On a formula that is not in “Grothendieck Topologies in Posets”

A.J. Lindenhovius Eduardo Ochs

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Let $\mathbf{P}$ be a downward-directed poset, that we will also regard as a category. Then $\mathbf{E} := \mathbf{Set}^{\mathbf{P}}$ is a topos and $H := \mathrm{Sub}(1_{\mathbf{E}})$ is a Heyting Algebra. Let’s denote the set of all Grothendieck topologies on $\mathbf{E}$ by $\mathsf{GrTops}(\mathbf{P})$, the set of all nuclei on H by $\mathsf{Nucs}(\mathbf{P})$, and the set of subsets of (the set of points of) $\mathbf{P}$ by $\mathcal{P}(\mathbf{P})$. In [Lin14] Bert Lindenhovius shows that when $\mathbf{P}$ is Artinian we have bijections between $\mathcal{P}(\mathbf{P})$, $\mathsf{Nucs}(\mathbf{P})$, and $\mathsf{GrTops}(\mathbf{P})$. He uses notational conventions in which J always denotes a Grothendieck topology, j always denotes a nucleus, and X always denotes a subset of $\mathbf{P}$, and he writes the components these bijections as $(j \mapsto J_j)$, $(J \mapsto j_J)$, and so on; we will also write them here as $(j \mapsto J)$, $(J \mapsto j)$, etc, and we will write the bijections as $(X \leftrightarrow j)$, $(X \leftrightarrow J)$, and $(j \leftrightarrow J)$. Let’s put all this in a diagram:

$$
\begin{tikzcd}
\mathcal{P}(\mathbf{P}) \arrow[r, shift left, "\text{C.4.2}"] \arrow[d, shift right, "\substack{\text{2.8,}\\ \text{C.4.1}}"'] & \mathsf{Nucs}(\mathbf{P}) \arrow[l, shift left, dashed, "\text{(e-mail)}"] \arrow[dl, leftrightarrow, "\substack{\text{B.8,}\\ \text{B.25}}"] \\
\mathsf{GrTops}(\mathbf{P}) \arrow[u, shift right, "\text{2.9}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, mapsto, shift left, "(X\mapsto j)"] \arrow[d, leftrightarrow, "\substack{(X\mapsto J),\\ (J\mapsto X)}"'] & j \arrow[l, mapsto, shift left, dashed, "(j\mapsto X)"] \arrow[dl, leftrightarrow, "\substack{(j\mapsto J),\\ (J\mapsto j)}"] \\
J &
\end{tikzcd}
$$

He defines the components of these bijections as:

$$
\begin{array}{rrcll}
(X \mapsto j): & j_X(S) & = & X \to S & \text{(C.4.2, C.2)} \\
(j \mapsto X): & X_j & = & \{\, p \in \mathbf{P} \;:\; p \notin j({\downarrow}p \backslash \{p\}) \,\} & \text{(e-mail)} \\
(X \mapsto J): & J_X(p) & = & \{\, S \in \mathcal{D}({\downarrow}p) \;:\; X \cap {\downarrow}p \subseteq S \,\} & \text{(2.8, C.4.1)} \\
(J \mapsto X): & X_J & = & \{\, p \in \mathbf{P} \;:\; J(p) = \{{\downarrow}p\} \,\} & \text{(2.9)} \\
(j \mapsto J): & J_j(p) & = & \{\, S \in \mathcal{D}({\downarrow}p) \;:\; p \in j(S) \,\} & \text{(B.8, B.25)} \\
(J \mapsto j): & j_J(S) & = & \{\, p \in \mathbf{P} \;:\; S \cap {\downarrow}p \in J(p) \,\} & \text{(B.8, B.25)}
\end{array}
$$

The annotations like "(C.4.2, C.2)" indicate where these components are defined. Note that one of the annotations says "(e-mail)"; this is because that formula doesn't appear explicitly in [Lin14], and so I (Eduardo) asked him (Bert) if that formula was what I guessed it would be, and he replied with a formula slightly shorter than my guess, and a proof...

These notes are just to make his formula and his proof available in a public place. All the mathematical content here is by Bert Lindenhovius, and all the typesetting was done by Eduardo Ochs, who found Bert's proof hard to follow and decided to typeset it in Natural Deduction form.

