

Notes on Saunders MacLane's [CWM], a.k.a.:

"Categories for the Working Mathematician (2nd ed.)" (Springer, 1998)

<https://link.springer.com/book/10.1007/978-1-4612-9839-7>

https://en.wikipedia.org/wiki/Categories_for_the_Working_Mathematician

<https://ncatlab.org/nlab/show/Categories+Work>

These notes are at:

<http://angg.twu.net/LATEX/2020cwm.pdf>

III. Universals and Limits

1. Universal Arrows

Definition. If $S : D \rightarrow C$ is a functor and c an object of C , a universal arrow from c to S is pair $\langle r, u \rangle$ such that... (see the diagram at the left below; formally and minus the types, $\forall d. \forall f. \exists! f'. sf' \circ u = f$).

Equivalently, $u : c \rightarrow Sr$ is universal from c to S when the pair $\langle r, u \rangle$ is an initial object in the comma category $(c \downarrow S)$... (diagram at the right below).

$$\begin{array}{ccc}
 & c & \\
 & \downarrow u & \\
 r \mapsto & Sr & \\
 \exists! f' \downarrow & \downarrow Sf' & \downarrow \forall f \\
 \forall d \mapsto & Sd & \\
 D \xrightarrow{S} & C & \\
 & & \left(\begin{array}{c} c \\ \downarrow u \\ r \mapsto Sr \end{array} \right) \\
 & & \downarrow \exists! f' \\
 & & \forall \left(\begin{array}{c} c \\ \downarrow f \\ d \mapsto Sd \end{array} \right) \\
 & & (c \downarrow S)
 \end{array}$$

(p.57):

The idea of universality is sometimes expressed in terms of “universal elements”. If D is a category and $H : D \rightarrow \mathbf{Set}$ a functor, a *universal element* of the functor H is a pair $\langle r, e \rangle$ consisting of an object $r \in D$ and an element $e \in Hr$ such that for every pair $\langle d, x \rangle$ with $x \in Hd$ there is a unique arrow $f : r \rightarrow d$ of D with $(Hf)e = x$.

$$\begin{array}{ccc}
 & * & \\
 & \downarrow \lceil e \rceil & \\
 r \mapsto & Hr & \\
 \exists! f \downarrow & \downarrow Hf & \\
 \forall d \mapsto & Hd & \\
 D \xrightarrow{H} & \mathbf{Set} & \\
 & & \downarrow \forall \lceil x \rceil \\
 & & \left(\begin{array}{c} * \\ \downarrow \lceil e \rceil \\ r \mapsto Hr \end{array} \right) \\
 & & \downarrow \exists! f' \\
 & & \forall \left(\begin{array}{c} * \\ \downarrow \lceil x \rceil \\ d \mapsto Hd \end{array} \right) \\
 & & (c \downarrow S)
 \end{array}$$

(p.57):

Many familiar constructions (...) consider an equivalence relation E on a set S , the corresponding quotient set S/E (...) and the projection $p : S \rightarrow S/E$. (...)

Definitions (mine): if $E \subseteq S \times S$ then we say that

$f : S \rightarrow X$ respects E iff $\forall s, s' \in E. sEs' \rightarrow fs = fs'$, and

$H(X) := \{ f : S \rightarrow X \mid f \text{ respects } E \}$.

$$\begin{array}{ccc}
 \begin{array}{ccc}
 S & & * \\
 p \downarrow & & \downarrow \ulcorner p \urcorner \\
 S/E & \xrightarrow{\forall f \text{ resp. } E} & H(S/E) \\
 \exists! f' \downarrow & & \downarrow Hf' \\
 \forall X & \xrightarrow{\exists! f'} & \forall X \mapsto H(X)
 \end{array} & & \begin{array}{c}
 \left(\begin{array}{c} * \\ \downarrow \ulcorner p \urcorner \\ S/E \mapsto H(S/E) \end{array} \right) \\
 \downarrow \exists! f' \\
 \forall \left(\begin{array}{c} * \\ \downarrow \ulcorner f' \urcorner \\ X \mapsto H(X) \end{array} \right)
 \end{array} \\
 \text{Set} \xrightarrow{H} \text{Set} & & (* \downarrow H)
 \end{array}$$

