

Notes on Bart Jacobs's book

"Categorical Logic and Type Theory" (Elsevier, 1999).

<http://www.cs.ru.nl/B.Jacobs/CLT/bookinfo.html>

<http://www.math.mcgill.ca/rags/jacobs.html>

These notes are at:

<http://angg.twu.net/LATEX/2020jacobs.pdf>

See:

<http://angg.twu.net/LATEX/2020favorite-conventions.pdf>

<http://angg.twu.net/math-b.html#favorite-conventions>

I wrote these notes mostly to test if the conventions above are good enough.

## 4.8. Quotient types, categorically

(Page 291):

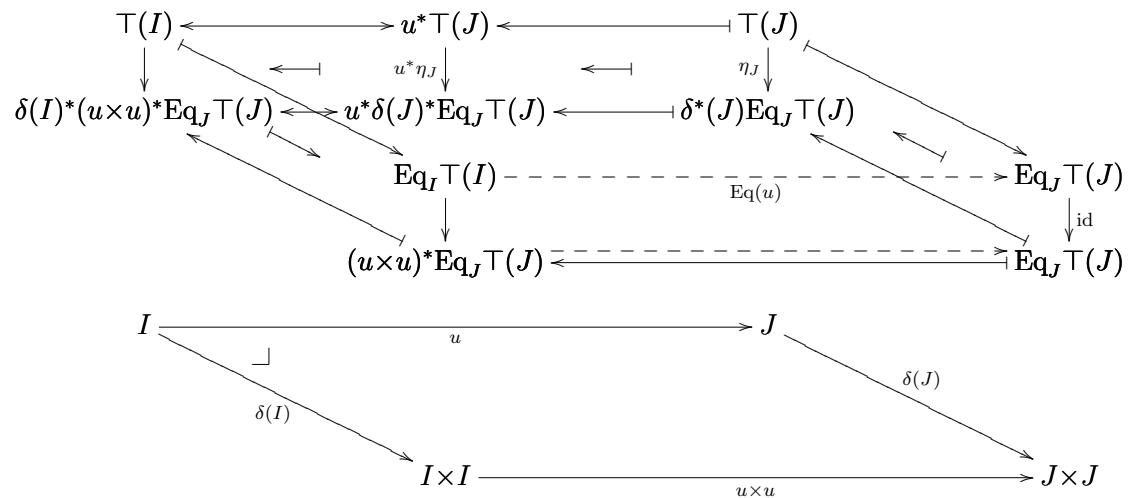
A morphism  $u : I \rightarrow J$  in  $\mathbb{B}$  is mapped to the composite

$$\text{Eq}(I) = \text{Eq}_I \top(I) \longrightarrow (u \times u)^* \text{Eq}_J \top(J) \longrightarrow \text{Eq}_J \top(J) = \text{Eq}(J)$$

where the first part of this map is obtained by transposing the following composite across the adjunction  $\text{Eq}_I \dashv \delta(I)^*$ :

$$\top(I) \cong u^* \top(J) \xrightarrow{u^* \eta_J} u^* \delta(J)^* \text{Eq}_J \top(J) \cong \delta(I)^* (u \times u)^* \text{Eq}_J \top(J)$$

Here is a diagram that explains that construction. Two morphisms in the construction above are morphisms in  $\mathbb{E}$  that are not-vertical; I drew them as ' $- - >$ 's.



From this point on all the diagrams are from notes that I wrote in 2008, using almost only the “downcased notation” from [IDARCT, sec.3]... *TODO*: create diagrams similar to these using the notation from Jacobs! See [FavC], section 2, conventions (COT) and (CNSh).

### Comprehension categories with unit

Jacobs, 10.4.7 (p.616):

A fibration  $p : \mathbb{E} \rightarrow \mathbb{B}$  with a terminal object functor  $1 : \mathbb{B} \rightarrow \mathbb{E}$  (where we know by lemma 1.8.8 that  $p \dashv 1$  and that  $\eta_I = \text{id}$ ) is *comprehension category with unit* if  $1$  has a right adjoint. We call this right adjoint  $\{-\}$ .

$$\begin{array}{ccccc} (b) & \longmapsto & (*) & \longmapsto & (e) \\ \downarrow p & \uparrow & \downarrow 1 & \uparrow & \downarrow \{-\} \\ a & \longrightarrow & c & \longrightarrow & d, e \end{array}$$

