

# What “is” the Yoneda Lemma?

(EBCT 2026 - 2026 may 27)

<https://anggtwu.net/math-b.html#2026-ebct>  
<https://anggtwu.net/2026-alguns-motivos-bel.html>

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## Abstract

The “Yoneda Embedding” is easy to remember, but to prove it we need to prove the “Yoneda Lemma” first; the “Yoneda Lemma” takes a lot of mental space.

Let me refer to that “Yoneda Lemma” as the “Yoneda Lemma (in the strict sense)”; the “Yoneda Lemma (in the wide sense)” will be the “Yoneda Lemma (in the strict sense)” plus the “Yoneda Embedding”. In the next paragraphs I will use the term “Yoneda Lemma” to mean the “Yoneda Lemma (in the wide sense)”.

What “is” the Yoneda Lemma? Let’s call that question Q1, and let’s split it into several subquestions in the usual way:

- Q2 What is the best notation, and what are the best diagrams, for remembering the Yoneda Lemma after we understand it well?
- Q3 What are the best motivating examples for the Yoneda Lemma? As an aside, what does “best motivating examples” mean? Can we formalize that idea? There is a partial answer in [OchsMD]; are there other ones?
- Q4 Imagine that you want to present the Yoneda Lemma for non-categorists, and imagine that you can prepare a video for your presentation, and complement it with a blog post. What are the figures and animations that you would choose? Which slogans would you choose? Why?

Note that I am sort of splitting “what is the Yoneda Lemma?” into several subquestions, and some of them are naturally associated to verbs: “remember”, “reconstruct”, “generalize”, “specialize”, “present”, “teach”; some others are nouns: “analogies”, “animations”, “slogans”.

I only started studying Mathematical Education (“ME”) very recently, and I was very surprised to see that this division into subquestions is studied a lot in ME. One extreme example is the book [Ma], that discusses “deep understanding” in the context of teaching Arithmetic to children, but practically all the ideas in that book are trivial to adapt to Category Theory. A less extreme example is [VanHiele], that discusses “levels of thinking”, mainly in the context of school geometry.

In my presentation I will show some texts and ideas from ME that I believe that would be especially useful to categorists. The “recommended readings” will be basically the references at the end of [BadFoundations].

## Introduction

I was a **very incompetent** Category Theorist.  
Everybody understood the Yoneda Lemma –  
except me.

Why? What was going on?

How could I diagnose the problem  
*in a way that would let me fix it?*

I knew that my *working memory* – see [Greer] –  
was relatively weak, and *sometimes* I could  
understand concepts from CT  
by drawing lots of diagrams,  
but knowing that wasn't enough...

Then I started to study Mathematical Education  
to understand some difficulties of my students,  
and the lots of ideas fell into place.

## Sfard

Some sections from [Sfard, pp.9–33]:

(“Thinking as Communicating – Human Development, the Growth of Discourses, and Mathematizing”)

- I.1. The Quandary of Number
- I.2. The Quandary of Abstraction (and Transfer)
- I.3. The Quandary of Misconceptions
- I.4. The Quandary of Learning Disability
- I.5. The Quandary of Understanding
- I.6. Puzzling about Thinking - in a Nutshell

## Introduction (2)

In short: mathematicians don't even know that Mathematical Education **exists**.

Here are some books, articles and chapters on ME:

- Anna Sfard: “Thinking as Communicating – Human Development, the Growth of Discourses, and Mathematizing”
- Anna Sierpinska: “Understanding in Mathematics”
- Pierre van Hiele: “Structure and Insight – A Theory of Mathematics Education”
- Dave Hewitt: “Arbitrary and Necessary, part 2: Assisting Memory”
- Drouhard/Teppo: “Symbols and Language”

They are canonical references (FSV of...), but even Eugenia Cheng and David Corfield don't cite them...

My first plan for this talk was to start it by apologizing profusely and by saying “people, my talk may look trivial but at least you will learn a bit about an area that mathematicians know nothing about”...