(p.58):

The notion “universal element” is a special case of the notion “universal arrow”. Indeed, if $*$ is the set with one point... Indeed if $S : D \rightarrow C$ is a functor and $c \in C$ is an object, then $\langle r, u : c \rightarrow Sr \rangle$ is a universal arrow from c to S is and only if the pair $\langle r, u \in C(c, Sr) \rangle$ is a universal element of the functor $H = C(c, S-)$. This is the functor which acts on objects d and arrows h of D by:

$$d \mapsto C(c, Sd), \quad h \mapsto C(c, Sh).$$

2. The Yoneda Lemma

Proposition 1:

$$\begin{array}{ccc}
 & c & \\
 & \downarrow u & \\
 r & \longrightarrow & Sr \\
 f' \downarrow & & \downarrow Sf' \\
 d & \longrightarrow & Sd \\
 g' \downarrow & & \downarrow Sg' \\
 d' & \longrightarrow & Sd
 \end{array}
 \quad
 \begin{array}{ccc}
 d & D(r, d) \xrightarrow{\varphi_r} & C(c, Sd) \\
 f' \downarrow & D(r, f') \downarrow & \downarrow C(c, Sf') \\
 d' & D(r, d') \xrightarrow{\varphi_d} & C(c, Sd')
 \end{array}
 \quad
 \begin{array}{ccc}
 f' & \longrightarrow & Sf' \circ u \\
 \downarrow & & \downarrow \\
 g' \circ f' & \longrightarrow & S(g \circ f) \circ u
 \end{array}$$

$$D \xrightarrow{S} C \qquad D(r, -) \xrightarrow{\varphi} C(c, S-)$$

$$D(r, -) \xrightarrow{\varphi} C(c, S-)$$

Definition. Let D have small hom-sets. A representation of a functor $K : D \rightarrow \mathbf{Set}$ is....

Proposition 2:

$$\begin{array}{ccc}
 & * & \\
 & \downarrow u & \\
 r & \longrightarrow & Kr \\
 f' \downarrow & & \downarrow Kf' \\
 d & \longrightarrow & Kd \\
 \downarrow & & \downarrow \\
 d' & \longrightarrow & Kd;
 \end{array}
 \quad
 \begin{array}{c}
 \downarrow \\
 K(f') \circ u = \\
 \lceil K(f')(u(*)) \rceil
 \end{array}$$

$$D \xrightarrow{K} \mathbf{Set}$$

$$\begin{array}{ccc}
 D(r, -) & \longrightarrow & \mathbf{Set}(*, K-) \\
 & \searrow \psi & \updownarrow \\
 & & K \\
 & & \text{(iso)}
 \end{array}$$

(Original text + missing diagrams)

III.1. Universal Arrows

(Page 55):

1. Universal Arrows

Definition. if $S : D \rightarrow C$ is a functor and c an object of C , a universal arrow from c to S is a pair $\langle r, u \rangle$ consisting of an object r of D and an arrow $u : c \rightarrow Sr$ of C , such that to every pair $\langle d, f \rangle$ with d an object of D and $f : c \rightarrow Sd$ an arrow of C , there is a unique arrow $f' : r \rightarrow d$ of D with $Sf' \circ u = f$. In other words, every arrow f to S factors uniquely through the universal arrow u , as in the commutative diagram

