Each closure operator induces a topology and vice-versa (“version for children”)

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July 26, 2021

Abstract

One of the main prerequisites for understanding sheaves on elementary toposes is the proof that a (Lawvere-Tierney) topology on a topos induces a closure operator on it, and vice-versa. That standard theorem is usually presented in a relatively brief way, with most details being left to the reader — see for example [Joh77, section 3.1], [McL92, chapter 21], [LM92, section V.1], [Bel88, chapter 5] — and with no hints on how to visualize some of the hardest axioms and proofs.

These notes are, on a first level, an attempt to present that standard theorem in all details and in a visual way, following the conventions in [FavC]; in particular, some properties, like stability by pullbacks, are always drawn in the same “shape”.

On a second level these notes are also an experiment on doing these proofs on “archetypal cases” in ways that makes all the proofs easy to lift to the “general case”. Our first archetypal case is a “topos with inclusions”. This is a variant of the “toposes with canonical subobjects” from [LS86, section 2.15]; all toposes of the form $\mathbf{Set}^{\mathbf{C}}$, where $\mathbf{C}$ is a small category, are toposes with inclusions, and when we work with toposes with inclusions concepts like subsets and intersections are very easy to formalize. We do all our proofs on the correspondence between closure operators and topologies in toposes with inclusions, and then we show how to lift all our proofs to proofs that work on any topos. Our second archetypal case is toposes of the form $\mathbf{Set}^{\mathbf{D}}$, where $\mathbf{D}$ is a finite two-column graph. We show a way to visualize all the Lawvere-Tierney topologies on toposes of the form $\mathbf{Set}^{\mathbf{D}}$, and we define formally a way to “add visual intuition to a proof about toposes”; this is related to the several techniques for doing “Category Theory for children” that are explained in the first sections of [FavC].

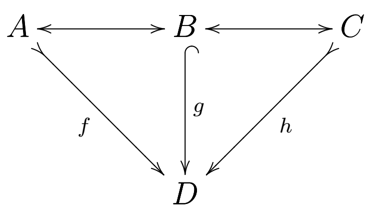
Contents

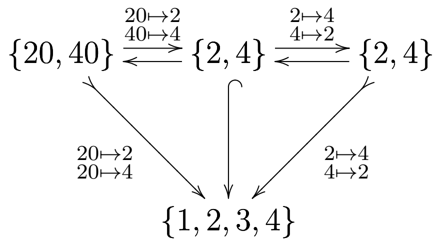
Status of these notes: this is not yet in final form. The last section still needs to be written, and there are a few proofs in the last sections are marked with "TODO" — but all things that still need to be done are clearly marked, so that's not so bad. For the most recent version look here:

`http://angg.twu.net/math-b.html#clops-and-tops`

1 Subobjects and inclusions

The *subobjects* of an object D of a topos $\mathbf{E}$ are the monics with codomain D *modulo isomorphism*. Here is an example in **Set**:





Here the monics $f : A \rightarrowtail D$, $g : B \rightarrowtail D$, $h : C \rightarrowtail D$ are all equivalent; in some texts they are "the same subobject". Let's make that precise. For us the elements of $\mathrm{Sub}(D)$ are the monics with codomain D. If $(f : A \rightarrowtail D)$, $(g : B \rightarrowtail D)$ are elements of $\mathrm{Sub}(D)$ then they are *equivalent* (notation: $f \equiv g$) iff there is an iso $A \leftrightarrow B$ making the obvious triangle commute. We write $[f]$ for the equivalence class made of an $f \in \mathrm{Sub}(D)$ and all other monics in $\mathrm{Sub}(D)$ equivalent to f, and we write $\overline{\mathrm{Sub}}(D)$ for $\mathrm{Sub}(D)$ modulo equivalence: so $[f] \in \overline{\mathrm{Sub}}(D)$.

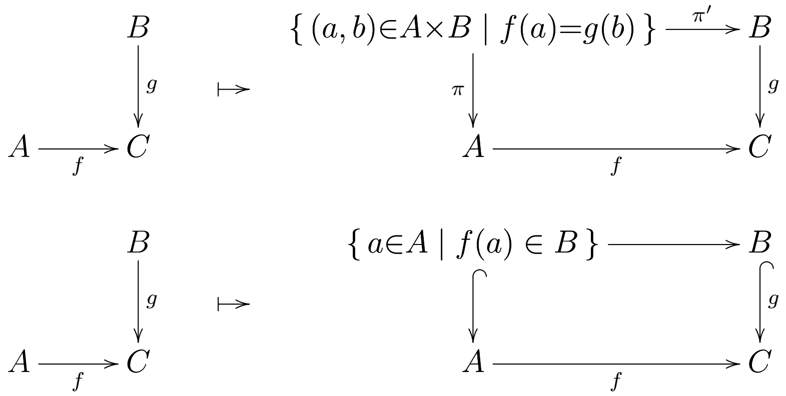
A monic $g : B \rightarrowtail D$ in **Set** is an *inclusion* if it obeys:

$$\forall b \in \mathrm{dom}(g).\, g(b) = b.$$

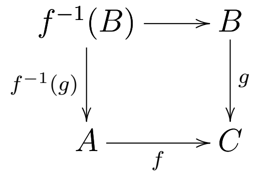
The usual way to formalize inclusions in toposes is via canonical subobjects. A topos $\mathbf{E}$ *has canonical subobjects* is it comes equiped with a class $\mathsf{CanSub}(\mathbf{E})$ of monics that obey a certain list of properties — see [LS86, p.200 onwards] — that are also obeyed by the inclusions in **Set**. Here we will do something similar but with a different list of properties, and in section 3 we will see how to translate our proofs, done in toposes with inclusions, to proofs in arbitrary toposes.

When $f : A \rightarrowtail C$ and $g : B \rightarrowtail C$ are subobjects of C we say that f *is contained in* g (notation: $f \subseteq g$) when there is a monic $m : A \rightarrowtail B$ making the obvious triangle commute. We call m the "mediating map".

In **Set** we have two different operations that take two maps f, g with a common codomain and produce pullbacks:

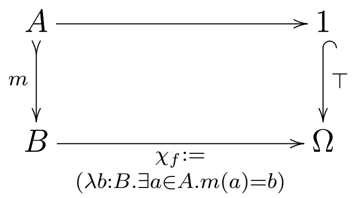


The second one only works when the right wall is an inclusion, but it produces pullbacks whose left walls are inclusions. In both cases we will write the left wall as $f^{-1}(g) : f^{-1}(B) \to A$,



and there will be no default name for the top wall. When the right wall is marked as an inclusion we will use the second pullback operation, otherwise the first one.

In **Set** the classifying map of a monic $m : A \rightarrowtail B$ is defined as:



and the "true" map $\top : 1 \hookrightarrow \Omega$ is the inclusion $\{1\} \hookrightarrow \{0, 1\}$.

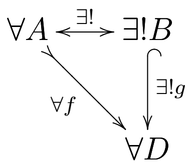
The *inclusion classified by a map* $f : B \to \Omega$ is the map $f^{-1}(\top)$; we will sometimes write it as $\sigma(f)$. Note that for any monic $m : A \rightarrowtail B$ we have $m \equiv \sigma(\chi_m)$, and we have $m = \sigma(\chi_m)$ if m is an inclusion; and for any $f : B \to \Omega$ we have $\chi_{\sigma(f)} = f$.

1.1 Inclusions, precisely

A *topos with inclusions* is a topos $\mathbf{E}$ endowed with a class of monics $\mathsf{Incs}(\mathbf{E})$, called the *inclusions*, and two pullback operations, as in the previous section, obeying the properties below:

Inc1) For any two object C and D of $\mathbf{E}$ there is at most one inclusion from C to D. When that inclusion map exists we write it as $C \hookrightarrow D$ — we don't need to name it — and we say that C *is a subset of* D (notation: $C \subseteq D$).

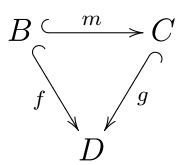
Inc2) Each $[f] \in \overline{\mathrm{Sub}}(D)$ contains exactly one inclusion map. This can be expressed as



in the variant of Freyd's diagrammatic language defined in [FavC, section 4.1]. We will say that this g is *the inclusion associated* (or: *equivalent*) *to* f, and write this as $\mathsf{can}(f) = g$.

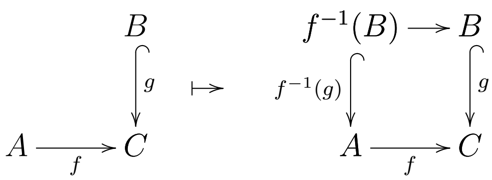
Inc3) The composite of two inclusions is an inclusion. Or, in the language of Inc1: if $B \subseteq C$ and $C \subseteq D$ then $B \subseteq D$, with $B \hookrightarrow D = B \hookrightarrow C \hookrightarrow D$.

Inc4) If $f : B \hookrightarrow D$ and $g : C \hookrightarrow D$ are inclusions with $f \subseteq g$ then the mediating map $m : B \rightarrowtail C$ is an inclusion. In the language of Inc1: $f \subseteq g$ implies $B \subseteq C$. We can visualize this as:

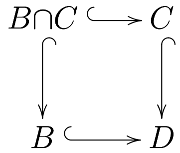


Inc5) The "true" map $\top : 1 \hookrightarrow \Omega$ is an inclusion.

Inc6) The second operation that produces pullbacks in $\mathbf{E}$ receives maps $f : A \to C$ and $g : B \hookrightarrow C$ and returns pullbacks whose left walls are inclusions. In a diagram:



Inc7) The *intersection* of two inclusions $B \hookrightarrow D$ and $C \hookrightarrow D$ is defined as their pullback:



Note that its upper wall is the mediating map from the composite $B \cap C \hookrightarrow C \hookrightarrow D$ to $C \hookrightarrow D$, so it is an inclusion.

Using Inc2 we can see that $B \cap C$ and $C \cap B$ are the *same* subset of D, not just isomorphic subobjects.