## Introduction (3)

...but then I found some nice technical things to present –  
 a family of operations **with omitted names** that work like this,

$$(F \xrightarrow{T} G) \left( \begin{array}{c} D \\ f \downarrow \\ E \end{array} \right) = \left( \begin{array}{ccc} FD & \xrightarrow{TD} & GD \\ Fg \downarrow & & \downarrow Gg \\ FE & \xrightarrow{TE} & GE \end{array} \right)$$

$$(F \rightarrow G) \left( \begin{array}{c} D \\ \downarrow \\ E \end{array} \right) = \left( \begin{array}{ccc} FD & \longrightarrow & GD \\ \downarrow & & \downarrow \\ FE & \longrightarrow & GE \end{array} \right)$$

$$(F \rightarrow G) \left( \begin{array}{c} D \\ \downarrow \\ E \end{array} \right) = \left( \begin{array}{ccc} (C \rightarrow D) & \longrightarrow & (A \rightarrow RD) \\ \downarrow & & \downarrow \\ (C \rightarrow E) & \longrightarrow & (A \rightarrow RE) \end{array} \right)$$

$$((C \rightarrow) \rightarrow (A \rightarrow R-)) \left( \begin{array}{c} D \\ \downarrow \\ E \end{array} \right) = \left( \begin{array}{ccc} (C \rightarrow)D & \longrightarrow & (A \rightarrow R-)D \\ \downarrow & & \downarrow \\ (C \rightarrow)E & \longrightarrow & (A \rightarrow R-)E \end{array} \right)$$

## Juxtapositions

From [BadFoundations, sec. 4.1]:

In some cases, like  $2(3+4)$ , the juxtaposition means an operation that was elided, but whose name we know, and we can write it explicitly:  $2 \cdot (3 + 4)$ . In other cases the operation doesn't have a standard notation, and we have to improvise. Let's use the symbol 'ap' for application, and 's' for substitute:

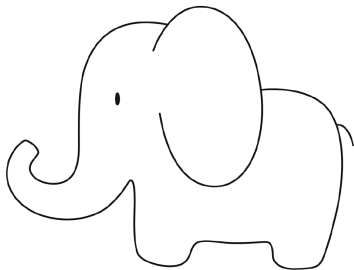
$$\begin{aligned}
 2(y + z) &\Rightarrow 2 \cdot (y + z) \\
 f(y + z) &\Rightarrow f \text{ ap } (y + z) \\
 (a + b)[a := 42] &\Rightarrow (a + b) \text{ s } [a := 42]
 \end{aligned}$$

## Variants of **ap**

$$(F \xrightarrow{T} G) \left( \begin{array}{c} D \\ f \downarrow \\ E \end{array} \right) = \left( \begin{array}{ccc} FD & \xrightarrow{TD} & GD \\ Fg \downarrow & & \downarrow Gg \\ FE & \xrightarrow{TE} & GE \end{array} \right)$$

$$(F \rightarrow G) \left( \begin{array}{c} D \\ \downarrow \\ E \end{array} \right) = \left( \begin{array}{ccc} FD & \longrightarrow & GD \\ \downarrow & & \downarrow \\ FE & \longrightarrow & GE \end{array} \right)$$

## What is an elephant?



This is a **drawing**  
of an elephant.

From [Bessis, p.253]:

But that's not all. When a child sees a real elephant for the first time, even if they've never seen a picture or heard it spoken of before, if you point your finger at the elephant and say, "That's an elephant" the child knows immediately what you're talking about.

That's not as obvious as it may seem. What keeps the child from thinking that what you're calling an elephant is simply the left front foot, or the trunk, or a piece of the trunk, or a fly sitting on the trunk?

**What is the Yoneda Lemma?**

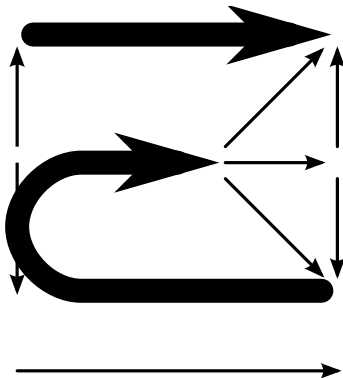
**What is a drawing of the Yoneda Lemma?**

## Reconstructive memory

From [Greer, p.23]:

He stressed the reconstructive nature of memory – what is remembered is not an exact copy but a stripped-down version of the original.