Jacobs, 10.4.7 (p.616):

Definition of the functor  $\mathbb{E} \rightarrow \mathbb{B}^\rightarrow$ : its action on objects is  $X \mapsto p\epsilon_X$ .

$$\begin{array}{ccc} 1\{X\} & \xrightarrow{\epsilon_X} & X \\ \uparrow & \uparrow & \uparrow \\ \{X\} & \xrightarrow{\text{id}} & \{X\} \xrightarrow{p\epsilon_X} pX & \xrightarrow{\quad} & \begin{array}{c} (*) \\ (d, e) \end{array} \xrightarrow{\quad} \begin{array}{c} (e) \\ d \end{array} \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ d, e & \xrightarrow{\quad} & d, e & \xrightarrow{\quad} & d \end{array}$$

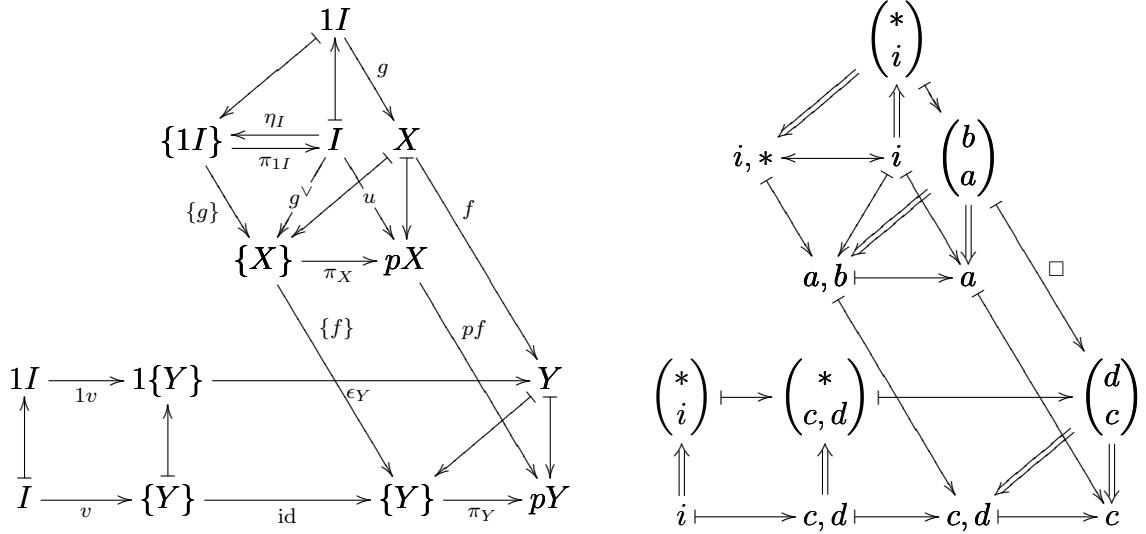
The functor  $\mathbb{E} \rightarrow \mathbb{B}^\rightarrow$  is a comprehension category, i.e., it takes cartesian morphisms to pullback squares.

We want to check that the image of a cartesian morphism is a pullback.

Given two maps  $i \mapsto a$  and  $i \mapsto c, d$  such that

$a \mapsto c$  is well-defined, we need to construct a

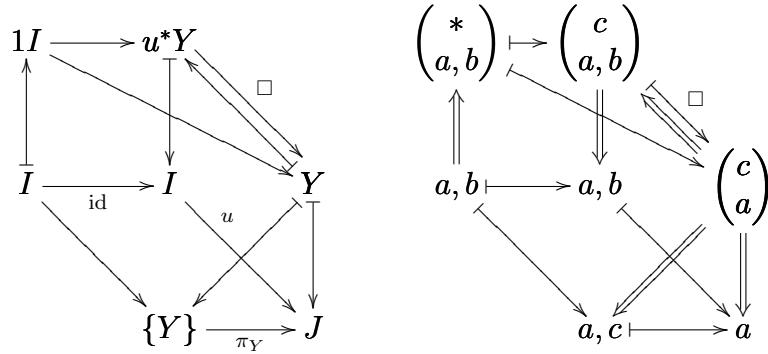
mediating map  $i \mapsto a, b$ .