We write $\mathsf{Incs}(D)$ for the class of inclusions with codomain D and $\mathsf{Subsets}(D)$ for the class of subsets of D. In a topos with inclusions we have:

$$\mathsf{Subsets}(D) \cong \mathsf{Incs}(D) \cong \overline{\mathrm{Sub}}(D) \cong \mathrm{Sub}(D),$$

where the first two '$\cong$'s are isomorphisms and the last one is just an "equivalence of categories": if we start with a monic f in $\mathrm{Sub}(D)$, take it to its equivalence class $[f]$ in $\overline{\mathrm{Sub}}(D)$, and then go back to $\mathrm{Sub}(D)$, what we get is $\mathsf{can}(f)$, and we have $f \equiv \mathsf{can}(f)$ but not necessarily $f = \mathsf{can}(f)$.

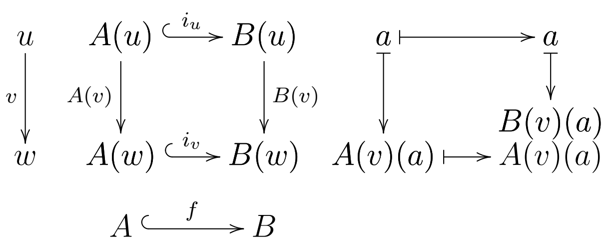
1.2 Inclusions in a topos of the form $\mathbf{Set}^{\mathbf{C}}$

From here onwards $\mathbf{C}$ will always denote a small category.

All toposes of the form $\mathbf{Set}^{\mathbf{C}}$ are toposes with inclusions. We define the class of inclusions of $\mathbf{Set}^{\mathbf{C}}$, $\mathsf{Incs}(\mathbf{Set}^{\mathbf{C}})$, as follows:

IncSC) A morphism $i : A \hookrightarrow B$ in $\mathbf{Set}^{\mathbf{C}}$ is an inclusion iff for every object u of $\mathbf{C}$ the map $i_u : A(u) \hookrightarrow B(u)$ is an inclusion in $\mathbf{Set}$; that is, if $\forall u \in \mathbf{C}. \forall a \in A(u). i_u(a) = a$.

Take an inclusion $i : A \hookrightarrow B$ and a morphism $v : u \to w$ in $\mathbf{C}$. As i is a natural transformation, these squares commute:



This means that $\forall a \in A(u). A(v)(a) = B(v)(a)$ – so $A(v) : A(u) \to A(w)$ is a restriction of the function $B(v) : B(u) \to B(w)$ to $A(u)$.

1.3 'And' and 'implies'

In section 2.2 we will need the "internal conjunction map", $(\land) : \Omega\times\Omega \to \Omega$, whose internal view is $(P,Q) \mapsto P{\land}Q$, and the "internal implication map", $(\to) : \Omega\times\Omega \to \Omega$, that works as $(P,Q) \mapsto (P{\to}Q)$. They are well explained in sections 13.3 and 13.4 of [McL92], but only in their forms "for adults", that work in arbitrary toposes. In this section I will just complement [McL92] by showing briefly how those definitions that hold in any topos are translations of definitions that make sense in **Set**.

The arrow $(\land)$ is built as the classifying map of the inclusion $\sigma(\land)$ in this diagram,

$$\begin{array}{ccc}
\{\,(P,Q)\in\Omega\times\Omega \mid P\land Q\,\} & \longrightarrow & 1 \\
{\scriptstyle\sigma(\land)}\big\downarrow & & \big\downarrow \\
\Omega\times\Omega & \xrightarrow[\land:=\chi_{\sigma(\land)}]{} & \Omega
\end{array}$$

and the inclusion $\sigma(\land)$ is built as an equalizer. We have:

$$\begin{array}{rl}
\multicolumn{2}{l}{\{\,(P,Q)\in\Omega\times\Omega \mid P\land Q\,\}} \\
= & \{\,(P,Q)\in\Omega\times\Omega \mid P=\top \land Q=\top\,\} \\
= & \{\,(P,Q)\in\Omega\times\Omega \mid \mathrm{id}_\Omega(P)=\top_\Omega(P)\land \mathrm{id}_\Omega(Q)=\top_\Omega(Q)\,\} \\
= & \{\,(P,Q)\in\Omega\times\Omega \mid (\mathrm{id}_\Omega\times\mathrm{id}_\Omega)(P,Q)=(\top_\Omega\times\top_\Omega)(P,Q)\,\} \\
= & \mathsf{Eq}((\mathrm{id}_\Omega\times\mathrm{id}_\Omega),(\top_\Omega\times\top_\Omega))
\end{array}$$

$$\{\,(P,Q)\in\Omega\times\Omega \mid P\land Q\,\} \xhookrightarrow{\sigma(\land)} \Omega\times\Omega \overset{(P,Q)\mapsto(P,Q)}{\underset{(P,Q)\mapsto(\top,\top)}{\rightrightarrows}} \Omega$$

$$\mathsf{Eq}((\mathrm{id}_\Omega\times\mathrm{id}_\Omega),(\top_\Omega\times\top_\Omega)) \xhookrightarrow{\mathsf{eq}((\mathrm{id}_\Omega\times\mathrm{id}_\Omega),(\top_\Omega\times\top_\Omega))} \Omega\times\Omega \overset{\mathrm{id}_\Omega\times\mathrm{id}_\Omega}{\underset{\top_\Omega\times\top_\Omega}{\rightrightarrows}} \Omega$$

Where the map $\top_\Omega$ is defined as:

$$A \xrightarrow{!_A} 1 \xrightarrow{\top} \Omega,\quad A \xrightarrow[\top_A:=\top\circ !_A]{} \Omega \qquad\qquad \Omega \xrightarrow{!_\Omega} 1 \xrightarrow{\top} \Omega,\quad \Omega \xrightarrow[\top_\Omega:=\top\circ !_\Omega]{} \Omega$$

The arrow $(\to)$ is the classifier of the inclusion $\sigma(\to)$, that is built as another equalizer:

$$\begin{array}{ccc}
\{\,(P,Q)\in\Omega\times\Omega \mid P\to Q\,\} & \longrightarrow & 1 \\
{\scriptstyle\sigma(\to)}\big\downarrow & & \big\downarrow \\
\Omega\times\Omega & \xrightarrow[(\to):=\chi_{\sigma(\to)}]{} & \Omega
\end{array}$$

$$\begin{array}{ll} & \{\,(P,Q) \in \Omega \times \Omega \mid P \to Q\,\} \\ = & \{\,(P,Q) \in \Omega \times \Omega \mid \top \le P \to Q\,\} \\ = & \{\,(P,Q) \in \Omega \times \Omega \mid \top \wedge P \le Q\,\} \\ = & \{\,(P,Q) \in \Omega \times \Omega \mid P \le Q\,\} \\ = & \{\,(P,Q) \in \Omega \times \Omega \mid P = P \wedge Q\,\} \\ = & \{\,(P,Q) \in \Omega \times \Omega \mid \pi(P,Q) = (\wedge)(P,Q)\,\} \\ = & \mathsf{Eq}(\pi, \wedge) \end{array}$$

$$\{\,(P,Q) \in \Omega \times \Omega \mid P \to Q\,\} \xhookrightarrow{\sigma(\to)} \Omega \times \Omega \underset{(P,Q)\mapsto(P\wedge Q)}{\overset{(P,Q)\mapsto P}{\rightrightarrows}} \Omega$$

$$\mathsf{Eq}(\pi, \wedge) \xhookrightarrow{\mathsf{eq}(\pi,\wedge)} \Omega \times \Omega \underset{\wedge}{\overset{\pi}{\rightrightarrows}} \Omega$$

Note that in **Set** we have:

$$\begin{array}{rcl} \{\,(P,Q) \in \Omega \times \Omega \mid P \wedge Q\,\} & = & \{(1,1)\}, \\ \{\,(P,Q) \in \Omega \times \Omega \mid P \to Q\,\} & = & \{(0,0),(0,1),(1,1)\}. \end{array}$$

2 Closure operators

A *closure operator* $\overline{(\cdot)}$ on a topos with inclusions $\mathbf{E}$ is a family of operations like this,

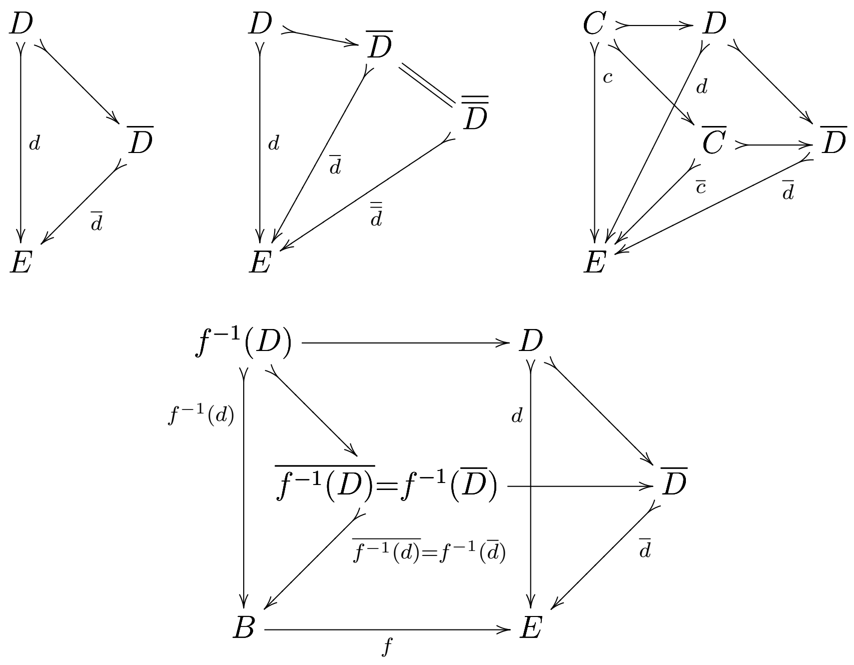
$$\begin{array}{rrcl} \overline{(\cdot)}_E : & \mathsf{Incs}(E) & \to & \mathsf{Incs}(E) \\ & (d : D \hookrightarrow E) & \mapsto & (\overline{d} : \overline{D} \hookrightarrow E), \end{array}$$

where we have one $\overline{(\cdot)}_E$ for each object E of the topos, and these '$\overline{(\cdot)}_E$'s obey:

C1) $d \subseteq \overline{d}$,
C2) $\overline{d} = \overline{\overline{d}}$,
C3) $c \subseteq d$ implies $\overline{c} \subseteq \overline{d}$,
C4) $\overline{c \cap d} = \overline{c} \cap \overline{d}$,
C5) $\overline{f^{-1}(d)} = f^{-1}(\overline{d})$,

for all inclusions $c : C \hookrightarrow E$ and $d : D \hookrightarrow E$ and for all maps $f : B \to E$.

