Notes on Saunders MacLane's [CWM], a.k.a.:
"Categories for the Working Mathematician (2nd ed.)" (Springer, 1998)
https://link.springer.com/book/10.1007/978-1-4612-9839-7
https://en.wikipedia.org/wiki/Categories_for_the_Working_Mathematician
https://ncatlab.org/nlab/show/Categories+Work

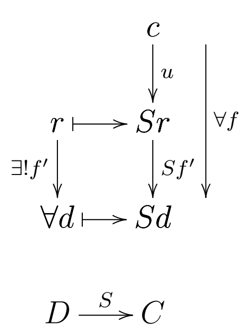
These notes are at:
http://angg.twu.net/LATEX/2020cwm.pdf

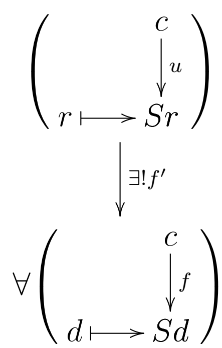
III. Universals and Limits

1. Universal Arrows

Definition. If $S : D \to C$ is a functor and c an object of C, a universal arrow from c to S is pair $\langle r, u \rangle$ such that... (see the diagram at the left below; formally and minus the types, $\forall d. \forall f. \exists ! f'. sf' \circ u = f$).

Equivalently, $u : c \to Sr$ is universal from c to S when the pair $\langle r, u \rangle$ is an initial object in the comma category $(c \downarrow S)$... (diagram at the right below).

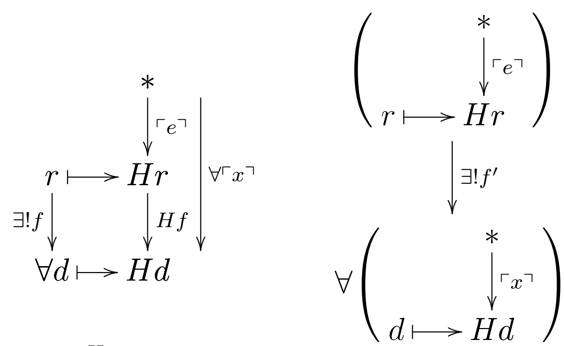




$(c \downarrow S)$

(p.57):

The idea of universality is sometimes expressed in terms of "universal elements". If D is a category and $H : D \to \mathbf{Set}$ a functor, a *universal element* of the functor H is a pair $\langle r, e \rangle$ consisting of an object $r \in D$ and an element $e \in Hr$ such that for every pair $\langle d, x \rangle$ with $x \in Hd$ there is a unique arrow $f : r \to d$ of D with $(Hf)e = x$.



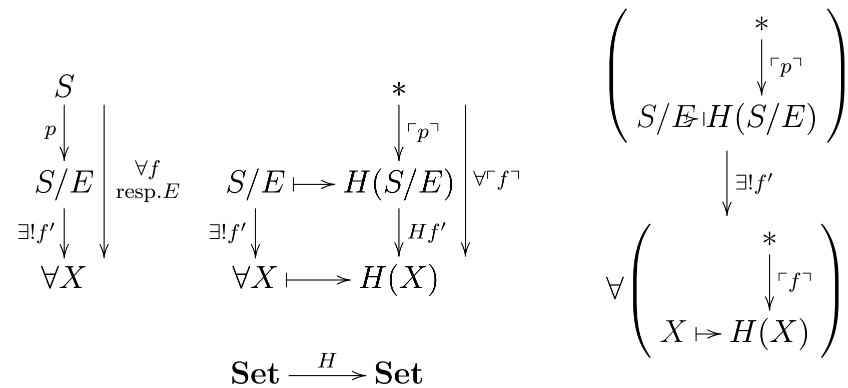
$D \xrightarrow{H} \mathbf{Set}$

$(c \downarrow S)$

(p.57):

Many familiar constructions (...) consider an equivalence relation E on a set S, the corresponding quotient set S/E (...) and the projection $p : S \to S/E$. (...)