### Comprehension categories with unit: a bijection

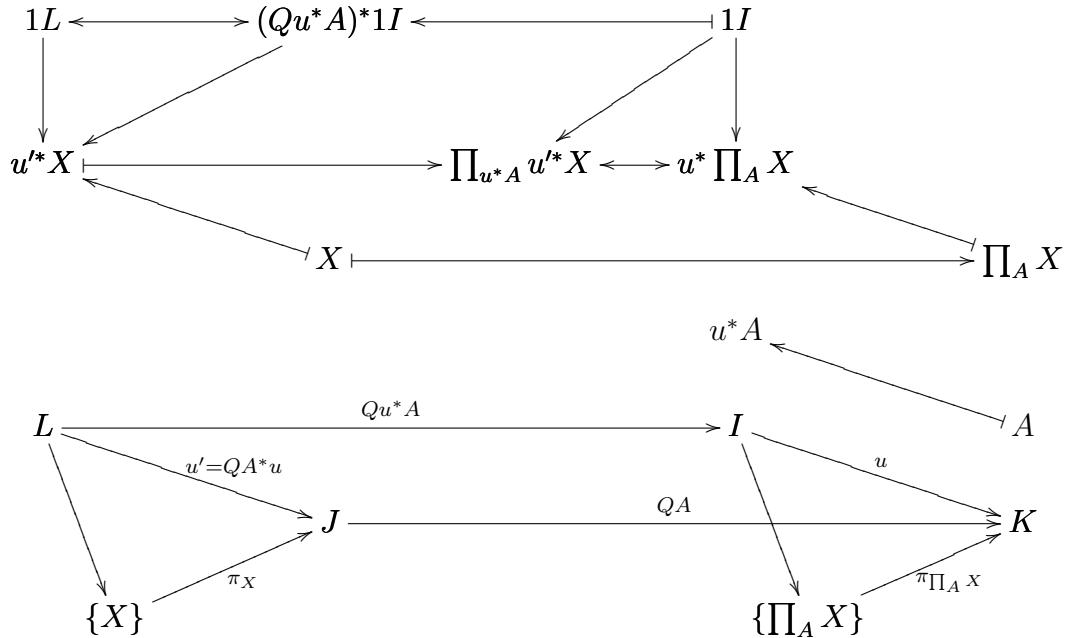
Jacobs, 10.4.9 (i):

In a CCw1, pack a morphism  $u : I \rightarrow J$  in the base category, and an object  $Y$  over  $J$ . Then the vertical morphisms  $1I \rightarrow u^*Y$  are in bijection with morphisms from  $u$  to  $\pi_Y$  in  $\mathbb{B}/J$ .



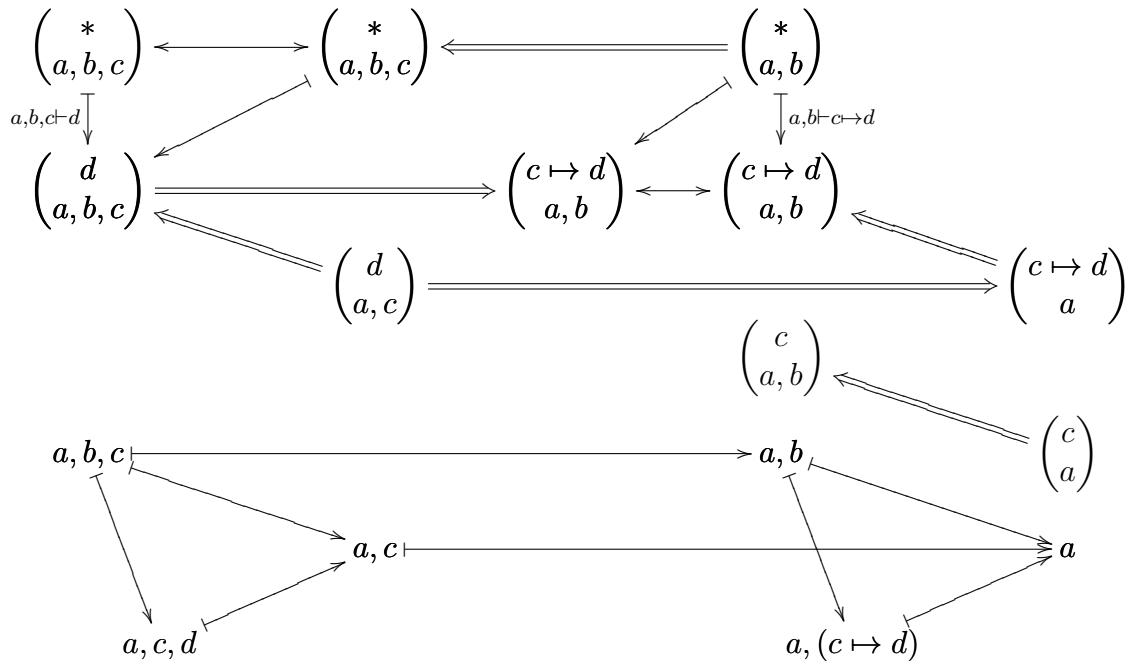
**Comprehension categories with unit: big bijection**