We will draw the properties C1, C2, C3, C5 as:



Where all the '$\rightarrowtail$'s in the diagrams are inclusions.

Important: in all diagrams from this section to section to 2.6 all the '$\rightarrowtail$'s will stand for inclusions. This is for typographical reasons, to make the diagrams a bit lighter. The distinction between '$\rightarrowtail$'s and '$\hookrightarrow$'s will reappear in section 3.

2.1 Topologies

A *(Lawvere-Tierney) Topology* on a topos $\mathbf{E}$ is a map $j : \Omega \to \Omega$ obeying:
LT1) $j \circ j = j$,
LT2) $j \circ \top = \top$,
LT3) $j \circ \wedge = \wedge \circ (j \times j)$.
We draw LT1, LT2, and LT3 as:

$$\begin{array}{ccc} \Omega & \xrightarrow{\top} & \Omega \\ & {}_{\top}\searrow & \downarrow{\scriptstyle j} \\ & & \Omega \end{array} \qquad \begin{array}{ccc} \Omega & \xrightarrow{j} & \Omega \\ & {}_{j}\searrow & \downarrow{\scriptstyle j} \\ & & \Omega \end{array} \qquad \begin{array}{ccc} \Omega \times \Omega & \xrightarrow{\wedge} & \Omega \\ {\scriptstyle j\times j}\downarrow & & \downarrow{\scriptstyle j} \\ \Omega \times \Omega & \xrightarrow{\wedge} & \Omega \end{array}$$

One way to grasp the intuitive meaning of LT1, LT2, and LT3 is to look at their internal views. If we have maps $p, q : A \to \Omega$, the internal views of

$$\begin{array}{ccccc} A & \xrightarrow{p} & \Omega & \xrightarrow{\top} & \Omega \\ & & & {}_{\top}\searrow & \downarrow{\scriptstyle j} \\ & & & & \Omega \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{p} & \Omega & \xrightarrow{j} & \Omega \\ & & & {}_{j}\searrow & \downarrow{\scriptstyle j} \\ & & & & \Omega \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{\langle p,q\rangle} & \Omega \times \Omega & \xrightarrow{\wedge} & \Omega \\ & & {\scriptstyle j\times j}\downarrow & & \downarrow{\scriptstyle j} \\ & & \Omega \times \Omega & \xrightarrow{\wedge} & \Omega \end{array}$$

are:

$$\begin{array}{ccccc} a & \longmapsto & P(a) & \mapsto & \top \\ & & & \searrow & \downarrow \\ & & & & \top^* \\ & & & & \top \end{array} \qquad \begin{array}{ccccc} a & \longmapsto & P(a) & \mapsto & P(a)^* \\ & & & \searrow & \downarrow \\ & & & & P(a)^{**} \\ & & & & P(a)^* \end{array}$$

$$\begin{array}{ccccc} a & \longmapsto & (P(a), Q(a)) & \longmapsto & P(a) \wedge Q(a) \\ & & \downarrow & & \downarrow \\ & & & & (P(a) \wedge Q(a))^* \\ & & (P(a)^*, Q(a)^*) & \longmapsto & P(a)^* \wedge Q(a)^* \end{array}$$

We are writing $j(P(a))$ as $P(a)^*$ to suggest a connection between topologies and the J-operators of [PH2]; we will develop this idea in section ....
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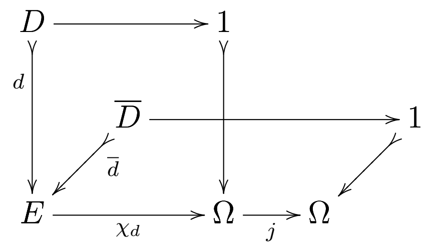
2.2 Topologies induce closure operators

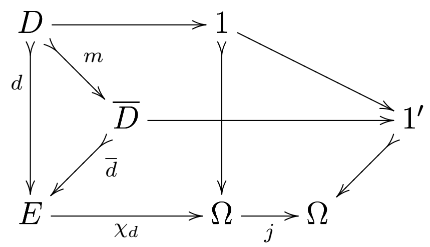
Theorem 2.2.1. Let $\mathbf{E}$ be a topos with inclusions, and let j be a topology on it. For each inclusion $d : D \hookrightarrow E$ let $\overline{d} : \overline{D} \hookrightarrow E$ be the inclusion that is classified by $j \circ \chi_d$, as in the diagram below:



Then this operation $d \mapsto \overline{d}$ is a closure operator — i.e., it obeys C1, C2, C3, C4, C5.

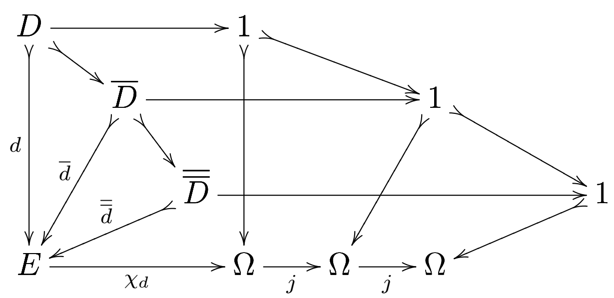
Proof.

For C1, rename the second 1 to $1'$ in the diagram above and draw the identity map $1 \to 1'$. The slanted rectangle with $\overline{D}$ in its upper left corner is a pullback. We can factor the maps $d : D \rightarrowtail E$ and $! : D \to 1'$ through it,



and this gives us a mediating map $m : D \rightarrowtail \overline{D}$. It is easy to check that this m is a monic and an inclusion.[1]

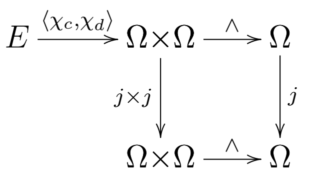
For C2, draw the diagram below:



[1] I thank David Michael Roberts for helping me with this.

The inclusion $\overline{d}$ is classified by $j \circ \chi_d$ and $\overline{\overline{d}}$ is classified by $j \circ j \circ \chi_d$. By LT1 we have $j \circ j = j$, and so $j \circ \chi_d = j \circ j \circ \chi_d$. This means that $\overline{d}$ and $\overline{\overline{d}}$ are two inclusions classified by the same map — so $\overline{d} = \overline{\overline{d}}$, and the inclusion $\overline{D} \hookrightarrow \overline{\overline{D}}$ is the identity.

To prove C4 we use the diagram below and the series of equalities at the right of it:



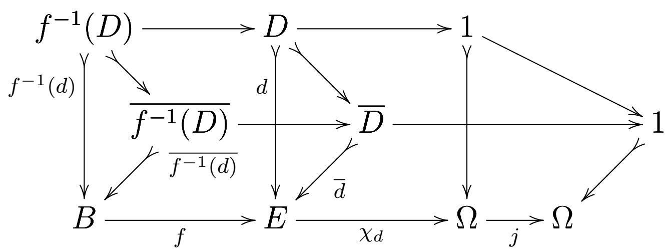
$$\begin{aligned} \chi_{(\overline{c \cap d})} &= j \circ \chi_{c \cap d} \\ &= j \circ \wedge \circ \langle \chi_c, \chi_d \rangle \\ &= \wedge \circ (j \times j) \circ \langle \chi_c, \chi_d \rangle \\ &= \wedge \circ \langle j \circ \chi_c, j \circ \chi_d \rangle \\ &= \wedge \circ \langle \chi_{\overline{c}}, \chi_{\overline{d}} \rangle \\ &= \chi_{(\overline{c} \cap \overline{d})} \end{aligned}$$

The inclusions $\overline{c \cap d}$ and $\overline{c} \cap \overline{d}$ are classified by the same map, so they are equal.

The proof of C3 is this series of inferences:

$$\dfrac{\dfrac{\dfrac{\dfrac{c \subseteq d}{c = c \wedge d}}{\overline{c} = \overline{c \wedge d}} \qquad \overline{c \wedge d} = \overline{c} \wedge \overline{d}}{\overline{c} = \overline{c} \wedge \overline{d}}}{\overline{c} \subseteq \overline{d}}$$

The proof of C5 is this diagram



plus these equalities:

$$\begin{aligned} \chi_{(\overline{f^{-1}(d)})} &= j \circ \chi_{f^{-1}(d)} \\ &= j \circ \chi_d \circ f \\ &= \chi_{\overline{d}} \circ f \\ &= \chi_{f^{-1}(\overline{d})} \end{aligned}$$

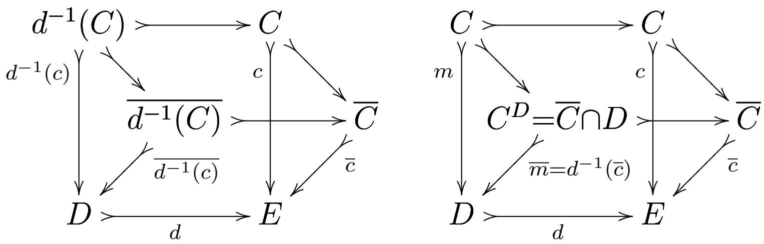
The inclusions $f^{-1}(d)$ and $f^{-1}(\overline{d})$ are classified by the same map, so they are the same inclusion.

2.3 Restricting a $\overline{(\cdot)}_E$

In this section we will see how a closure operation $\overline{(\cdot)}_E$ can be "restricted" to a subset $D \subseteq E$.

Theorem 2.3.1. Let $\mathbf{E}$ be a topos with inclusions, with a closure operator $\overline{(\cdot)}$. If $C \subseteq D \subseteq E$ in it, then the closure of $m : C \hookrightarrow D$ can be calculated from the closures of $c : C \hookrightarrow D$ and $d : D \hookrightarrow E$ — and we have $\overline{m} = d^{-1}(\overline{c})$ and $\mathrm{dom}(m) = \overline{C} \cap D$.

Proof. draw the diagram at the left below, that is the diagram for C5 with some things renamed. The pullback of $c : C \hookrightarrow E$ and $d : D \hookrightarrow E$ is $C \cap D$, which is C; so $\overline{d}^{-1}(c) = m$, and we have the diagram at the right.