Definitions (mine): if $E \subseteq S \times S$ then we say that
$f : S \to X$ respects E iff $\forall s, s' \in E. sEs' \to fs = fs'$, and
$H(X) := \{\, f : S \to X \mid f \text{ respects } E \,\}$.



$(* \downarrow H)$

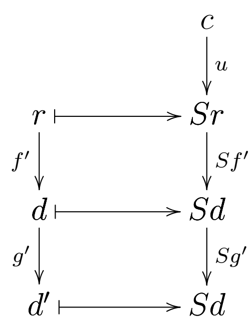
(p.58):

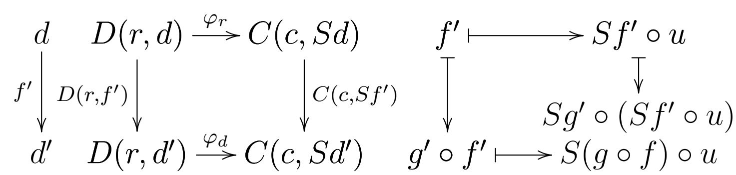
The notion "universal element" is a special case of the notion "universal arrow". Indeed, if $*$ is the set with one point... Indeed if $S : D \to C$ is a functor and $c \in C$ is an object, then $\langle r, u : c \to Sr \rangle$ is a universal arrow from c to S is and only if the pair $\langle r, u \in C(c, Sr) \rangle$ is a universal element of the functor $H = C(c, S-)$. This is the functor which acts on objects d and arrows h of D by:

$$d \mapsto C(c, Sd) \ , \qquad h \mapsto C(c, Sh) \ .$$

2. The Yoneda Lemma

Proposition 1:





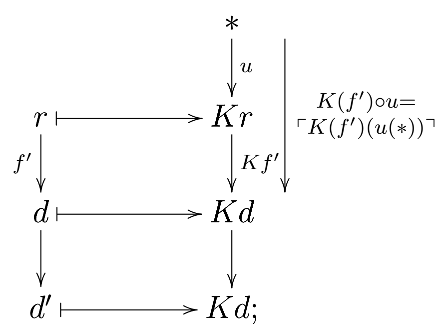
$$D(r,-) \xrightarrow{\varphi} C(c,S-)$$

$$D \xrightarrow{S} C$$

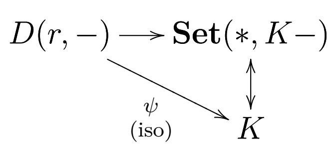
$$D(r,-) \xrightarrow{\varphi} C(c,S-)$$

Definition. Let D have small hom-sets. A representation of a functor $K : D \to \mathbf{Set}$ is....

Proposition 2:



$$D \xrightarrow{K} \mathbf{Set}$$



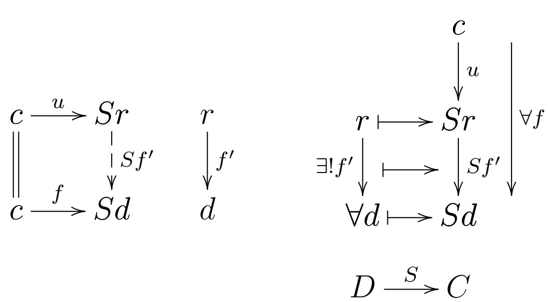
(Original text + missing diagrams)

III.1. Universal Arrows

(Page 55):

1. Universal Arrows

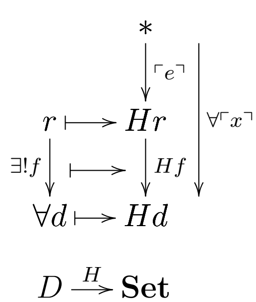
Definition. if $S : D \to C$ is a functor and c an object of C, a universal arrow from c to S is a pair $\langle r, u \rangle$ consisting of an object r of D and an arrow $u : c \to Sr$ of C, such that to every pair $\langle d, f \rangle$ with d an object of D and $f : c \to Sd$ an arrow of C, there is a unique arrow $f' : r \to d$ of D with $Sf' \circ u = f$. In other words, every arrow f to S factors uniquely through the universal arrow u, as in the commutative diagram