$$\begin{array}{ccc}
 \begin{array}{ccc}
 c & \xrightarrow{u} & Sr \\
 \parallel & & \downarrow Sf' \\
 c & \xrightarrow{f} & Sd
 \end{array} & \begin{array}{c}
 r \\
 \downarrow f' \\
 d
 \end{array} & \begin{array}{ccc}
 & & c \\
 & & \downarrow u \\
 r & \mapsto & Sr \\
 \exists! f' \downarrow & \mapsto & \downarrow Sf' \\
 \forall d \mapsto & & Sd
 \end{array} \begin{array}{c}
 \downarrow \forall f \\
 \\
 \\
 \\
 \\
 \end{array} & \langle r, u : c \rightarrow Sr \rangle \text{ is a} \\
 & & \text{universal arrow} \\
 & & \text{from } c \text{ to } S \\
 & & \\
 & & D \xrightarrow{S} C
 \end{array}$$

(Page 57):

The idea of universality is sometimes expressed in terms of “universal elements”. If D is a category and $H : D \rightarrow \mathbf{Set}$ a functor, a *universal element* of the functor H is a pair $\langle r, e \rangle$ consisting of an object $r \in D$ and an element $e \in Hr$ such that for every pair $\langle d, x \rangle$ with $x \in Hd$ there is a unique arrow $f : r \rightarrow d$ of D with $(Hf)e = x$.

$$\begin{array}{ccc}
 & & * \\
 & & \downarrow \lceil e \rceil \\
 r & \mapsto & Hr \\
 \exists! f \downarrow & \mapsto & \downarrow Hf \\
 \forall d \mapsto & & Hd
 \end{array} \begin{array}{c}
 \downarrow \forall x \\
 \\
 \\
 \\
 \\
 \end{array} \begin{array}{c}
 \langle r, e \in Hr \rangle \text{ is a} \\
 \text{universal element} \\
 \text{of the functor } H
 \end{array} \\
 \\
 D \xrightarrow{H} \mathbf{Set}
 \end{array}$$

(Page 58, middle paragraph):

The notion “universal element” is a special case of the notion “universal arrow”. Indeed, if $*$ is the set with one point, then any element $e \in Hr$ can be regarded as an arrow $e : * \rightarrow Hr$ in **Ens**. Thus a universal element $\langle r, e \rangle$ for H is exactly a universal arrow from $*$ to H .

$$\begin{array}{ccc}
 & * & \\
 & \downarrow e & \\
 r \mapsto & Hr & \downarrow \forall f \\
 \exists!g \downarrow \mapsto & \downarrow Hg & \\
 \forall d \mapsto & Hd & \\
 D \xrightarrow{H} & \mathbf{Set} &
 \end{array}
 \quad \langle r, e \in Hr \rangle \text{ is a universal element of the functor } H
 \quad \Rightarrow \quad
 \langle r, e : * \rightarrow Hr \rangle \text{ is a universal arrow from } r \text{ to } H$$

(Page 58, middle paragraph, 2nd part):

Conversely, if C has small hom-sets, the notion “universal arrow” is a special case of the notion “universal element”. Indeed, if $S : D \rightarrow C$ is a functor and $c \in C$ is an object, then $\langle r, u : c \rightarrow Sr \rangle$ is a universal arrow from c to S if and only if the pair $\langle r, u \in C(c, Sr) \rangle$ is a universal element of the functor $H = C(c, S-)$.

$$\begin{array}{ccc}
 & c & \\
 & \downarrow u & \\
 r \mapsto & Sr & \downarrow \forall f \\
 \exists!g \downarrow \mapsto & \downarrow Sg & \\
 \forall d \mapsto & Sd & \\
 D \xrightarrow{S} & C &
 \end{array}
 \quad
 \begin{array}{ccc}
 & * & \\
 & \downarrow \ulcorner u \urcorner & \\
 r \mapsto & C(c, Sr) & \downarrow \forall \ulcorner f \urcorner \\
 \exists!g \downarrow \mapsto & \downarrow Sg & \\
 \forall d \mapsto & C(c, Sd) & \\
 D \xrightarrow{C(c, S-)} & \mathbf{Set} &
 \end{array}$$

