Notes on Valeria de Paiva's Technical Report (1991)

Eduardo Ochs - RCN/PURO/UFF
Psicopata do CEFET
https://anggtwu.net/math-b.html

Proposition 1

Proposition 1, p.4,
a.k.a: "composition needs checking".

Lets suppose that we are in **Set**.
Let's omit the types when they can be inferred from the diagrams, *and let's use dependent variables.*

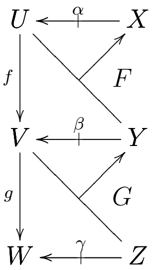
So $A \equiv (U, X, \alpha)$ is a triple
$(U \in \mathbf{Sets}, X \in \mathbf{Sets}, \alpha : U \times X \to \{0,1\})$,
So $B \equiv (V, Y, \beta)$ is a triple
$(V \in \mathbf{Sets}, Y \in \mathbf{Sets}, \beta : V \times Y \to \{0,1\})$,

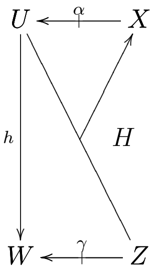
and a morphism $(f, F) : A \to B$
is actually a triple

$$\begin{pmatrix} f : U \to V, \\ F : U \times Y \to X, \\ [\forall (u,y).\alpha \to \beta] : \ldots \end{pmatrix}$$

in which the $[\forall (u,y).\alpha \to \beta]$ is a guarantee/witness for:

$$\begin{aligned} & \forall (u,y).\alpha \to \beta \\ \equiv\ & \forall (u,y).\alpha(u,x) \to \beta(v,y) \\ \equiv\ & \forall (u,y).\alpha(u,x(u,y)) \to \beta(v(u),y) \\ \equiv\ & \forall (u,y).\alpha(u,F(u,y)) \to \beta(f(u),y) \end{aligned}$$





We want to compose the
$(f, F, [\forall (u,y).\alpha \to \beta])$ with the
$(g, G, [\forall (v,z).\beta \to \gamma])$,
and obtain a:

$$(h, H, [\forall (u,z).\alpha \to \gamma]).$$

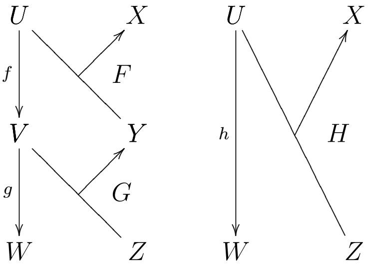
Proposition 1: compatibility conditions

Let's forget α, β and γ for a while, and let's consider f, F, g and G as compatibility conditions...

I will write $\begin{pmatrix} u, & x, \\ v, & y, \\ w, & z \end{pmatrix}$, with commas, for a tuple that only obeys $u \in U$, $x \in X$, $\ldots$, $z \in Z$, and will write $\begin{pmatrix} u & x \\ v & y \\ w & z \end{pmatrix}$, without commas, for $\begin{pmatrix} u, & x, \\ v, & y, \\ w, & z \end{pmatrix}$ plus guarantees for the "obvious" compatibility conditions, that in this case are:

$$\begin{array}{l} [v = v(u)], \\ [x = x(u,y)], \\ [w = w(v)], \\ [y = y(v,z)] \end{array} \quad \text{i.e.,} \quad \begin{array}{l} [v = f(u)], \\ [x = F(u,y)], \\ [w = g(v)], \\ [y = G(v,z)] \end{array}$$

So:
$\begin{pmatrix} u & x \\ v & y \\ w & z \end{pmatrix}$ has 4 compatibility conditions,
$\begin{pmatrix} u & x \\ v & y \end{pmatrix}$ has 2 compatibility conditions,
$\begin{pmatrix} v & y \\ w & z \end{pmatrix}$ has 2 compatibility conditions,
$\begin{pmatrix} u & \\ & y \end{pmatrix}$ has 0 compatibility conditions...