The figure at the right is from [OchsIDARCT, sec.1] – it is how I remember the Frobenius Property... I forget the letters, and I remember a **shape** and **movement**.



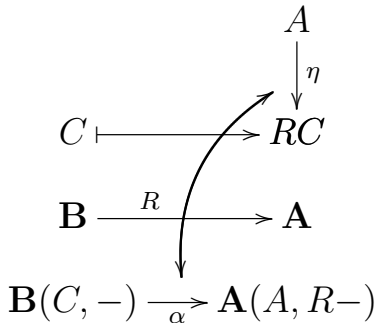
## On the the Missing Diagrams in Category Theory

From [OchsMD, sec.2]:

(CD): Our diagrams are made of components that are nodes and arrows. The nodes can contain arbitrary expressions. The arrows work as connectives, and each arrow can be interpreted as the top-level connective in the smallest subexpression that contains it. For example, the curved arrow in the diagram at the right can be interpreted as:

$$(A \xrightarrow{\eta} RC) \leftrightarrow (\mathbf{B}(C, -) \xrightarrow{T} \mathbf{A}(A, R-)).$$

(CAI): “Above” usually means “inside”, or “internal view”. In the diagram above the morphism  $\eta : A \rightarrow RC$  is in  $\mathbf{A}$  and  $C$  is an object of  $\mathbf{B}$ . Also, the arrow  $C \mapsto RC$  is above  $\mathbf{B} \xrightarrow{R} \mathbf{A}$ , and this means that it is an internal view of the functor  $R$ . Note that *usually* is not *always* — and  $\mathbf{B} \xrightarrow{R} \mathbf{A}$  is not an internal view of  $\mathbf{B}(C, -) \xrightarrow{T} \mathbf{A}(A, R-)$ .



## More conventions

From [OchsMD, sec.2]:

(COT): We use a notation as close to the original text as possible, especially when we are trying to draw the missing diagrams for some existing text. If we were drawing the missing diagrams for the Proposition 1 of [CWM, Section III.2] our diagram would be the one at the right...

...but I hate Mac Lane's choice of letters, so I decided to use another notation here (i.e., in [OchsMD]).

(CNSh): A translation of a diagram  $D$  to another notation is drawn with the same shape as  $D$ .

$$\begin{array}{ccc}
 & & c \\
 & & \downarrow u \\
 r & \xrightarrow{\quad} & Sr \\
 D & \xrightarrow{S} & C \\
 & \searrow & \downarrow \\
 D(r, -) & \xrightarrow{\varphi} & C(c, S-)
 \end{array}$$

## Some Yonedas

From [CWM, p.61]:

**Lemma (Yoneda).**

If  $K : D \rightarrow \mathbf{Set}$  is a functor from  $D$   
and  $r$  an object in  $D$

(for  $D$  a category with small hom-sets),  
there is a bijection

$$y : \mathbf{Nat}(D(r, -), K) \cong Kr$$

which sends each natural transformation  $\alpha : D(r, -) \rightarrow K$  to  $\alpha_r 1_r$ ,  
the image of the identity  $r \rightarrow r$ .

From [SmithCat3, p.351]:

**Theorem 187 (The Core Yoneda Lemma).**

For any object  $A$  of the locally small category  $\mathbf{C}$ ,  
and any functor  $F : \mathbf{C} \rightarrow \mathbf{Set}$ ,

$$\mathbf{Nat}(\mathbf{C}(A, -), F) \cong FA.$$

From [SmithCat3, p.356]:

**Theorem 190 (Full Yoneda Lemma).**

For any locally small category  $\mathbf{C}$ , object  $A$  in  $\mathbf{C}$ ,  
and covariant functor  $F : \mathbf{C} \rightarrow \mathbf{Set}$ ,

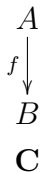
$$\mathbf{Nat}(\mathbf{C}(A, -), F) \cong FA,$$

both naturally in  $A$   
and naturally in  $F$ .

## Convention: Type Theory

From [OchsMD, sec.2]:

(CTT): Our diagrams should be close to Type Theory: it should be possible to use them as a scaffolding for formalizing our text in (pseudocode for) a proof assistant.



