Cálculo 3 - 2025.1

Aulas 4 e ?: curvas de Bézier

Eduardo Ochs - RCN/PURO/UFF
http://anggtwu.net/2025.1-C3.html

Links

Versões anteriores deste PDFzinho:
3dT25 (2021.2) Curvas de Bézier

O vídeo
Assista este vídeo aqui,
https://www.youtube.com/watch?v=aVwxzDHniEw
só no trecho entre 1:55 e 6:17.

O vídeo foi feito pela Freya Holmér
e o título dele é "The Beauty of Bézier Curves".
As animações do vídeo foram feitas no Unity.

Ele não tem legendas em português, mas eu vou copiar
as legendas em inglês dele pra próxima página –
pergunte no Telegram o significado dos trechos
que você não entender.

00:01:56.160 –> 00:02:01.200
Let’s say you have two points: P0 and P1
connected by a line segment

00:02:02.640 –> 00:02:06.160
Now imagine a third point P
between these two points

00:02:06.160 –> 00:02:09.680
The position of P could be described
by what is called a t-value

00:02:10.240 –> 00:02:13.520
a value between 0 to 1, similar to a percentage

00:02:13.520 –> 00:02:18.800
where t-values at 1 moves it to P1
and t-values at 0 moves it to P0

00:02:19.440 –> 00:02:22.480
and any values in between
are a blend between the two

00:02:23.440 –> 00:02:27.760
This function is called linear interpolation
or lerp for short

00:02:27.760 –> 00:02:30.320
Mathematically, you can write it as (1-t)P0*tP1

00:02:33.760 –> 00:02:35.840
Now, what if we add another point?

00:02:36.880 –> 00:02:40.800
We now have two interpolated points
one on each line segment

00:02:40.800 –> 00:02:45.120
”lerping” on their respective line
based on the same t-value we saw earlier

00:02:45.680 –> 00:02:50.240
We can then connect these two
points with another line segment

00:02:50.240 –> 00:02:54.880
If we then add a point on that line that
also lerps based on the same t-value

00:02:55.680 –> 00:02:58.080
you can see that it follows a very specific path

00:02:58.800 –> 00:03:03.840
This path is a quadratic bézier curve

00:03:06.320 –> 00:03:07.840
but we don’t have to stop here

00:03:07.840 –> 00:03:09.120
what if we add another point?

00:03:09.760 –> 00:03:11.440
we repeat the same process

00:03:12.160 –> 00:03:13.600
add three points

00:03:13.600 –> 00:03:14.640
connect them

00:03:14.640 –> 00:03:15.920
add two points

00:03:15.920 –> 00:03:17.120
connect those

00:03:17.120 –> 00:03:18.320
and add the last point

00:03:20.000 –> 00:03:23.760
and this point will now follow
the path of the cubic bézier curve

00:03:25.440 –> 00:03:30.000
what’s beautiful about this construction is that
it works no matter what points we use

00:03:30.560 –> 00:03:32.720
We can change the shape to anything

00:03:32.720 –> 00:03:36.000
and following the same rules it
will give us this smooth path

00:03:38.880 –> 00:03:42.800
We’re going to focus mostly on the cubic
bézier curve for the rest of this video

00:03:42.800 –> 00:03:45.840
since it’s the most common one

00:03:57.600 –> 00:04:00.800
This particular method of
getting a point in a bézier curve

00:04:00.800 –> 00:04:02.400
based on nested lerps

00:04:02.960 –> 00:04:08.320
where each point along the way is calculated
from lerps from the points that came before it

00:04:08.320 –> 00:04:10.480
eventually forming the path of the bézier curve

00:04:11.040 –> 00:04:14.320
is called De Casteljau’s Algorithm

00:04:14.320 –> 00:04:17.360
I personally love it because
of its numerical stability

00:04:17.360 –> 00:04:19.600
and just how easy it is to remember

00:04:19.600 –> 00:04:22.000
it’s just lerps all the way down

00:04:22.000 –> 00:04:24.720
But there is another way we can interpret this

00:04:25.840 –> 00:04:28.720
Let’s start by writing out the math
for all of our lerps

00:04:33.280 –> 00:04:35.920
then, let’s expand this formula entirely

00:04:37.120 –> 00:04:39.760
let’s also color code the points for readability

00:04:40.800 –> 00:04:42.480
What you might be able to see is that

00:04:42.480 –> 00:04:45.600
we can rearrange this formula
in terms of each point

00:04:51.360 –> 00:04:55.840
First, each point can be visualized
as a vector from the origin

00:05:00.160 –> 00:05:02.240
But this is where it gets interesting

00:05:02.240 –> 00:05:06.640
Each of them are multiplied by four polynomials
based on our t-value

00:05:07.360 –> 00:05:10.640
This is what they look like
for all cubic bézier curves

00:05:12.560 –> 00:05:14.880
You might be able to tell
that the values are sort of

00:05:14.880 –> 00:05:24.000
trading off with each other
as weights as we change t

00:05:24.000 –> 00:05:26.400
in the beginning, the first weight is 1

00:05:26.400 –> 00:05:30.720
but as the t-value increases
the values shift across the points

00:05:30.720 –> 00:05:32.960
until the last weight has a value of 1

00:05:33.760 –> 00:05:35.520
This is called a weighted sum

00:05:35.520 –> 00:05:39.600
where each of these weights together add up to 1
at any given t value