...and $\begin{pmatrix} u & x \\ w & z \end{pmatrix}$ has these compatibility conditions:

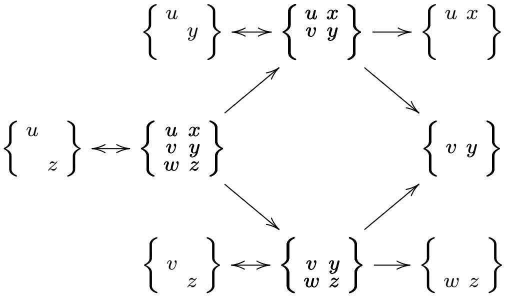
$$\begin{array}{rcl} w & = & w(v(u)) \\ & = & g(f(u)) \\ \\ x & = & x(u,y) \\ & = & x(u,y(v,z)) \\ & = & x(u,y(v(u),z)) \\ & = & F(u,G(f(u),z)) \end{array}$$

Proposition 1: basic pullbacks

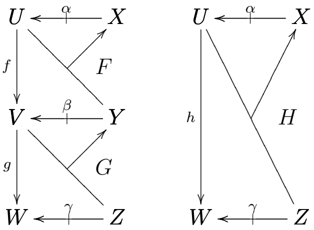
Now let's write $\left\{\begin{smallmatrix} u & x \\ v & y \\ w & z \end{smallmatrix}\right\}$
for the space of all (compatible) tuples of the form $\left(\begin{smallmatrix} u & x \\ v & y \\ w & z \end{smallmatrix}\right)$,
and do the same for all other combinations of letters...

We have the maps at the right, and the diamond at the middle is a pullback.

Note that:
α is defined on $\left\{\begin{smallmatrix} u & x \\ & \\ & \end{smallmatrix}\right\}$,
β is defined on $\left\{\begin{smallmatrix} & \\ v & y \\ & \end{smallmatrix}\right\}$,
γ is defined on $\left\{\begin{smallmatrix} & \\ & \\ w & z \end{smallmatrix}\right\}$,
$\alpha \to \beta$ is defined on $\left\{\begin{smallmatrix} u & \\ & y \\ & \end{smallmatrix}\right\}$,
$\beta \to \gamma$ is defined on $\left\{\begin{smallmatrix} & \\ v & \\ & z \end{smallmatrix}\right\}$, and
$\alpha \to \gamma$ is defined on $\left\{\begin{smallmatrix} u & \\ & \\ & z \end{smallmatrix}\right\}$...



Proposition 1: the implication



Remember that want to compose the $(f, F, [\forall(u,y).\alpha \to \beta])$ with the $(g, G, [\forall(v,z).\beta \to \gamma])$, and obtain a: $(h, H, [\forall(u,z).\alpha \to \gamma])$.

We saw that
$h(u) = g(f(u))$ and
$H(u,z) = F(u, G(f(u), z))$.

We still need to check this:

$$\left(\begin{array}{l} \left(\forall \begin{pmatrix} u & \\ & y \end{pmatrix} .\alpha \to \beta\right) \quad \wedge \\ \left(\forall \begin{pmatrix} v & \\ & z \end{pmatrix} .\beta \to \gamma\right) \end{array} \right) \to$$
$$\left(\forall \begin{pmatrix} u & \\ & z \end{pmatrix} .\alpha \to \gamma\right)$$

Sketch of the proof:

$$\dfrac{\dfrac{\dfrac{\forall \begin{pmatrix} u & \\ & y \end{pmatrix} .\alpha \to \beta}{\forall \begin{pmatrix} u & x \\ v & y \end{pmatrix} .\alpha \to \beta}}{\forall \begin{pmatrix} u & x \\ v & y \\ w & z \end{pmatrix} .\alpha \to \beta} \qquad \dfrac{\dfrac{\forall \begin{pmatrix} v & \\ & z \end{pmatrix} .\beta \to \gamma}{\forall \begin{pmatrix} v & y \\ w & z \end{pmatrix} .\beta \to \gamma}}{\forall \begin{pmatrix} u & x \\ v & y \\ w & z \end{pmatrix} .\beta \to \gamma}}{\dfrac{\forall \begin{pmatrix} u & x \\ v & y \\ w & z \end{pmatrix} .\alpha \to \gamma}{\forall \begin{pmatrix} u & \\ & z \end{pmatrix} .\alpha \to \gamma}}$$