$\mathbf{C}$  is a category  
 $A \in \text{Objs}_{\mathbf{C}}$   
 $B \in \text{Objs}_{\mathbf{C}}$   
 $A \rightarrow B$  is  $\text{Hom}(A, B)$ , i.e.,  
 $\text{Hom}_{\mathbf{C}}(A, B)$   
 $f : A \rightarrow B$ , i.e.,  
 $f \in A \rightarrow B$ , i.e.,  
 $f \in \text{Hom}_{\mathbf{C}}(A, B)$

So the ‘ $\downarrow$ ’ at the right is a hom-set, and the diagram would still makes sense if we deleted the  $f$ !



$\mathbf{C}$  is a category  
 $A \in \text{Objs}_{\mathbf{C}}$   
 $B \in \text{Objs}_{\mathbf{C}}$   
 $A \rightarrow B$  is  $\text{Hom}(A, B)$ , i.e.,  
 $\text{Hom}_{\mathbf{C}}(A, B)$

**Two adjunctions:**

$L \dashv R$  and

$(\times B) \dashv (B \rightarrow)$

$$\begin{array}{ccc}
 LA & \longleftarrow \dashv & A \\
 \downarrow & \longleftrightarrow & \downarrow \\
 B & \dashv \longrightarrow & RB
 \end{array}$$

$$\mathbf{B} \begin{array}{c} \xleftarrow{L} \\ \xrightarrow{R} \end{array} \mathbf{A}$$

$$\begin{array}{ccc}
 A \times B & \longleftarrow \dashv & A \\
 \downarrow & \longleftrightarrow & \downarrow \\
 C & \dashv \longrightarrow & B \rightarrow C
 \end{array}$$

$$\mathbf{Set} \begin{array}{c} \xleftarrow{(\times B)} \\ \xrightarrow{(B \rightarrow)} \end{array} \mathbf{Set}$$

## An adjunction with functions, units, etc

$$\begin{array}{ccccccc}
 & & LA' \leftarrow A' & & & & \\
 & & \downarrow Lf \quad \leftarrow \downarrow f & & & & \\
 LR & LRB & LA \leftarrow A & A & \text{id}_A & & \\
 \varepsilon \downarrow & \varepsilon_B \downarrow & \downarrow h^b \quad \leftarrow \downarrow h & \downarrow \eta_A & \downarrow \eta & & \\
 \text{id}_B & B & B \xrightarrow{g} RB & RLA & LR & & \\
 & & \downarrow k \quad \xrightarrow{\quad} \downarrow Rk & & & & \\
 & & B' \xrightarrow{\quad} RB' & & & & \\
 & & \mathbf{B} \xrightleftharpoons[R]{L} \mathbf{A} & & & & 
 \end{array}$$

# Reconstructing the functors in Yoneda

$$\begin{array}{ccc}
 & A & \\
 & \downarrow \eta & \\
 C & \longrightarrow & RC \\
 \downarrow f & \longmapsto & \downarrow \\
 D & \longrightarrow & RD \\
 \downarrow g & \longmapsto & \downarrow \\
 E & \longrightarrow & RE \\
 & & \downarrow \\
 ? & \xrightarrow{?} & ?
 \end{array}$$

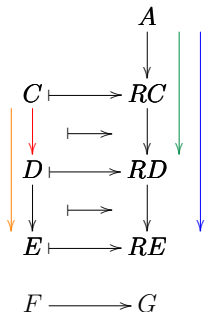
$$\begin{array}{ccc}
 (C \rightarrow D) & \longrightarrow & (A \rightarrow RD) \\
 \downarrow & & \downarrow \\
 (C \rightarrow E) & \longrightarrow & (A \rightarrow RE)
 \end{array}$$

$$\begin{array}{ccc}
 f & \longrightarrow & \eta; Rf \\
 \downarrow & & \downarrow \\
 f; g & \longrightarrow & (\eta; Rf); Rg \\
 & & \downarrow \\
 & & \eta; R(f; g)
 \end{array}$$