2. The Yoneda Lemma

(Page 59):

Proposition 1. For a functor $S : D \rightarrow C$ a pair $\langle r, u : c \rightarrow Sr \rangle$ is universal from c to S if and only if the function sending each $f' : r \rightarrow d$ into $Sf' \circ u : c \rightarrow Sd$ is a bijection of hom-sets

$$D(r, d) \cong C(c, Sd).$$

$$\begin{array}{ccc}
 & & c \\
 & & \downarrow u \\
 r & \xrightarrow{\quad} & Sr \\
 \exists! f' \downarrow & \xrightarrow{\quad} & \downarrow Sf' \\
 d & \xrightarrow{\quad} & Sd \\
 h \downarrow & \xrightarrow{\quad} & \downarrow Sh \\
 d' & \xrightarrow{\quad} & Sd' \\
 \\
 D & \xrightarrow{S} & C \\
 \\
 D(r, d) & \longleftrightarrow & C(c, Sd) \\
 \\
 f' & \xrightarrow{\quad} & Sf' \circ u \\
 f' & \longleftarrow & \downarrow f
 \end{array}$$

(Page 59, Proposition 1, cont.)

This bijection is natural in D .

$$\begin{array}{ccccc}
 & & c & & \\
 & & \downarrow u & & \\
 r & \mapsto & Sr & & \\
 f' \downarrow & \mapsto & \downarrow Sf' & & \\
 d & \mapsto & Sd & \quad d \mapsto D(r, d) & \quad f' \\
 h \downarrow & \mapsto & \downarrow Sh & \quad h \downarrow \mapsto \downarrow D(r, h) & \quad \downarrow \\
 d' & \mapsto & Sd' & \quad d' \mapsto D(r, d') & \quad h \circ f' \\
 D \xrightarrow{S} C & & & D \xrightarrow{D(r, -)} \mathbf{Set} &
 \end{array}$$

$$\begin{array}{ccccc}
 & & c & & \\
 & & \downarrow u & & \\
 r & \mapsto & Sr & & \\
 f' \downarrow & \mapsto & \downarrow Sf' & & \\
 d & \mapsto & Sd & \quad d \quad D(r, d) \xrightarrow{\varphi_d} C(c, Sd) & \quad f' \mapsto Sf' \circ u \\
 h \downarrow & \mapsto & \downarrow Sh & \quad h \downarrow D(r, h) \downarrow \quad \downarrow C(c, Sh) & \quad \downarrow \quad \downarrow \\
 d' & \mapsto & Sd' & \quad d' \quad D(r, d') \xrightarrow{\varphi_{d'}} C(c, Sd') & \quad h \circ f' \mapsto S(h \circ f') \circ u \\
 D \xrightarrow{S} C & & & D(r, -) \xrightarrow{\varphi} C(c, S-) &
 \end{array}$$

3. Coproducts and colimits

(Page 67):

Cone and limiting cone:

We call μ the *limiting cone* or the *universal cone* (from F).

$$\begin{array}{ccc}
 & F & \\
 & \downarrow \mu & \\
 \text{Colim } F = \text{Lim } F \dashrightarrow \Delta(\text{Lim } F) & \xrightarrow{\quad} & \text{Colim } F \dashleftarrow F \\
 \exists! t' \downarrow \quad \dashrightarrow \quad \downarrow \Delta t' & & \downarrow \quad \dashleftarrow \quad \downarrow \\
 \forall c \dashrightarrow \Delta c & & c \dashrightarrow \Delta c \\
 C \xrightarrow{\Delta} C^J & & C \xrightleftharpoons[\Delta]{\text{Colim}} C^J
 \end{array}$$

4. Products and limits

(Page 68):

A *limit* for a functor $F : J \rightarrow C$ is a universal arrow $\langle r, \nu \rangle$ from Δ to F .