Our notation for the domain of the closure of an $m : C \hookrightarrow D$ when the name $\overline{C}$ is taken will be C^D, for "the closure of C in D"; the operation '$\cdot^D$' will generalize the '$\cdot^*$' of [PH2]. As C^D is the pullback of $\overline{c} : \overline{C} \hookrightarrow E$ and $d : D \hookrightarrow E$ we have $C^D = \overline{C} \cap D = C^E \cap D$.

Theorem 2.3.2. Let $\mathbf{E}$ be a topos with inclusions with closure operator $\overline{(\cdot)}$. If $D \subseteq E$ in $\mathbf{E}$, then $\overline{(\cdot)}_D$ can be obtained from $\overline{(\cdot)}_E$ in the following way:

$$\begin{array}{rrcl} \overline{(\cdot)}_D : & \mathsf{Incs}(D) & \to & \mathsf{Incs}(D) \\ & (m : C \hookrightarrow D) & \to & (\overline{m} : C^D \to D) \\ & & & := (d^{-1}(\overline{c}) : \overline{C} \cap D \hookrightarrow D) \end{array}$$

where c is $c : C \hookrightarrow E$ and $\overline{c}$ is its closure, $\overline{c} : \overline{C} \hookrightarrow E$.

Proof. This is an easy corollary of Theorem 2.3.

2.4 Dense and closed

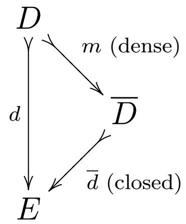
For the next theorems we need some definitions:
An inclusion $c : C \to D$ is *dense* iff $\overline{c} = \mathrm{id}_D$.
An inclusion $d : D \to E$ is *closed* iff $\overline{d} = d$.

Theorem 2.4.1. If an inclusion $a : A \hookrightarrow B$ is dense and closed then it is the identity.

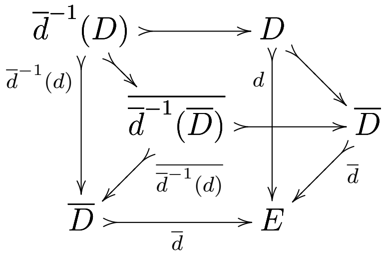
Proof:

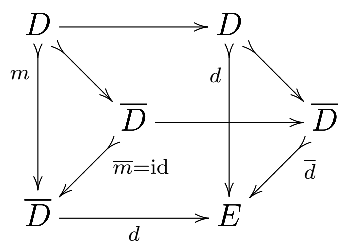
$$\dfrac{\dfrac{a \text{ dense}}{\overline{a} = \mathrm{id}_B} \qquad \dfrac{a \text{ closed}}{\overline{a} = a}}{a = \mathrm{id}_B}$$

Theorem 2.4.2. In a topos with inclusions **E** with closure operator $\overline{(\cdot)}$, for any inclusion $d : D \hookrightarrow E$ we have:



Proof. $\overline{\overline{d}} = \overline{d}$, so $\overline{d}$ is closed. To see that $m : D \hookrightarrow \overline{D}$ is dense, we build the diagram at the left below:

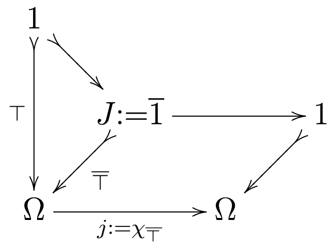




we have $\overline{d}^{-1}(D) = \overline{D} \cap D = D$ and $\overline{d}^{-1}(\overline{D}) = \overline{D} \cap \overline{D} = \overline{D}$, so we can rewrite it as the diagram at the right above, and we get that $\overline{m} = \mathrm{id}$.

2.5 Closure operators induce topologies

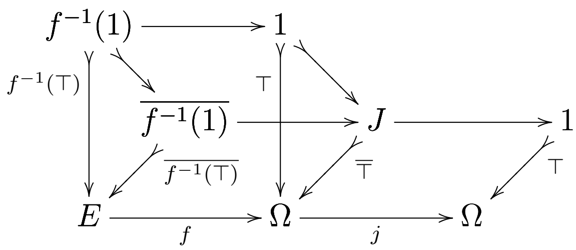
Let **E** be a topos with inclusions, and $\overline{(\cdot)}$ a closure operator on it. Build this diagram on it:



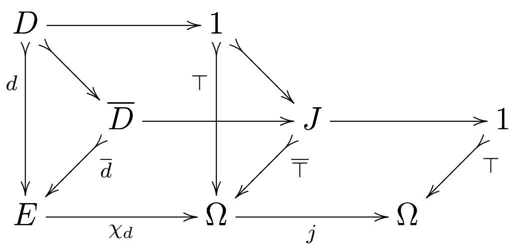
Here the closure of $\top : 1 \to \Omega$ is $\overline{\top} : \overline{1} \to \Omega$, and J is an alternate name for this $\overline{1}$; and $j := \chi_{\overline{\top}}$ is the map that classifies $\overline{\top}$.

Theorem 2.5.1. For every inclusion $d : D \hookrightarrow E$ we have $\chi_{\overline{d}} = j \circ \chi_d$, where j is the map above.

Proof. Take a map $f : E \to \Omega$, and add to the diagram above the diagram for $\overline{f^{-1}(\top)} = f^{-1}(\overline{\top})$. We get this:



This map f is the classifying map for some inclusion; let's call it $d : D \hookrightarrow E$, and rewrite f as χ_d. We get:

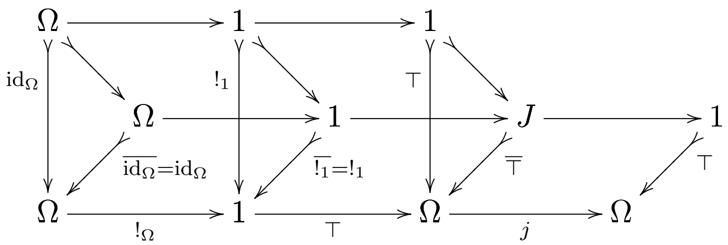


We have $\overline{d} = \overline{f^{-1}(\top)} = f^{-1}(\overline{\top}) = {\chi_d}^{-1}(\overline{\top}) = {\chi_d}^{-1}(j^{-1}(\top)) = (j \circ \chi_d)^{-1}(\top)$, and so $\chi_{\overline{d}} = \chi_{((j \circ \chi_d)^{-1}(\top))} = j \circ \chi_d$.

Theorem 2.5.2. The map j defined above is a topology.

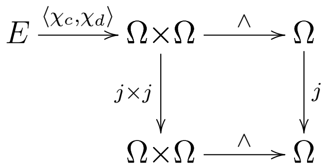
Proof. To prove LT1 we have to see that $j = j \circ j$. We have $\overline{d} = \overline{\overline{d}}$ for all inclusions d; so $\chi_{\overline{d}} = \chi_{\overline{\overline{d}}}$ always. We have $\chi_{\overline{d}} = j \circ \chi_d$ and $\chi_{\overline{\overline{d}}} = j \circ j \circ \chi_d$, so $j \circ \chi_d = j \circ j \circ \chi_d$ always holds. There is a way to make $\chi_d = \mathrm{id}$ here — which is when $d : D \hookrightarrow E$ is $\top : 1 \hookrightarrow \Omega$ — and so a particular case of $j \circ \chi_d = j \circ j \circ \chi_d$ is $j \circ \mathrm{id} = j \circ j \circ \mathrm{id}$, which gives us $j = j \circ j$.

To prove LT2 we have to see that $\top_\Omega = j \circ \top_\Omega$, i.e., that $\top \circ !_\Omega = j \circ \top \circ !_\Omega$. To do this we draw this diagram,



and check that its two upright squares and its three lower slanted squared are pullbacks. With this we get that both $\top \circ !_\Omega$ and $j \circ \top \circ !_\Omega$ classify id_Ω, so $\top \circ !_\Omega = j \circ \top \circ !_\Omega$.

To prove LT3 we start by choosing any two inclusions with the same codomain, $c : C \hookrightarrow E$ and $d : D \hookrightarrow E$. From the maps $\chi_c, \chi_d : E \to \Omega$ we build a map $\langle \chi_c, \chi_d \rangle : \Omega \to \Omega \times \Omega$, and we plug it on the diagram for LT3. We get:

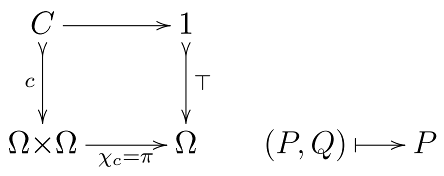


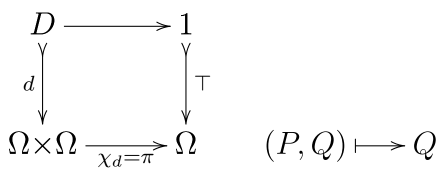
We have

$$
\begin{aligned}
\wedge \circ (j \times j) \circ \langle \chi_c, \chi_d \rangle &= \chi_{\overline{c} \cap \overline{d}} \\
j \circ \wedge \circ \langle \chi_c, \chi_d \rangle &= \chi_{\overline{c \cap d}}
\end{aligned}
$$

and C4 tells us that $\overline{c} \cap \overline{d} = \overline{c \cap d}$; so it is always true that $\wedge \circ (j \times j) \circ \langle \chi_c, \chi_d \rangle = j \circ \wedge \circ \langle \chi_c, \chi_d \rangle$. We can make $\langle \chi_c, \chi_d \rangle$ be the identity map if we take $E := \Omega \times \Omega$,

$\langle \chi_c, \chi_d \rangle = \mathrm{id}_{\Omega\times\Omega} = \langle \pi, \pi' \rangle$. The internal views of χ_c and χ_d are:

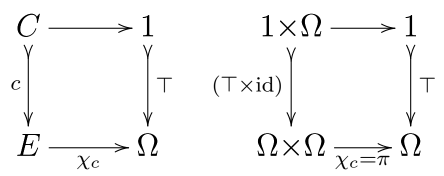


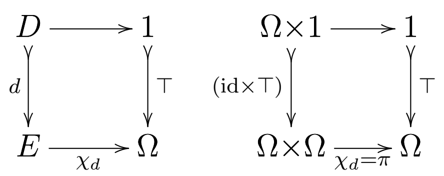


In **Set** we can construct the subsets C and D as:

$$\begin{array}{rcl} C & = & \{ (P, Q) \in \Omega \times \Omega \mid P = \top \} \\ & = & \{\top\} \times \Omega \\ D & = & \{ (P, Q) \in \Omega \times \Omega \mid Q = \top \} \\ & = & \Omega \times \{\top\} \end{array}$$

This *suggests* that we can generalize that construction to any topos as:

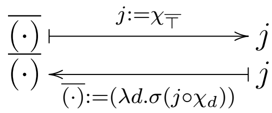




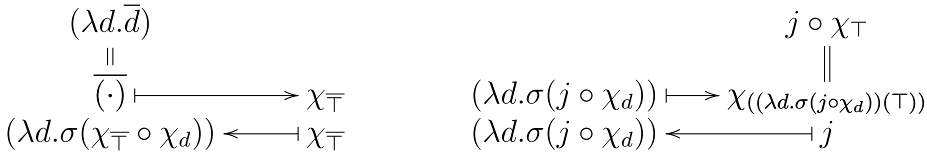
These constructions do work, but I will skip the details of the proof. So: with $c = (\top \times \mathrm{id})$ and $d = (\mathrm{id} \times \top)$ we have $\langle \chi_c, \chi_d \rangle = \mathrm{id}_{\Omega\times\Omega}$, and in this particular case our equality $\wedge \circ (j \times j) \circ \langle \chi_c, \chi_d \rangle = j \circ \wedge \circ \langle \chi_c, \chi_d \rangle$ reduces to $\wedge \circ (j \times j) = j \circ \wedge$ — and this proves LT3.