$\langle r, u : c \to Sr \rangle$ is a universal arrow from c to S

(Page 57):

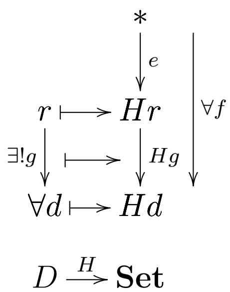
The idea of universality is sometimes expressed in terms of "universal elements". If D is a category and $H : D \to \mathbf{Set}$ a functor, a *universal element* of the functor H is a pair $\langle r, e \rangle$ consisting of an object $r \in D$ and an element $e \in Hr$ such that for every pair $\langle d, x \rangle$ with $x \in Hd$ there is a unique arrow $f : r \to d$ of D with $(Hf)e = x$.



$\langle r, e \in Hr \rangle$ is a universal element of the functor H

(Page 58, middle paragraph):

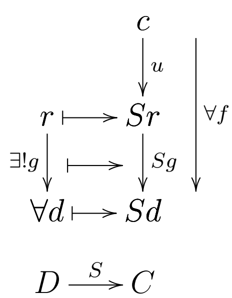
The notion "universal element" is a special case of the notion "universal arrow". Indeed, if $*$ is the set with one point, then any element $e \in Hr$ can be regarded as an arrow $e : * \to Hr$ in **Ens**. Thus a universal element $\langle r, e \rangle$ for H is exactly a universal arrow from $*$ to H.

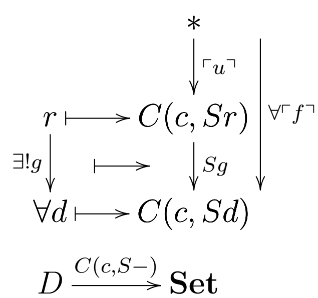


$\langle r, e \in Hr \rangle$ is a universal element of the functor H $\Rightarrow$ $\langle r, e : * \to Hr \rangle$ is a universal arrow from r to H

(Page 58, middle paragraph, 2nd part):

Conversely, if C has small hom-sets, the notion "universal arrow" is a special case of the notion "universal element". Indeed, if $S : D \to C$ is a functor and $c \in C$ is an object, then $\langle r, u : c \to Sr \rangle$ is a universal arrow from c to S if and only if the pair $\langle r, u \in C(c, Sr) \rangle$ is a universal element of the functor $H = C(c, S-)$.

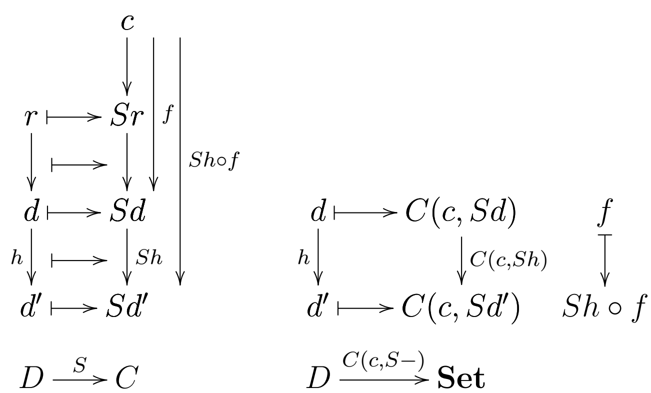




(Page 58, middle paragraph, 3rd part):

This — *i.e., the functor* $C(c, S-)$ — is the functor which acts on objects d and arrows h of D by

$$d \mapsto C(c, Sd), \qquad h \mapsto C(c, Sh).$$

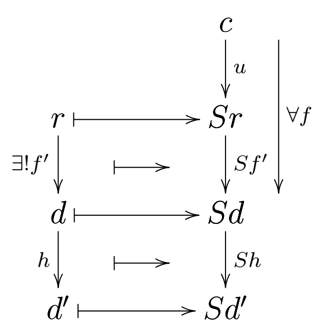


2. The Yoneda Lemma

(Page 59):