$$\begin{array}{ccc}
 \Delta c \dashleftarrow \forall c & & \Delta c \dashleftarrow c \\
 \downarrow \forall? \quad \dashleftarrow \quad \downarrow \Delta? & & \downarrow \quad \dashleftarrow \quad \downarrow \\
 \Delta r \dashrightarrow r = \text{Lim } F & & F \dashrightarrow \text{Lim } F \\
 \nu \downarrow & & \\
 F & & \\
 C^J \xleftarrow{\Delta} C & & C^J \xrightleftharpoons[\text{Lim}]{\Delta} C
 \end{array}$$

(Page 69):

...and its limiting cone $\nu : \text{Lim } F \rightarrow F$...

(or more precisely $\nu : \Delta(\text{Lim } F) \rightarrow F$)

V. Limits

5. Adjoint on Limits

(Page 118):

One of the most useful properties of adjoints is this: A functor which is a right adjoint preserves all the limits which exist in its domain:

Theorem 1. *If the functor $G : A \rightarrow X$ has a left adjoint, while the functor $T : J \rightarrow A$ has a limiting cone $\tau : a \rightarrow T$ in A , then GT has the limiting cone $G\tau : Ga \rightarrow GT$ in X .*

$$\begin{array}{ccc}
 \Delta a & \longrightarrow & \Delta Ga \\
 \text{(univ)} \downarrow \tau & \longrightarrow & \downarrow G\tau \\
 & & \text{(univ)} \\
 T & \longrightarrow & GT \\
 \\
 A & \xleftarrow{F} & X \\
 & \xrightarrow{G} &
 \end{array}$$

This proof can also be cast in a more sophisticated form by using the fact that Lim is right adjoint to the diagonal functor Δ . In fact, given an adjunction...

$$\begin{array}{ccccc}
 \Delta LA = L^{\mathbf{I}} \Delta A & \longleftarrow & & \longrightarrow & \Delta A \\
 \downarrow & \swarrow & & \searrow & \downarrow \\
 D & \longrightarrow & & \longrightarrow & R^{\mathbf{I}} D \\
 \downarrow & \searrow & & \swarrow & \downarrow \\
 & & LA & \longleftarrow & A \\
 & & \downarrow & & \downarrow \\
 \lim D & \longrightarrow & R(\lim D) & & \lim(R^{\mathbf{I}} D)
 \end{array}$$

$$\begin{array}{ccccc}
 X^J & \xleftarrow{F^J} & & \xrightarrow{G^J} & A^J \\
 \swarrow \Delta & & & & \swarrow \Delta \\
 \text{Lim} \longleftarrow & & X & \xleftarrow{F} & A \\
 & & & \xrightarrow{G} &
 \end{array}$$

X. Kan Extensions

3. The Kan Extension

$$\begin{array}{ccc}
 SK & \longleftarrow & \forall S \\
 \sigma K \downarrow & \longleftarrow & \downarrow \exists! \sigma. \alpha = \epsilon \cdot \sigma K \\
 RK & \longleftarrow & R := \text{Ran}_K T \\
 \downarrow \epsilon & & \downarrow \\
 T & & \\
 A^M & \xleftarrow{A^K} & A^C \\
 & \xrightarrow{\text{Ran}_K} & \\
 M & \xrightarrow{K} & C
 \end{array}$$

$$\begin{array}{ccc}
 & C & \\
 K \nearrow & & \searrow \forall S \\
 M & \xrightarrow{T} & A \\
 & \downarrow \epsilon & \downarrow \exists! \sigma \\
 & & R
 \end{array}$$

$$\begin{array}{ccc}
 \text{Nat}(SK, T) & \cong & \text{Nat}(S, \text{Ran}_K T) \\
 \epsilon \cdot \sigma K & \longleftarrow & \sigma
 \end{array}$$

$$\begin{array}{l}
 \forall S. \\
 \forall \alpha : SK \rightarrow T. \\
 \exists! \sigma. \alpha = \epsilon \cdot \sigma K
 \end{array}$$