2.6 A bijection

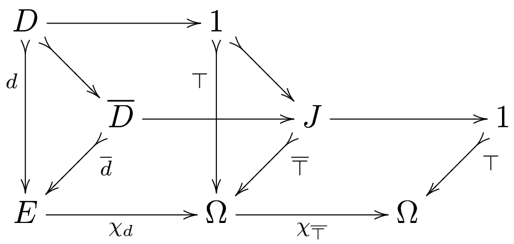
We saw that a closure operator induces a topology and that a topology induces a closure operator. Now we need to check that these two operations, that we can abbreviate as $\overline{(\cdot)} \mapsto j$ and $j \mapsto \overline{(\cdot)}$, as below,



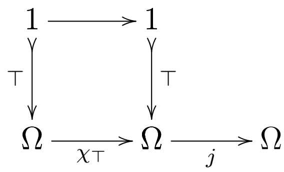
are inverses to one another — i.e., that the composites $\overline{(\cdot)} \mapsto j \mapsto \overline{(\cdot)}$ and $j \mapsto \overline{(\cdot)} \mapsto j$ are identity maps. We will organize all this visually as:



To prove that $\overline{(\cdot)} \mapsto j \mapsto \overline{(\cdot)}$ is the identity we need to check that in any topos with inclusions with a closure operator $\overline{(\cdot)}$ we have that $\overline{(\cdot)}$, i.e., $(\lambda d.\overline{d})$, is equal to $(\lambda d.\sigma(\chi_{\overline{\top}} \circ \chi_d))$. It is enough that check that we have $\overline{d} = \sigma(\chi_{\overline{\top}} \circ \chi_d)$ for any inclusion d. Look at the diagram below...



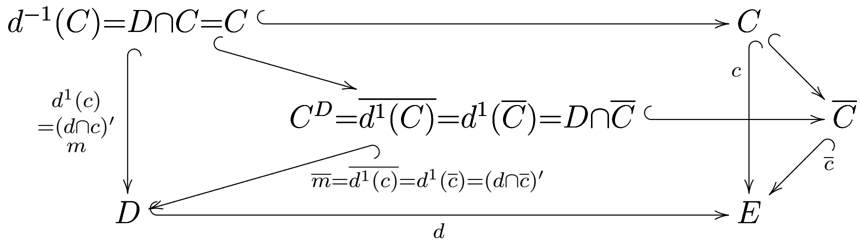
To prove that $j \mapsto \overline{(\cdot)} \mapsto j$ is the identity we need to check that in any topos with inclusions with a topology j we have $j = j \circ \chi_T$. Look at the diagram below:



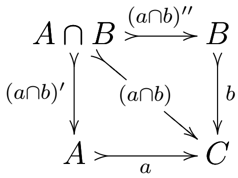
We have $\chi_\top = \mathrm{id}_\Omega$, and so $j \circ \chi_T = j \circ \mathrm{id} = j$.

3 Translating all this to toposes without inclusions

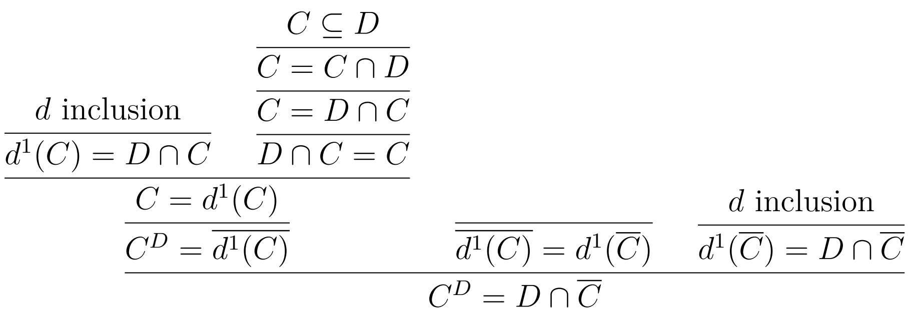
Let's start by an example – we will translate Theorem 2.3. This diagram condenses the two diagrams of the original proof into a single one:



where $(d\cap c)'$ and $(d\cap\overline{c})'$ are maps in intersection pullbacks. The convention is that if $a : A \rightarrowtail C$ and $b : B \rightarrowtail C$ are monics then the components of the diagram for $A \cap B$ are named like this:



This is how I would start to structure the proof above to implement it in a proof assistant. Most nodes in this tree



state that two inclusions with different constructions are isomorphic, and so they are the same morphism. For example, "$C = d^1(C)$" is an abbreviation for this:

$$(m : C \hookrightarrow D) = (d^1(c) : D^1(C) \hookrightarrow D)$$

The properties of inclusions let us omit the codomains and the names of the arrows in many cases, and write only their domains.

We can regard the tree above as a *proof* of this *equality* of *inclusions* that appears at the root node:

$$(\overline{m}:C^D\hookrightarrow D)=((d\cap\overline{c})':D\cap\overline{C}\hookrightarrow D)$$

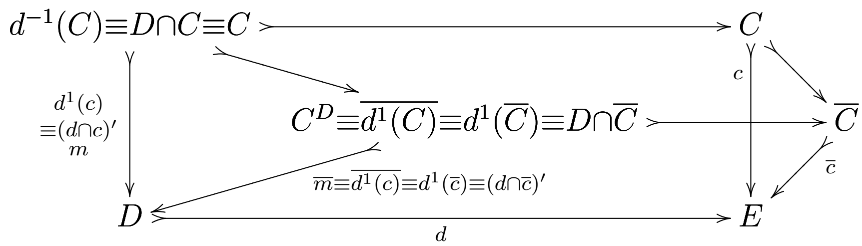
We can translate it to a *construction* of this *isomorphism* of *monics*:

$$(\overline{m}:C^D\rightarrowtail D)\equiv((d\cap\overline{c})':D\cap\overline{C}\rightarrowtail D)$$

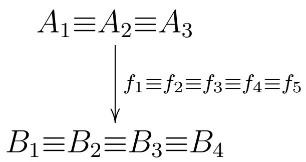
Now the names of the morphisms are primary and the names of the objects secondary. I prefer write both, otherwise I feel that the translated tree becomes unreadable. Here is the translation of the upper left part of the previous tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{d\text{ monic}}{(d^1(c):d^1(C)\rightarrowtail D)\equiv(\mathrm{id}\cap m:D\cap C\rightarrowtail D)}\quad\dfrac{\dfrac{\dfrac{(m:C\rightarrowtail D)\subseteq(\mathrm{id}:D\rightarrowtail D)}{(m:C\rightarrowtail D)\equiv(m\cap\mathrm{id}:C\cap D\rightarrowtail D)}}{(m:C\rightarrowtail D)\equiv(\mathrm{id}\cap m:D\cap C\rightarrowtail D)}}{(\mathrm{id}\cap m:D\cap C\rightarrowtail D)\equiv(m:C\rightarrowtail D)}}{(d^1(c):d^1(C)\rightarrowtail D)\equiv(m:C\rightarrowtail D)}}{(m:C\rightarrowtail D)\equiv(d^1(c):d^1(C)\rightarrowtail D)}}{(\overline{m}:C^D\rightarrowtail D)\equiv(\overline{d^1(c)}:\overline{d^1(C)}\rightarrowtail D)}$$

I tried to draw a diagram with all the morphisms in the tree above following my usual conventions, and I found the result too messy. But if we translate the original diagram to this,



and we define in the right way how to interpret the '≡'s in it, then everything works. In



the "object" $A_1\equiv A_2\equiv A_3$ means that we have three objects with known, but unnamed, isos between each one and the next, like this: $A_1\leftrightarrow A_2\leftrightarrow A_3$, and the "arrow" $f_1\equiv f_2\equiv f_3\equiv f_4\equiv f_5$ is in fact five "isomorphic" arrows, and each

f_i goes from some A_j to some B_k, but the diagram does not say what are these 'j's and 'k's; in this context "the 'f_i's are isomorphic" means that the diagram made by $A_1 \leftrightarrow A_2 \leftrightarrow A_3$, $B_1 \leftrightarrow B_2 \leftrightarrow B_3 \leftrightarrow B_4$, and all the '$f_i$'s commutes.

The translation sketched above works for all constructions and proofs in sections 2–2.6. It may be possible to characterize the class of constructions and proofs on which it works, but this is far beyond the scope of these notes.

4 Toposes of the form $\mathbf{Set}^{\mathbf{D}}$

In sec.1.2 we conventioned that $\mathbf{C}$ would always denote a small category. In this section we will need many other conventions like that, especially to define Grothendieck topologies in sections 4.6 and 4.7.

From here onwards $\mathbf{D}$ will always denote a finite DAG regarded as posetal category. We will consider that $\mathbf{D}$ is downward directed: if $u, v \in \mathbf{D}$ then we say that u is *above* v, or v is *below* u, when we have an arrow $u \to v$. We will say that u is *strictly above* v when we have an arrow $u \to v$ and $u \neq v$. Note that every $u \in \mathbf{D}$ is above itself.