Jacobs, 10.4.9 (ii):



$$a, c; b \vdash d$$

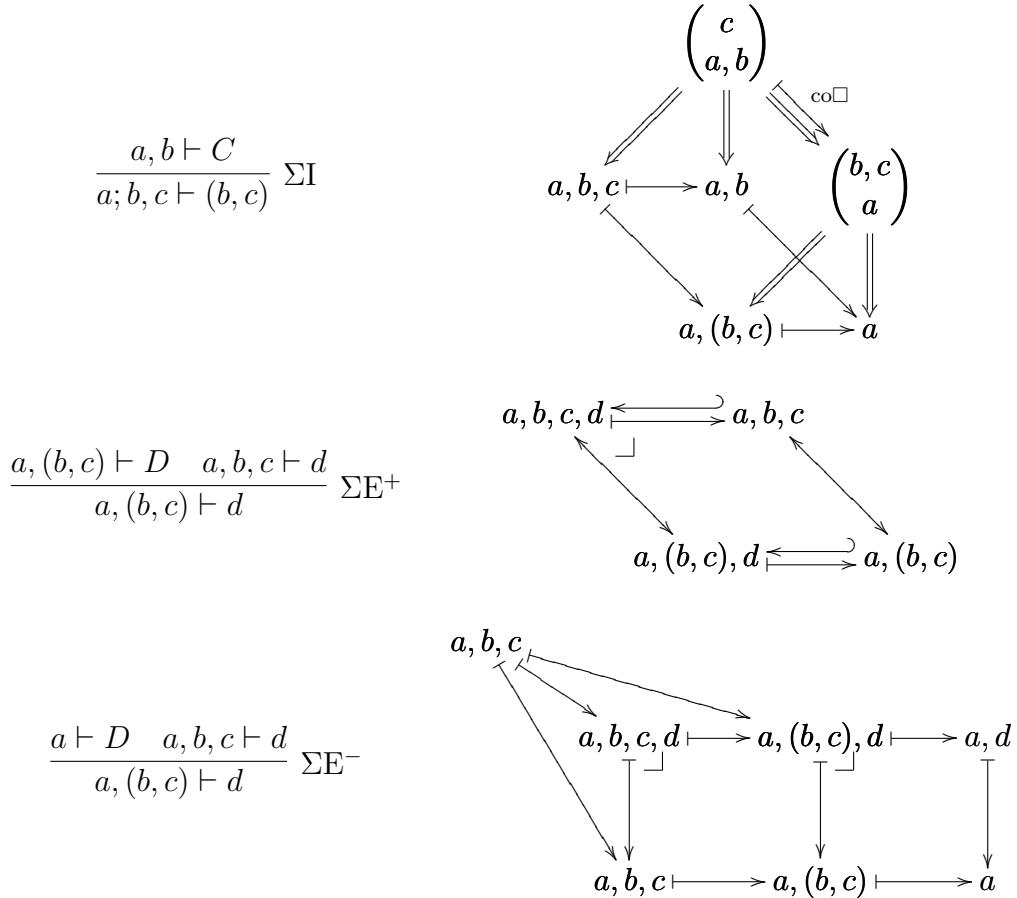
$$a; b \vdash c \rightarrow d$$



### Comprehension categories with unit: three rules

Jacobs, 10.3.3:

The categorical interpretation of  
the rules for dependent sums:



(Oops, the diagram for  $\Sigma E^-$  is wrong)

