseteq S \}$$

$$S^{\mathrm{int}} := \bigcup_{\substack{\alpha \in X \\ \downarrow\alpha \subseteq S}} \downarrow\alpha = \bigcup \{ U \in \mathcal{O}(X) \mid U \subseteq S \}$$

THE TWO EQUALITIES HOLD IN ALEXANDROFF SPACES

NOTE THAT $S^{\mathrm{int}} = \bigcup_{\substack{\alpha \in X \\ \downarrow\alpha \subseteq S}} \downarrow\alpha$ WORKS BECAUSE THE "$\downarrow\alpha$"S FORM A BASIS.

AN EXAMPLE: FOR OUR TOPOLOGICAL SPACE

$$\mathbb{V} = \begin{tikzcd}[sep=small] \alpha \arrow[dr] & & \beta \arrow[dl] \\ & \gamma & \end{tikzcd},$$

$$(\mathbb{V}, \mathcal{O}(\mathbb{V})) = \left(\left\{ \substack{\alpha,\ \beta, \\ \gamma} \right\}, \left\{ \substack{00\\0}, \substack{00\\1}, \substack{01\\1}, \substack{10\\0}, \substack{11\\1} \right\} \right)$$

WE HAVE:

$$\begin{aligned}
\left(\substack{01\\0}\right)^{\mathrm{coint}} &= \{\beta\}^{\mathrm{coint}} \\
&= \bigcap \{ U \in \mathcal{O}(\mathbb{V}) \mid U \supseteq \{\beta\} \} \\
&= \bigcap \left\{ \substack{01\\1}, \substack{11\\1} \right\} \\
&= \substack{01\\1} \cap \substack{11\\1} \\
&= \substack{01\\1} \\
&= \{\beta, \gamma\}
\end{aligned}$$

AND ALSO:

$$\begin{aligned}
\bigcup_{\alpha \in \substack{01\\0}} \downarrow\alpha &= \downarrow\beta \\
&= \substack{01\\1} \\
&= \{\beta, \gamma\}.
\end{aligned}$$