In contexts in which we have a category $\mathbf{D}$ our (default) topos will be the topos $\mathbf{E} = \mathbf{Set}^{\mathbf{D}}$, with the inclusions given by the definitions in sec.1.2. We will use the conventions from [FavC, section 7.12] to draw its objects; for example, if

$$\mathbf{D} = \left(\begin{array}{ccc} \bullet & & \bullet \\ \downarrow & \searrow & \downarrow \\ \bullet & & \bullet \end{array} \right) = \left(\begin{array}{ccc} 2\_ & & \_2 \\ \downarrow & \searrow & \downarrow \\ 1\_ & & \_1 \end{array} \right)$$

then these are two objects of $\mathbf{Set}^{\mathbf{D}}$:

$$A = \left(\begin{array}{ccc} \emptyset & & \{4\} \\ \downarrow & \searrow & \downarrow \\ \{5\} & & \{7\} \end{array} \right), \qquad B = \left(\begin{array}{ccc} \{1,2\} & & \{3,4\} \\ {\scriptstyle 1\mapsto 5 \atop 2\mapsto 6}\downarrow & \searrow & \downarrow \\ \{5,6\} & & \{7\} \end{array} \right).$$

There is an inclusion $f : A \hookrightarrow B$ between them.

We will use a notation with expressions like '$(a \in A(u))$'s to make certain calculations easier to visualize. For example,

$$\begin{array}{rll} A\left(\begin{smallmatrix} \_2 \\ \downarrow \\ \_1 \end{smallmatrix}\right)(4 \in A(\_2)) = (7 \in A(\_1)) & \text{will mean} & A\left(\begin{smallmatrix} \_2 \\ \downarrow \\ \_1 \end{smallmatrix}\right) : A(\_2) \to A(\_1) \\ & \text{and} & A\left(\begin{smallmatrix} \_2 \\ \downarrow \\ \_1 \end{smallmatrix}\right)(4) = 7, \quad \text{and} \\ f(4 \in A(\_2)) = (4 \in B(\_2)) & \text{will mean} & f(\_2) : A(\_2) \to B(\_2) \\ & \text{and} & f(\_2)(4) = 4. \end{array}$$

Note that the way to expand the expressions at the left is different for functors (first case) and for natural transformations (second case).

4.1 The Heyting Algebra H

A $\mathbf{Set}^{\mathbf{D}}$ has several diffent terminal objects. The symbol 1, or $1_{\mathbf{E}}$, will by default denote the one that has $1(u) = \{*\}$ for every $u \in \mathbf{D}$. Its class of subsets always forms a set; for example, when $D = \left(\begin{smallmatrix} \bullet & & \bullet \\ & \searrow & \\ \bullet & & \bullet \end{smallmatrix}\right)$ we have:

$$\mathsf{Subsets}(1_{\mathbf{E}}) = \left\{ {}^{0}_{0}{\searrow}^{0}_{0},\ {}^{0}_{0}{\searrow}^{0}_{1},\ {}^{0}_{1}{\searrow}^{0}_{0},\ {}^{0}_{1}{\searrow}^{0}_{1},\ {}^{0}_{0}{\searrow}^{1}_{1},\ {}^{0}_{1}{\searrow}^{1}_{1},\ {}^{1}_{1}{\searrow}^{0}_{1},\ {}^{1}_{1}{\searrow}^{1}_{1} \right\}$$

Note that here the small '0's mean '$\emptyset$' and the small '1's mean '$\{*\}$'. We will use the notations for piles from [PH1, sec.15] to draw this as:

$$H \;=\; \mathsf{Subsets}(1_{\mathbf{E}}) = \begin{array}{ccc} & 22 & \\ 21 & & 12 \\ & 11 & & 02 \\ 10 & & 01 \\ & 00 & \end{array}$$

but with a difference: in [PH1, sec.15] two-digit numbers are interpreted as subsets of the set of points of the current default two-column graph — so, there we would have $21 = \left\{ \begin{smallmatrix} 2\_, & \\ 1\_, & \_2 \end{smallmatrix} \right\} \subset \mathbf{D}_0$ — while here they will be subsets of $1_{\mathbf{E}}$; so, here this holds:

$$21 = \left({}^{1}_{1}{\searrow}^{0}_{1} \right) = \left(\begin{smallmatrix} \{*\} & & \emptyset \\ \downarrow & \searrow & \downarrow \\ \{*\} & & \{*\} \end{smallmatrix} \right) \subset \left(\begin{smallmatrix} \{*\} & & \{*\} \\ \downarrow & \searrow & \downarrow \\ \{*\} & & \{*\} \end{smallmatrix} \right) = 1_{\mathbf{E}}\ .$$

In contexts in which we have a category $\mathbf{D}$ the letter H will denote this set $\mathsf{Subsets}(1_{\mathbf{E}})$, regarded as a Heyting Algebra, and we will refer to its elements as *truth-values*. In contexts in which our $\mathbf{D}$ is a 2-column graph this H will be a *Planar* Heyting Algebra — in the sense of sections 4 and 17 of [PH1] — and we will denote these truth-values by two-digit numbers.

4.2 Some definitions on posets

Suppose that $\mathbf{P}$ is a downward-directed poset; we will denote its set of points as $\mathbf{P}_0$. A subset $\mathcal{A} \subset \mathbf{P}_0$ is a *down-set* iff it obeys this:

$$\forall u, v \in \mathbf{P}_0.\ \begin{pmatrix} u \\ \text{above} \\ v \end{pmatrix} \to \begin{pmatrix} u \in \mathcal{A} \\ \downarrow \\ v \in \mathcal{A} \end{pmatrix}$$

We will denote the set of all down-sets of $\mathbf{P}$ by $\mathsf{D}(\mathbf{P})$, and if $\mathcal{A} \subset \mathbf{P}_0$ then we will denote the down-set generated by $\mathcal{A}$ — i.e., the smallest down-set of

$\mathbf{P}$ containing $\mathcal{A}$ — by $\downarrow_{\mathbf{P}}\mathcal{A}$, or by $\downarrow\mathcal{A}$. When $u \in \mathbf{P}_0$ we will usually write $\downarrow\{u\}$ as $\downarrow u$; so, if

$$\mathbf{D} = \left(\substack{\bullet\searrow\bullet\\ \bullet\quad\bullet}\right) = \begin{pmatrix} 2\_ & & \_2 \\ \downarrow & \searrow & \downarrow \\ 1\_ & & \_1 \end{pmatrix}$$

then:

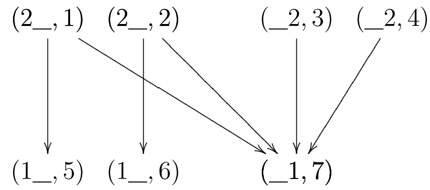
$$\begin{aligned} \downarrow_{\mathbf{D}}\{1\_,\ \_2\} &= \downarrow_{\mathbf{D}}\left\{\substack{\quad\ \ \_2,\\ 1\_\quad\ \ }\right\} = \left\{\substack{\quad\ \ \_2,\\ 1\_,\ \_1}\right\}, \\ \downarrow_{\mathbf{D}} 2\_ = \downarrow_{\mathbf{D}}\{2\_\} &= \downarrow_{\mathbf{D}}\left\{\substack{2\_\quad\ \ \\ \quad\quad}\right\} = \left\{\substack{2\_,\quad\ \ \\ 1\_,\ \_1}\right\}. \end{aligned}$$

We will use the poset $\mathbf{D}$ above in all examples in this section.

Every $\downarrow_{\mathbf{P}}\mathcal{A}$ can be seen as a poset — it inherits the order from $\mathbf{P}$.

Every object B of a $\mathbf{Set}^{\mathbf{D}}$ can be transformed into a poset $\mathsf{Po}(B)$ whose points are the pairs (u, b) in which $u \in \mathbf{D}$ and $b \in B(u)$. For example:

$$B = \begin{pmatrix} \{1,2\} & & \{3,4\} \\ {\scriptstyle 1\mapsto 5 \atop 2\mapsto 6}\downarrow & \searrow & \downarrow \\ \{5,6\} & & \{7\} \end{pmatrix} \qquad \mathsf{Po}(B) \ =$$



The operation that does the inverse of 'Po' is called 'Ob'. For every object B in a $\mathbf{Set}^{\mathbf{D}}$ there is a bijection between the subsets of B in $\mathbf{Set}^{\mathbf{D}}$ and the down-sets of $\mathsf{Po}(B)$. Let's define:

$$\begin{array}{rrcl} & \downarrow(b \in B(u)) & := & \mathsf{Ob}(\downarrow_{\mathsf{Po}(B)}(u, b)), \\ \text{so:} & \downarrow(1 \in B(2\_)) & = & \mathsf{Ob}(\downarrow_{\mathsf{Po}(B)}(2\_, 1)) \\ & & = & \mathsf{Ob}\begin{pmatrix} (2\_,1) & & \\ \downarrow & \searrow & \\ (1\_,5) & & (\_1,7) \end{pmatrix} \\ & & = & \begin{pmatrix} \{1\} & & \emptyset \\ \downarrow & \searrow & \downarrow \\ \{5\} & & \{7\} \end{pmatrix}. \end{array}$$

We will also use this other shorthand: if $u \in \mathbf{D}$, then

$$\downarrow u \ = \ \downarrow(* \in 1_{\mathbf{E}}(u)).$$

This looks ambiguous. We have both

$$\begin{array}{rrcl} & \downarrow 2\_ & = & \downarrow(* \in 1_{\mathbf{E}}(2\_)) \ = \ \left(\substack{1\searrow 0\\ 1\quad 1}\right) \ = \ 21 \ \subset \ 1_{\mathbf{E}} \\ \text{and} & \downarrow 2\_ & = & \downarrow_{\mathbf{D}}\{2\_\} \ = \ \left\{\substack{2\_,\quad\ \ \\ 1\_,\ \_1}\right\} \ \subset \ \mathbf{D}, \end{array}$$

but will always use the first meaning.

4.3 The classifier Ω

The classifier object on a $\mathbf{Set}^{\mathbf{D}}$ is the object $\Omega \in \mathbf{Set}^{\mathbf{D}}$ whose action on objects is $\Omega(u) = \mathsf{D}(\downarrow u)$ and whose action on morphisms is $\Omega\left(\begin{smallmatrix} u \\ \downarrow \\ v \end{smallmatrix}\right)(\mathcal{S}) = \mathcal{S} \cap \downarrow v$. Let's decypher this — it contains an abuse of language.