### Interpreting III and ΠE in a CCompC

(Jacobs, 10.5.3)

$$\begin{array}{ccc}
 \begin{array}{ccc}
 1\{X\} & \xleftarrow{\quad} & 1I \\
 f \downarrow & \longrightarrow & \downarrow \lambda_X f \\
 Y & \xrightarrow{\quad} & \prod_X Y
 \end{array} & \quad &
 \begin{array}{ccc}
 \binom{*}{a,b} & \Longleftarrow & \binom{*}{a} \\
 \downarrow a,b \vdash c & \longmapsto & \downarrow a \vdash (b \rightarrow c) \\
 \binom{c}{a,b} & \Longrightarrow & \binom{b \rightarrow c}{a}
 \end{array} \\
 \\[10pt]
 \begin{array}{ccc}
 & X & \\
 & \swarrow \quad \searrow & \\
 \{X\} & \xrightarrow{\pi_X} & I
 \end{array} & \quad &
 \begin{array}{ccc}
 & \binom{b}{a} & \\
 a,b & \rightrightarrows & a \\
 & \Downarrow & \\
 & a &
 \end{array} \\
 \\[10pt]
 \begin{array}{ccc}
 1I & \xleftarrow{\quad} & \pi_X^* 1I & \xleftarrow{\quad} & 1I \\
 gh \downarrow & \longleftarrow & g^\wedge \downarrow & \longleftarrow & \downarrow g \\
 h^{\vee *} Y & \xleftarrow{\quad} & Y & \xrightarrow{\quad} & \prod_X Y
 \end{array} & \quad &
 \begin{array}{ccc}
 \binom{*}{a} & \Longleftarrow & \binom{*}{a,b} & \Longleftarrow & \binom{*}{a} \\
 \downarrow a \vdash c & \longleftarrow & \downarrow & \longleftarrow & \downarrow a \vdash (b \rightarrow c) \\
 \binom{c}{a} & \Longleftarrow & \binom{c}{a,b} & \Longrightarrow & \binom{b \rightarrow c}{a}
 \end{array} \\
 \\[10pt]
 \begin{array}{ccc}
 1I & \xrightarrow{h} & X \\
 \uparrow & \downarrow & \downarrow \\
 I & \xrightarrow[h^\vee]{} & \{X\} & \xrightarrow{\pi_X} & I
 \end{array} & \quad &
 \begin{array}{ccc}
 \binom{*}{a} & \xrightarrow{a \vdash b} & \binom{b}{a} \\
 \uparrow a & \downarrow & \Downarrow \\
 a & \rightrightarrows & a, b & \xrightarrow{\quad} & a
 \end{array}
 \end{array}$$

In the top left vertex of the diagram for ΠE we have omitted an iso to keep the diagram shorter:  $1I \cong h^{\vee *} \pi_X^* 1I$ .

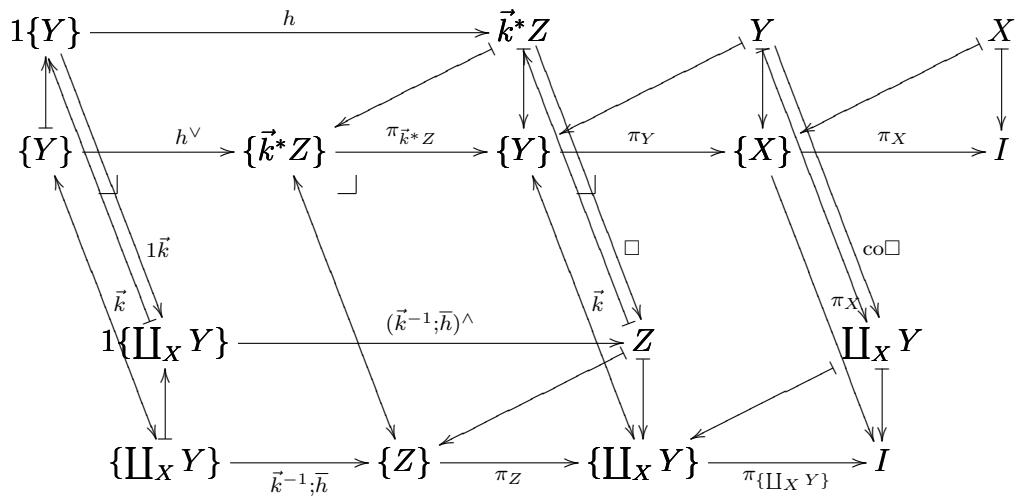
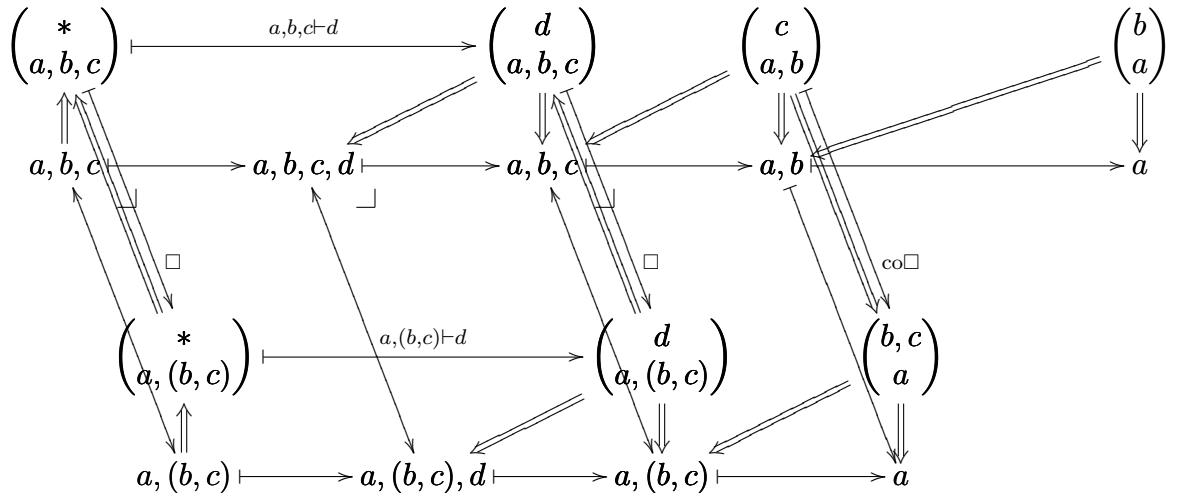
**Interpreting  $\Sigma I$  in a CCompC**  
 (Jacobs, 10.5.3)