Proposition 1. For a functor $S : D \to C$ a pair $\langle r, u : c \to Sr \rangle$ is universal from c to S if and only if the function sending each $f' : r \to d$ into $Sf' \circ u : c \to Sd$ is a bijection of hom-sets

$$D(r, d) \cong C(c, Sd).$$



$$D \xrightarrow{S} C$$

$$D(r, d) \longleftrightarrow C(c, Sd)$$

$$f' \longmapsto Sf' \circ u$$
$$f' \longleftarrow\!\mapsfromchar f$$

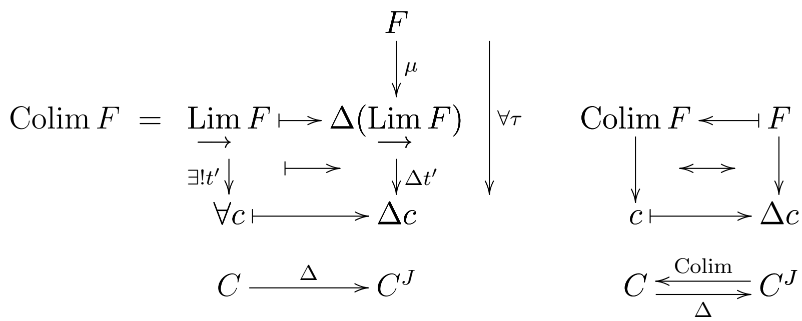
(Page 59, Proposition 1, cont.)
This bijection is natural in D.

$$
\begin{array}{ccccccccc}
 & & c & & & & & & \\
 & & \downarrow{\scriptstyle u} & & & & & & \\
r & \longmapsto & Sr & & & & & & \\
{\scriptstyle f'}\downarrow & \longmapsto & \downarrow{\scriptstyle Sf'} & & & & & & \\
d & \longmapsto & Sd & \quad & d & \longmapsto & D(r,d) & \quad & f' \\
{\scriptstyle h}\downarrow & \longmapsto & \downarrow{\scriptstyle Sh} & & {\scriptstyle h}\downarrow & \longmapsto & \downarrow{\scriptstyle D(r,h)} & & \downarrow \\
d' & \longmapsto & Sd' & & d' & \longmapsto & D(r,d') & & h\circ f' \\
 & & & & & & & & \\
D & \xrightarrow{S} & C & & D & \xrightarrow{D(r,-)} & \mathbf{Set} & &
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & c & & & & & & & & \\
 & & \downarrow{\scriptstyle u} & & & & & & & & \\
r & \longmapsto & Sr & & & & & & & & \\
{\scriptstyle f'}\downarrow & \longmapsto & \downarrow{\scriptstyle Sf'} & & & & & & & & \\
d & \longmapsto & Sd & \quad & d & D(r,d) & \xrightarrow{\varphi_d} & C(c,Sd) & \quad & f' & \longmapsto Sf'\circ u \\
{\scriptstyle h}\downarrow & \longmapsto & \downarrow{\scriptstyle Sh} & & {\scriptstyle h}\downarrow & {\scriptstyle D(r,h)}\downarrow & & \downarrow{\scriptstyle C(c,Sh)} & & \downarrow & \downarrow \\
 & & & & & & & & & & Sh\circ(Sf'\circ u) \\
d' & \longmapsto & Sd' & & d' & D(r,d') & \xrightarrow{\varphi_{d'}} & C(c,Sd') & & h\circ f' & \longmapsto S(h\circ f')\circ u \\
 & & & & & & & & & & \\
D & \xrightarrow{S} & C & & & D(r,-) & \xrightarrow{\varphi} & C(c,S-) & & &
\end{array}
$$

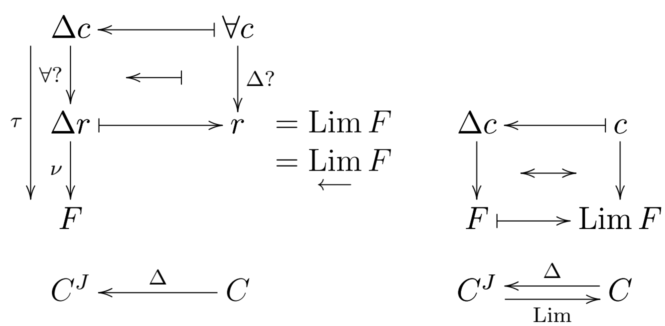
3. Coproducts and colimits

(Page 67):
Cone and limiting cone:
We call μ the *limiting cone* or the *universal cone* (from F).