Let $\mathbf{D} = \left(\begin{smallmatrix} \bullet & & \bullet \\ & \searrow & \\ \bullet & & \bullet \end{smallmatrix}\right)$ and $u = 2\_$. Then $\downarrow 2\_ = \left(\begin{smallmatrix} 1 & & 0 \\ & \searrow & \\ 1 & & 1 \end{smallmatrix}\right) \subset 1_{\mathbf{E}}$. We will convert that $\left(\begin{smallmatrix} 1 & & 0 \\ & \searrow & \\ 1 & & 1 \end{smallmatrix}\right)$ to a poset $\left(\begin{smallmatrix} \bullet & & \cdot \\ & \searrow & \\ \bullet & & \bullet \end{smallmatrix}\right)$, and $\mathsf{D}(\downarrow 2\_)$ will be the set of down-sets of this new poset, which is:

$$\begin{array}{rcl} \mathsf{D}(\downarrow 2\_) & = & \left\{ \begin{smallmatrix} 0 & \cdot \\ 0 \searrow & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & \cdot \\ 0 \searrow & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & \cdot \\ 1 \searrow & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & \cdot \\ 1 \searrow & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & \cdot \\ 1 \searrow & 1 \end{smallmatrix} \right\} \\ & = & \{00, 01, 10, 11, 21\} \\ & = & \left(\begin{smallmatrix} & \cdot & & \\ 21 & & \cdot & \\ & 11 & & \cdot \\ 10 & & 01 & \\ & 00 & & \end{smallmatrix}\right) \\ & \subset & H \end{array}$$

Each '·' means "this is out of the domain", but the precise details vary according to the context.

If we do the same as above for 1_, _1, and _2, we get:

$$\mathsf{D}(\downarrow 2\_) = \left(\begin{smallmatrix} & \cdot & & \\ 21 & & \cdot & \\ & 11 & & \cdot \\ 10 & & 01 & \\ & 00 & & \end{smallmatrix}\right) \qquad \mathsf{D}(\downarrow \_2) = \left(\begin{smallmatrix} & \cdot & & \\ \cdot & & \cdot & \\ & \cdot & & 02 \\ \cdot & & 01 & \\ & 00 & & \end{smallmatrix}\right)$$

$$\mathsf{D}(\downarrow 1\_) = \left(\begin{smallmatrix} & \cdot & & \\ \cdot & & \cdot & \\ & \cdot & & \cdot \\ 10 & & \cdot & \\ & 00 & & \end{smallmatrix}\right) \qquad \mathsf{D}(\downarrow \_1) = \left(\begin{smallmatrix} & \cdot & & \\ \cdot & & \cdot & \\ & \cdot & & \cdot \\ \cdot & & 01 & \\ & 00 & & \end{smallmatrix}\right)$$

When $\mathbf{D} = \left(\begin{smallmatrix} \bullet & & \bullet \\ & \searrow & \\ \bullet & & \bullet \end{smallmatrix}\right)$ we can draw the classifier of $\mathbf{Set}^{\mathbf{D}}$ as:

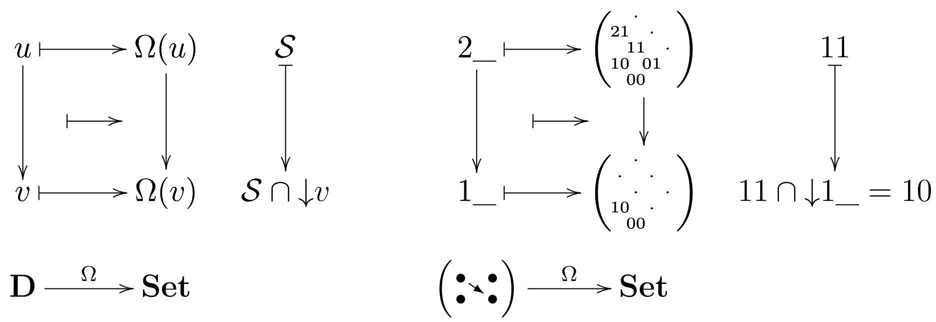
$$\Omega \quad = \quad \left(\begin{array}{ccc} \begin{smallmatrix} & \cdot & & \\ 21 & & \cdot & \\ & 11 & & \cdot \\ 10 & & 01 & \\ & 00 & & \end{smallmatrix} & & \begin{smallmatrix} & \cdot & & \\ \cdot & & \cdot & \\ & \cdot & & 02 \\ \cdot & & 01 & \\ & 00 & & \end{smallmatrix} \\ \downarrow & \searrow & \downarrow \\ \begin{smallmatrix} & \cdot & & \\ \cdot & & \cdot & \\ & \cdot & & \cdot \\ 10 & & \cdot & \\ & 00 & & \end{smallmatrix} & & \begin{smallmatrix} & \cdot & & \\ \cdot & & \cdot & \\ & \cdot & & \cdot \\ \cdot & & 01 & \\ & 00 & & \end{smallmatrix} \end{array}\right)$$

When $\mathbf{D}$ is a 2-column graph the classifier $\Omega \in \mathbf{Set}^{\mathbf{D}}$ can always be drawn in this (nice) way.

We still need to understand what the action of Ω on morphisms “means”. We can use the definition to calculate a particular case,

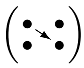
$$\begin{array}{rcl} \Omega\begin{pmatrix} u \\ \downarrow \\ v \end{pmatrix}(\mathcal{S}) & = & \mathcal{S} \cap {\downarrow} v \\ \Omega\begin{pmatrix} 2\_ \\ \downarrow \\ 1\_ \end{pmatrix}(11) & = & 11 \cap {\downarrow} 1\_ \\ & = & 11 \cap 10 \;=\; 10 \end{array}$$

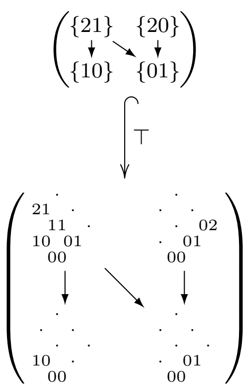
and can use a diagram like the one in [FavC, section 5.2], but with the particular case in its right half, to put that in a more categorical form:



4.4 The ‘true’ map $\top : 1 \hookrightarrow \Omega$

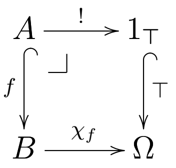
The ‘true’ map $\top : 1 \hookrightarrow \Omega$ will not be an inclusion if we take 1 as the default terminal. We can fix this by defining $1_\top$ as the terminal that takes each $u \in \mathbf{D}$ to $\{{\downarrow}u\}$, and by using this as our ‘true’ map: $\top : 1_\top \hookrightarrow \Omega$. Our *default* meaning for 1 is still the terminal that takes each u to $\{*\}$, but in contexts like “$\top : 1 \hookrightarrow \Omega$” the default meaning for the ‘1’ will change to ‘$1_\top$’.

When $\mathbf{D} =$  the ‘true’ map will be this one:

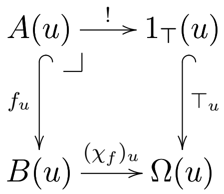


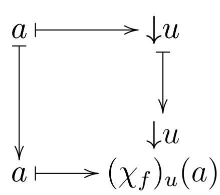
4.5 The classifying map χ_f of an inclusion f

Suppose that this diagram is a pullback in a $\mathbf{Set}^{\mathbf{D}}$:



then for every $u \in \mathbf{D}$ this is also a pullback:





so, for all $u \in \mathbf{D}$:

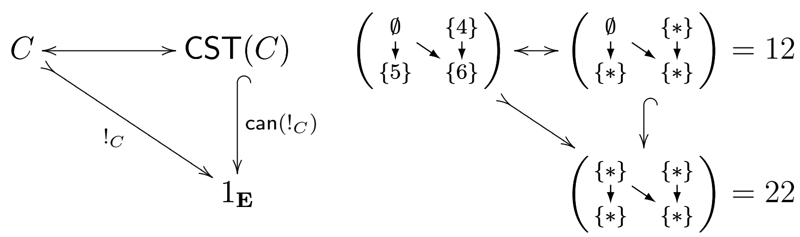
$$A(u) \;=\; \{\, b \in B(u) \mid (\chi_f)_u(b) = {\downarrow}u \,\}$$

This gives us an elementary way to start from just $\chi_f : B \to \Omega$ and construct the pullback in a way that makes the left wall an inclusion.

The other direction is harder. Suppose that we have an inclusion $f : A \hookrightarrow B$ in a $\mathbf{Set}^{\mathbf{D}}$, and we want to construct the map $\chi_f : B \to \Omega$ that completes the pullback. For every $u \in \mathbf{D}$ and $b \in B(u)$ we will define:

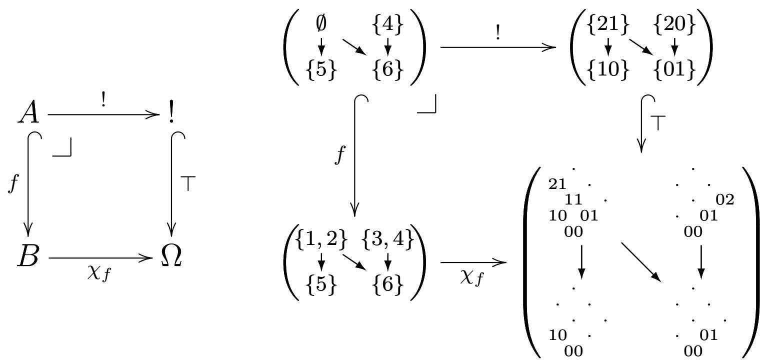
$$\chi_f(b \in B(u)) \;=\; \mathsf{CST}(A \cap {\downarrow}(b \in B(u)))$$

The operation CST is new: it "canonicalizes subobjects of the terminal". If $C \in \mathbf{Set}^{\mathbf{D}}$, then C is a subobject of the terminal iff the (unique) map $!_C : C \to 1$ is a monic. $\mathsf{CST}(C)$ is defined if an only if C is a subobject of the terminal, and is defined as $\mathsf{CST}(C) := \mathrm{dom}(\mathsf{can}(!_C))$. Here are a diagram for the general case and an example:



See the property Inc2 of toposes with inclusions in sec.1.1.