The figure consists of four separate commutative diagrams:

- Top Left Diagram:** A square with vertices  $1I$ ,  $f^{\vee*}X$ ,  $Y$ , and  $\Sigma_X Y$ . The top edge is labeled  $g$ . The left edge is labeled  $\langle f, g \rangle$ . The right edge is labeled  $\text{id}$ . The bottom edge is labeled  $\pi_X^* \Sigma_X Y$ . There are two horizontal arrows between  $f^{\vee*}X$  and  $Y$ , and two horizontal arrows between  $Y$  and  $\Sigma_X Y$ .
- Top Right Diagram:** A complex commutative diagram involving three rows of boxes. The top row has boxes  $\binom{*}{a}$ ,  $\binom{a \vdash c}{a \vdash c}$ , and  $\binom{b, c}{a}$ . The middle row has boxes  $\binom{c}{a}$ ,  $\binom{c}{a, b}$ , and  $\binom{b, c}{a}$ . The bottom row has boxes  $\binom{b, c}{a}$ ,  $\binom{b, c}{a, b}$ , and  $\binom{b, c}{a}$ . Arrows connect corresponding boxes in adjacent columns.
- Bottom Left Diagram:** A square with vertices  $1I$ ,  $X$ ,  $\{X\}$ , and  $I$ . The top edge is labeled  $f$ . The left edge is labeled  $I \succ_{f^\vee} \{X\}$ . The right edge is labeled  $I \prec_{\pi_X} \{X\}$ . The bottom edge is labeled  $\{X\} \succ_{\pi_X} I$ . There are two diagonal arrows from  $\{X\}$  to  $X$ .
- Bottom Right Diagram:** A square with vertices  $\binom{*}{a}$ ,  $\binom{b}{a}$ ,  $a, b \vdash$ , and  $a$ . The top edge is labeled  $a \vdash b$ . The left edge is labeled  $\binom{*}{a} \vdash$ . The right edge is labeled  $\vdash a$ . The bottom edge is labeled  $a, b \vdash \vdash a$ . There are two diagonal arrows from  $a, b \vdash$  to  $a$ .

Interpreting  $\Sigma E^+$  in a CCompC

(Jacobs, 10.5.3)



## References

- [FavC] E. Ochs. “On my favorite conventions for drawing the missing diagrams in Category Theory”. <http://angg.twu.net/math-b.html#favorite-conventions>. 2020.

- [IDARCT] E. Ochs. “Internal Diagrams and Archetypal Reasoning in Category Theory”. In: *Logica Universalis* 7.3 (Sept. 2013). <http://angg.twu.net/math-b.html#idarct>, pp. 291–321.