If we combine $(j \mapsto J)$ and $(J \mapsto X)$ we get this:

$$
\begin{array}{rrcl}
(j \mapsto J): & J_j(p) & = & \{\, S \in \mathcal{D}({\downarrow}p) \;:\; p \in j(S) \,\} \\
(J \mapsto X): & X_J & = & \{\, p \in \mathbf{P} \;:\; J(p) = \{{\downarrow}p\} \,\} \\
(j \mapsto J \mapsto X): & X_{J_j} & = & \{\, p \in \mathbf{P} \;:\; \{\, S \in \mathcal{D}({\downarrow}p) \;:\; p \in j(S) \,\} = \{{\downarrow}p\} \,\} \\
 & & = & \{\, p \in \mathbf{P} \;:\; \forall S \in \mathcal{D}({\downarrow}p).\; ((p \in j(S)) \leftrightarrow (S = {\downarrow}p)) \,\} \\
(j \mapsto X): & X_j & = & \{\, p \in \mathbf{P} \;:\; p \notin j({\downarrow}p \backslash \{p\}) \,\}
\end{array}
$$

It is not obvious at all that $X_{J_j} = X_j$. We will prove that $p \in X_j$ iff $p \in X_{J_j}$, where:

$$
\begin{array}{rcl}
(p \in X_j) & = & (p \notin j({\downarrow}p \backslash \{p\})) \\
(p \in X_{J_j}) & = & \forall S \in \mathcal{D}({\downarrow}p).\; (p \in j(S)) \leftrightarrow (S = {\downarrow}p)
\end{array}
$$

Look:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\downarrow p\backslash\{p\} \in \mathcal{D}(\downarrow p)} \qquad \dfrac{p \in X_{J_j}}{\forall S \in \mathcal{D}(\downarrow p).(p \in j(S)) \leftrightarrow (S = \downarrow p)}}{(p \in j(\downarrow p\backslash\{p\})) \leftrightarrow (\downarrow p\backslash\{p\} = \downarrow p)}}{(p \notin j(\downarrow p\backslash\{p\})) \leftrightarrow (\downarrow p\backslash\{p\} \neq \downarrow p)} \qquad \dfrac{}{\downarrow p\backslash\{p\} \neq \downarrow p}}{\dfrac{p \notin j(\downarrow p\backslash\{p\})}{p \in X_j}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[S \in \mathcal{D}(\downarrow p)]^2}{S \subset \downarrow p} \qquad [S \neq \downarrow p]^1}{p \notin S} \qquad \dfrac{[S \in \mathcal{D}(\downarrow p)]^2}{S \subset \downarrow p}}{S \subset \downarrow p\backslash\{p\}}}{j(S) \subset j(\downarrow p\backslash\{p\})} \qquad \dfrac{p \in X_j}{p \notin j(\downarrow p\backslash\{p\})}}{p \notin j(S)}}{(S \neq \downarrow p) \to (p \notin j(S))}\;1}{(p \in j(S)) \to (S = \downarrow p)} \qquad \dfrac{}{(S = \downarrow p) \to (p \in j(S))}}{(p \in j(S)) \leftrightarrow (S = \downarrow p)}}{\forall S \in \mathcal{D}(\downarrow p).(p \in j(S)) \leftrightarrow (S = \downarrow p)}\;2}{p \in X_{J_j}}$$

This proves that X_j and X_{J_j} are equal.

Now let's check that X_j and X'_j are equal,
where X'_j is defined as:

$$\begin{array}{rcl} X_j & = & \{\, p \in \mathbf{P} \;:\; p \notin j(\downarrow p \backslash \{p\}) \,\} \\ X'_j & = & \{\, p \in \mathbf{P} \;:\; j(\downarrow p) \neq j(\downarrow p \backslash \{p\}) \,\} \end{array}$$

Proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p \in j(\downarrow p\backslash\{p\})]^1}{\downarrow p \subseteq j(\downarrow p\backslash\{p\})}}{j(\downarrow p) \subseteq (j \circ j)(\downarrow p\backslash\{p\})} \quad \dfrac{\dfrac{}{\downarrow p\backslash\{p\} \subseteq \downarrow p}}{j(\downarrow p\backslash\{p\}) \subseteq j(\downarrow p)}}{j(\downarrow p) \subseteq (j \circ j)(\downarrow p\backslash\{p\})) = j(\downarrow p\backslash\{p\}) \subseteq j(\downarrow p)}}{j(\downarrow p) = j(\downarrow p\backslash\{p\})} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\downarrow p \subseteq j(\downarrow p)} \quad [j(\downarrow p) = j(\downarrow p\backslash\{p\})]^1}{\downarrow p \subseteq j(\downarrow p) = j(\downarrow p\backslash\{p\})}}{\downarrow p \subseteq j(\downarrow p\backslash\{p\})}}{p \in j(\downarrow p\backslash\{p\})}}{\dfrac{(p \in j(\downarrow p\backslash\{p\})) \leftrightarrow (j(\downarrow p) = j(\downarrow p\backslash\{p\}))}{\dfrac{(p \notin j(\downarrow p\backslash\{p\})) \leftrightarrow (j(\downarrow p) \neq j(\downarrow p\backslash\{p\}))}{p \in X_j \leftrightarrow p \in X'_j}}}\;1$$

The formula of X'_j above is the one that I asked if it was correct; Bert answered that yes, and showed that it is equivalent to the slighty shorter formula for X_j.

References

[Lin14] A.J. Lindenhovius. "Grothendieck Topologies on Posets". https://arxiv.org/pdf/1405.4408v2.pdf. 2014.