

Notes on John MacDonald and Manuela Sobral’s “Aspects of Monads”:  
<https://doi.org/10.1017/CB09781107340985.008>  
 a chapter of “Categorical Foundations: Special Topics in Order, Topology, Algebra, and Sheaf Theory”, edited by Maria Cristina Pedicchio and Walter Tholen, Cambridge, 2003.  
 These notes are at:  
<http://angg.twu.net/LATEX/2020macdonaldsobral.pdf>

## 1.2. Monads

(Page 216):

A monad on a category  $\mathbf{X}$  is a system  $(T, \eta, \mu)$ ...

(Page 217):

Proposition: any adjunction  $(F, G, \eta, \epsilon) : \mathbf{X} \rightleftarrows \mathbf{A}$  determines a monad...

$$\begin{array}{ccccc}
 FGA & FX \leftarrow X & X & 1 & \\
 \epsilon A \downarrow & \downarrow & \downarrow \eta_X & \downarrow \eta & \\
 A & A \xrightarrow{G} GA & GFX & T & \\
 & \uparrow G\epsilon FX & \uparrow \mu & & \\
 \mathbf{A} \xrightleftharpoons[G]{F} \mathbf{X} & GF GFX & T^2 & & \\
 & & & & T \xrightarrow{T\eta} T^2 \xleftarrow{T\mu} T^3 \\
 & & & & \eta T \downarrow \searrow 1_{\mathbf{X}} \downarrow \mu \downarrow \mu T \\
 & & & & T^2 \xrightarrow{\mu} T \xleftarrow{\mu} T^2
 \end{array}$$

## 1.3. The Eilenberg-Moore Construction

(Page 217):

...category of T-algebras which will be denoted by  $\mathbf{X}^T$ .

$$\begin{array}{ccc}
 (X, \xi) & \begin{array}{ccc} X \xrightarrow{\eta^X} TX \xleftarrow{\mu^X} T^2 X \\ \searrow 1_X \downarrow \xi \downarrow T\xi \\ X \xleftarrow{\xi} TX \end{array} \\
 \\
 (X, \xi) & \begin{array}{ccc} X \xleftarrow{\xi} TX \\ \downarrow f \downarrow Tf \\ (Y, \Theta) \quad Y \xleftarrow{\Theta} TY \end{array} \\
 \mathbf{X}^T & & \mathbf{X}
 \end{array}$$

## The Eilenberg-Moore adjunction

(Page 218):

Proposition: For a monad  $T = (T, \eta, \mu)$  on  $\mathbf{X}$  there is a free-forgetful adjunction

$$\mathbf{X}^T \begin{array}{c} \xrightarrow{G^T} \\ \xleftarrow{F^T} \end{array} \mathbf{X}$$

which induces the monad  $T$  in  $\mathbf{X}$ .

$$\begin{array}{c}
 \begin{array}{ccc}
 (TX', \mu X') \leftarrow X' & & \\
 Tf \downarrow & & \downarrow f \\
 (TX, \mu X) \leftarrow TX & & \\
 \epsilon^T(X, \xi) := \downarrow & & \\
 \xi & & \\
 (X, \xi) & & \\
 & & (Y, \theta) \dashrightarrow Y \\
 & & \downarrow \\
 & & (Y', \theta') \dashrightarrow Y' \\
 & & \\
 & & \mathbf{X}^T \begin{array}{c} \xrightarrow{F^T} \\ \xleftarrow{G^T} \end{array} \mathbf{X}
 \end{array}
 \end{array}
 \quad
 \begin{array}{ccc}
 & TX & \\
 \bar{f} \swarrow & \downarrow Tf & \\
 Y \leftarrow TY & & 
 \end{array}
 \quad
 \begin{array}{ccc}
 & X & \\
 \eta^T X := \downarrow & & \\
 \eta^X & & \\
 (\text{univ}) & & \downarrow \forall f \\
 (TX, \mu) \dashrightarrow TX & & \\
 \exists! \bar{f} := \downarrow & & \downarrow \bar{f} \\
 \theta \cdot Tf & & (Y, \theta) \dashrightarrow Y
 \end{array}$$