4. Products and limits

(Page 68):
A *limit* for a functor $F : J \to C$ is a universal arrow $\langle r, \nu \rangle$ from Δ to F.



(Page 69):
...and its limiting cone $\nu : \mathrm{Lim}\, F \to F$...
(or more precisely $\nu : \Delta(\mathrm{Lim}\, F) \to F$)

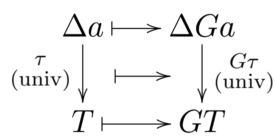
V. Limits

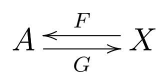
5. Adjoints on Limits

(Page 118):

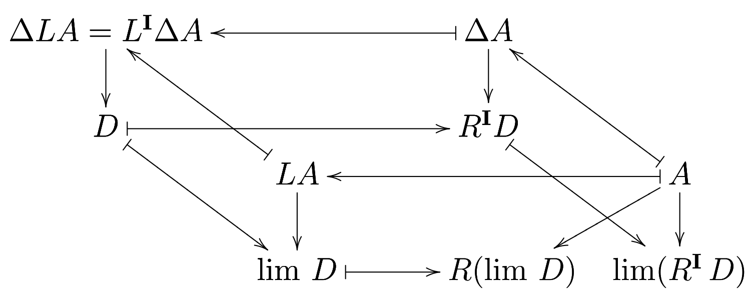
One of the most useful properties of adjoints is this: A functor which is a right adjoint preserves all the limits which exist in its domain:

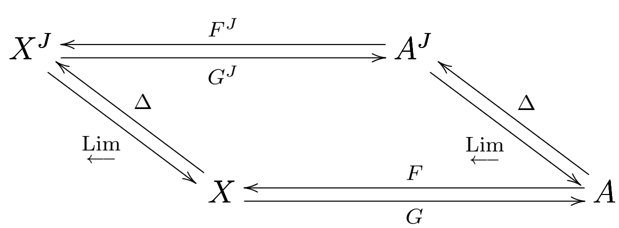
Theorem 1. *If the functor $G : A \to X$ has a left adjoint, while the functor $T : J \to A$ has a limiting cone $\tau : a \stackrel{\bullet}{\to} T$ in A, then GT has the limiting cone $G\tau : Ga \stackrel{\bullet}{\to} GT$ in X.*





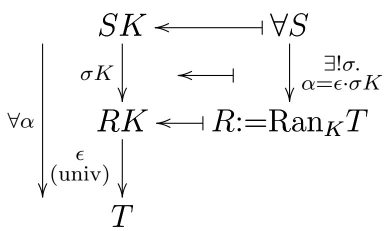
This proof can also be cast in a more sophisticated form by using the fact that Lim is right adjoint to the diagonal functor .1. In fact, given an adjunction...

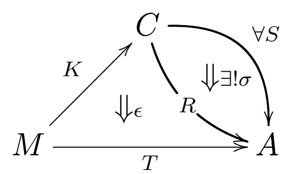


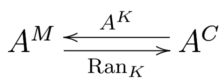


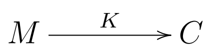
X. Kan Extensions

3. The Kan Extension









$$
\begin{array}{rcl}
\mathrm{Nat}(SK, T) & \cong & \mathrm{Nat}(S, \mathrm{Ran}_K T) \\
\epsilon \cdot \sigma K & \leftarrow\!\shortmid & \sigma
\end{array}
$$

$\forall S.$
$\forall \alpha : SK \to T.$
$\exists! \sigma.\ \alpha = \epsilon \cdot \sigma K$