00:05:40.960 –> 00:05:43.760
Let’s apply these weights
to the vectors of the points

00:05:44.480 –> 00:05:47.760
As you can see, they trade off
weights exactly the same way

00:05:48.880 –> 00:05:51.840
so let’s add them together

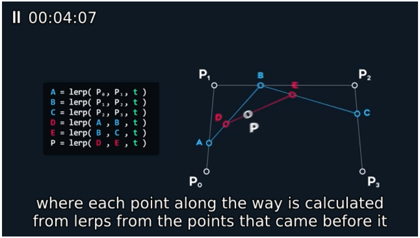
00:05:54.480 –> 00:05:57.840
What we get is exactly the
same behavior as with the lerps

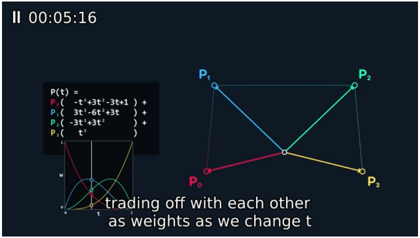
00:05:57.840 –> 00:06:00.720
but using a different
interpretation of the same math

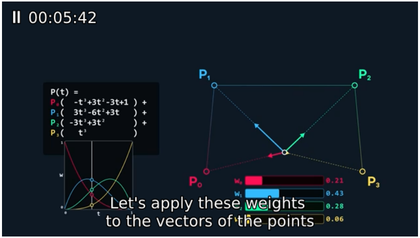
00:06:01.440 –> 00:06:05.840
This follows the very same bézier
curve we got with the lerps

00:06:14.160 –> 00:06:17.360
This is called
the Bernstein Polynomial Form of bézier curves

Alguns frames







Exercício 1

Neste exercício vamos usar

$P_0 = (1,1)$, $P_1 = (1,2)$, $P_2 = (3,2)$, $P_2 = (3,5)$.

Marque estes 4 pontos no $\mathbb{R}^2$ e escreva do lado de cada um dos pontos o nome dele. Neste exercício vamos chamar o gráfico com esses 4 pontos e os nomes dele de "diagrama básico", e você vai precisar de várias cópias do diagrama básico, uma pra cada item.

a) Seja $t = \frac{1}{2} = 0.5$. Encontre os pontos A, B, C, D, E, P da construção do frame 4:07 usando este valor de t e marque esses pontos na sua primeira cópia do diagrama básico. Escreva do lado de cada ponto o nome dele.

b) Idem, mas na segunda cópia do diagrama básico e com $t = \frac{1}{4}$.

c) Idem, mas na segunda cópia do diagrama básico e com $t = \frac{3}{4}$.

Exercício 1 (cont.)

d) Idem, mas em outra cópia, e usando $t = \frac{1}{8}$. Agora só escreva o nome do ponto P

e) Idem, mas usando $t = \frac{3}{8}$.

f) Idem, mas usando $t = \frac{5}{8}$.

g) Idem, mas usando $t = \frac{7}{8}$.

Exercício 2

Agora veja se você consegue refazer todos os exercícios anteriores num gráfico só sem desenhar os pontos auxiliares. Mais precisamente: comece com $t = \frac{1}{2}$, descubra no olho onde estão os pontos A, B, C, D, E, P para este valor de t sem desenhá-los, e desenhe só o ponto P, escrevendo "$P_{\frac{1}{2}}$" do lado dele. Depois faça a mesma coisa para $P_{\frac{1}{4}}$ e $P_{\frac{3}{4}}$, e depois para $P_{\frac{1}{8}}$, $P_{\frac{3}{8}}$, $P_{\frac{5}{8}}$, $P_{\frac{7}{8}}$. Dica: você pôr um dedo em cada um dos pontos A, B, C, D, E se ajudar.

Exercício 3.

No PDF sobre vetores tangentes você fez três exercícios de desenhar trajetórias curvas e os vetores tangentes delas em certos pontos. Eram os exercícios 2, 3 e 4 daqui:

http://angg.twu.net/LATEX/2021-2-C3-vetor-tangente.pdf

Agora que o seu olhômetro está bem melhor nós vamos ver um modo de desenhar aproximações pra esses trajetórias usando quase só desenhos e fazendo pouquíssimas contas.

Exercício 3 (cont.)

No exercício 2 do PDF de vetores tangentes nós tínhamos $P(t) = (\cos t, \operatorname{sen} t)$; no exercício 3 tínhamos $P(t) = (\cos t, \operatorname{sen} 2t)$, e no exercício 4 tínhamos $P(t) = (\cos 2t, \operatorname{sen} t)$. Vamos mudar os nomes para:

$$\begin{array}{ll} P(t) = (\cos t, \operatorname{sen} t), & P'(t) = \overrightarrow{(-\operatorname{sen} t, \cos t)} \\ Q(t) = (\cos t, \operatorname{sen} 2t), & Q'(t) = \overrightarrow{(-\operatorname{sen} t, 2\cos 2t)} \\ R(t) = (\cos 2t, \operatorname{sen} t), & R'(t) = \overrightarrow{(-2\operatorname{sen} 2t, \cos t)} \end{array}$$

Se usarmos quatro cores diferentes conseguimos representar pra cada trajetória as componentes x e y dela e as componentes x e y da derivada dela num gráfico só...