## Reconstructing the functors in Yoneda (2)

$$\begin{array}{ccc}
 & A & \\
 & \downarrow & \\
 C & \xrightarrow{\quad} & RC \\
 \downarrow & \lrcorner & \downarrow \\
 D & \xrightarrow{\quad} & RD \\
 \downarrow & \lrcorner & \downarrow \\
 E & \xrightarrow{\quad} & RE \\
 \\ 
 F & \longrightarrow & G
 \end{array}
 \quad
 (F \rightarrow G) \left( \begin{array}{c} D \\ \downarrow \\ E \end{array} \right) = \left( \begin{array}{ccc} (C \rightarrow D) & \longrightarrow & (A \rightarrow RD) \\ \downarrow & & \downarrow \\ (C \rightarrow E) & \longrightarrow & (A \rightarrow RE) \end{array} \right)$$

## Reconstructing the functors in Yoneda (3)



$$(F \rightarrow G) \left( \begin{array}{c} D \\ \downarrow \\ E \end{array} \right) = \left( \begin{array}{ccc} (C \rightarrow D) & \longrightarrow & (A \rightarrow RD) \\ \downarrow & & \downarrow \\ (C \rightarrow E) & \longrightarrow & (A \rightarrow RE) \end{array} \right)$$

$$((C \rightarrow) \rightarrow (A \rightarrow R-)) \left( \begin{array}{c} D \\ \downarrow \\ E \end{array} \right) = \left( \begin{array}{ccc} (C \rightarrow)D & \longrightarrow & (A \rightarrow R-)D \\ \downarrow & & \downarrow \\ (C \rightarrow)E & \longrightarrow & (A \rightarrow R-)E \end{array} \right)$$

## Ausubel

From: [AusubelEP, p.16]

### **The Interdependence of Theories of Learning and Theories of Teaching**

Although knowledge of causation does not imply immediate discovery of control procedures, it aids in the search for such procedures. For one thing, it narrows the field; for another, it enables one to try procedures that have proved successful in controlling related conditions. Knowing that tuberculosis was caused by a microorganism, for example, did not immediately provide us with a cure or a preventative. But it enabled us to try approaches—such as vaccines, immune sera, antisepsis, quarantine, and chemotherapy—that had been used successfully in treating other infectious diseases. In the same sense, knowledge of the cause of cancer would help immeasurably in discovering a cure, and knowledge of the nature and relevant variables involved in concept acquisition would be of invaluable assistance in devising effective methods of teaching concepts.

## References

- [AusubelEP] D. P. Ausubel, J. D. Novak, and H. Hanesian. *Educational Psychology, a Cognitive View - 2nd ed.* Holt, Rinehart, Winston, 1978.
- [BadFoundations] E. Ochs. “Bad Foundations and Manipulable Objects”. <https://anggtwu.net/math-b.html#2026-bad-foundations>. 2026.
- [Bessis] D. Bessis. *Mathematica: a Secret World of Intuition and Curiosity*. Yale, 2024.
- [CWM] S. Mac Lane. *Categories for the Working Mathematician (2nd ed.)* Springer, 1997.

- [Greer] B. Greer. “Cognitive Psychology and Mathematical Thinking”. In: *For the Learning of Mathematics* 1.3 (1982), pp. 19–26.
- [Ma] L. Ma. *Knowing and Teaching Elementary Mathematics (Anniversary Edition)*. Routledge, 2010.
- [OchsIDARCT] E. Ochs. “Internal Diagrams and Archetypal Reasoning in Category Theory”. In: *Logica Universalis* 7.3 (2013). <https://anggtwu.net/math-b.html#idarct>.
- [OchsMD] E. Ochs. *On the Missing Diagrams in Category Theory (First-Person Version)*. <https://anggtwu.net/math-b.html#2022-md>. 2022. arXiv: 2401.12345.

- [Sfard] A. Sfard. *Thinking as Communicating – Human Development, the Growth of Discourses, and Mathematizing*. Cambridge, 2008.
- [SmithCat3] P. Smith. *Introducing Category Theory, 3rd ed.* Cambridge, 2026.
- [VanHiele] P. M. van Hiele. *Structure and Insight - A Theory of Mathematics Education*. Academic Press, 1986.