Let's use an example to understand how the definition of $\chi_f(b \in B(u))$ above works. The diagram below defines objects $A, B \in \mathbf{Set}^{\mathbf{D}}$ and an inclusion $f : A \hookrightarrow B$, and the map $\top : 1 \hookrightarrow \Omega$ at its right wall is the one that we saw in sec.4.4:

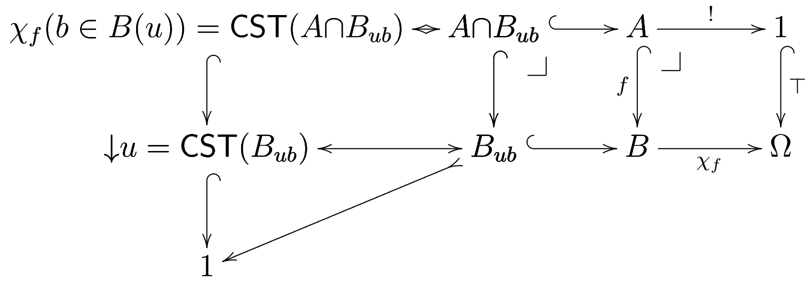


It *does not* define the map χ_f, but we can use the formula above to calculate $\chi_f(b \in B(u))$ for each pair (u, b) with $u \in \mathbf{D}$ and $b \in B(u)$. Let's do the case $u = 2\_$, $b = 2$:

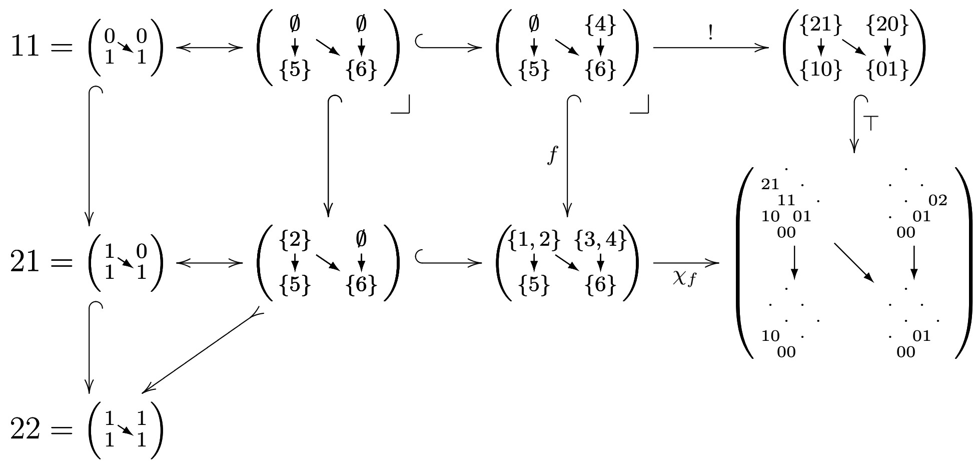
$$
\begin{array}{rcl}
\chi_f(b \in B(u)) & = & \mathsf{CST}(A \cap {\downarrow}(b \in B(u))) \\
\chi_f(2 \in B(2\_)) & = & \mathsf{CST}(A \cap {\downarrow}(2 \in B(2\_))) \\
& = & \mathsf{CST}\left(\begin{pmatrix} \emptyset & & \{4\} \\ \downarrow & \searrow & \downarrow \\ \{5\} & & \{6\} \end{pmatrix} \cap \begin{pmatrix} \{2\} & & \emptyset \\ \downarrow & \searrow & \downarrow \\ \{5\} & & \{6\} \end{pmatrix}\right) \\
& = & \mathsf{CST}\left(\begin{pmatrix} \emptyset & & \emptyset \\ \downarrow & \searrow & \downarrow \\ \{5\} & & \{6\} \end{pmatrix}\right) \\
& = & \begin{pmatrix} 0 & \searrow & 0 \\ 1 & & 1 \end{pmatrix} \\
& = & 11.
\end{array}
$$

We can also visualize the whole construction at once by drawing the diagram in the next page.

Note that we are abbreviating $\downarrow(b \in B(u))$ as B_{ub}, and that the left half of diagram will be different for each choice of $u \in \mathbf{D}$ and $b \in B(u)$.



When $u = 2\_$ and $b = 2$ the diagram above becomes:



4.6 The canonical Grothendieck topology J_{can}

Here's how to define the canonical Grothendieck topology on a topological space $(X, \mathcal{O}(X))$. We will denote open sets of $\mathcal{O}(X)$ by letters like U, V, W, and sets of open sets by letters like $\mathcal{A}, \mathcal{S}, \mathcal{U}$ — so $U, V, W \in \mathcal{O}(X)$ and $\mathcal{A}, \mathcal{S}, \mathcal{U} \subset \mathcal{O}(X)$. We will regard $\mathcal{O}(X)$ as a downward-directed poset, so these are equivalent:

$$\begin{array}{c} U \\ \textsf{above} \\ V \end{array}, \quad \begin{array}{c} U \\ \downarrow \\ V \end{array}, \quad \text{and} \quad \begin{array}{c} U \\ \cup\!\!\mid \\ V \end{array}.$$

A subset $\mathcal{S} \subset \mathcal{O}(X)$ will be called a *sieve* if it is downward-closed. A sieve $\mathcal{S} \subset \mathcal{O}(X)$ will be called a *sieve on* U if $\mathcal{S} \subset \mathcal{O}(U)$. We say that a sieve $\mathcal{S} \subset \mathcal{O}(U)$ *covers* U, or is a *covering sieve on* U, if $\bigcup \mathcal{S} = U$, and we say that a sieve $\mathcal{S}$ *r-covers* U if its restriction to $\mathcal{O}(U)$, $\mathcal{S} \cap \mathcal{O}(U)$, covers U.

The set of all sieves on U will be denoted by $\Omega(U)$, and the set of all covering sieves on U will be denoted by $J_{\text{can}}(U)$. Some extra notational conventions: The letters $\mathcal{S}$ and $\mathcal{R}$ will always denote sieves, and $\mathcal{U}$ will always denote a covering sieve (on U).

This diagram shows all these notional conventions, plus a few more:

$$\begin{array}{ccccccccc}
U & \in & \mathcal{O}(X) \\
\cup & & \cup \\
V & \in & \mathcal{S} & \in & \Omega(U) & = & \mathsf{D}(\downarrow U) & \subset & \mathcal{P}(\downarrow U) \\
& & & & \cup & & \cup \\
W & \in & \mathcal{U} & \in & J_{\text{can}}(U) & = & \{\, \mathcal{S} \in \mathsf{D}(\downarrow U) \mid \bigcup \mathcal{S} = U \,\} \\
& & & & \cup & & \cup \\
& & & & 1_{\top}(U) & = & \{\, \mathcal{S} \in \mathsf{D}(\downarrow U) \mid \mathcal{S} = \downarrow U \,\}
\end{array}$$

This J_{can} has this three properties:

$$\begin{array}{rlr}
\mathsf{hasmax}_{J_{\text{can}}}: & \forall U \in \mathcal{O}(X). & \downarrow U \text{ covers } U\,, \\
\mathsf{stab}_{J_{\text{can}}}: & \forall \begin{pmatrix} U \\ \textsf{above} \\ V \end{pmatrix} \in \mathcal{O}(X). \quad \forall \mathcal{S} \in \Omega(U). & \begin{pmatrix} \mathcal{S} \text{ covers } U \\ \downarrow \\ \mathcal{S} \text{ r-covers } V \end{pmatrix}, \\
\mathsf{trans}_{J_{\text{can}}}: & \forall U \in \mathcal{O}(X). \forall \mathcal{U} \in J(U). \quad \forall \mathcal{S} \in \Omega(U). & \begin{pmatrix} \mathcal{S} \text{ covers } U \\ \uparrow \\ \forall V \in \mathcal{U}.\ \mathcal{S} \text{ r-covers } V \end{pmatrix}.
\end{array}$$

4.7 Other Grothendieck topologies

Here's how to define what is a Grothendieck topology on an arbitrary downward-directed posed $\mathbf{D}$. The topology $\mathcal{O}(X)$ of the previous case will become the poset $\mathbf{D}$; we will refer to this $\mathbf{D}$ as our *ex-topology* and to the points $u, v, w \in \mathbf{D}$ as *ex-open sets*. These are equivalent:

$$\begin{matrix} u \\ \textsf{above} \\ v \end{matrix} \qquad \text{and} \qquad \begin{matrix} u \\ \downarrow \\ v \end{matrix} \;.$$

We will denote ex-open sets by letters like u, v, w, and sets of ex-open sets by letters like $\mathcal{A}$, $\mathcal{S}$, and $\mathcal{U}$. A subset $\mathcal{S} \subset \mathbf{D}$ will be called a *sieve* if it is downward-closed. A sieve $\mathcal{S} \subset \mathbf{D}$ will be called a *sieve on* u if $\mathcal{S} \subset {\downarrow}u$. The down-sets $\mathcal{O}(U)$ of the previous section will be replaced by ${\downarrow}u$ here.

In this context the set of all sieves is exactly H (see sec.4) and the set of all sieves on u is exactly $\Omega(u)$ (see sec.4.3).

A *Grothendieck topology* is an object $J \in \mathbf{Set}^{\mathbf{D}}$ obeying $1_{\top} \subset J \subset \Omega$ and having the properties $\textsf{hasmax}_J$, $\textsf{stab}_J$, and $\textsf{trans}_J$, that are defined as:

$$\begin{array}{rlll}
\textsf{hasmax}_J: & \forall u \in \mathbf{D}. & & {\downarrow}u \; J\textsf{-covers}\; u \;, \\
\textsf{stab}_J: & \forall \begin{pmatrix} u \\ \textsf{above} \\ v \end{pmatrix} \in \mathbf{D}. & \forall \mathcal{S} \in \Omega(u). & \begin{pmatrix} \mathcal{S}\; J\textsf{-covers}\; u \\ \downarrow \\ \mathcal{S}\; \textsf{r-}J\textsf{-covers}\; v \end{pmatrix} \;, \\
\textsf{trans}_J: & \forall u \in \mathbf{D}. \forall \mathcal{U} \in J(u). & \forall \mathcal{S} \in \Omega(u). & \begin{pmatrix} \mathcal{S}\; J\textsf{-covers}\; u \\ \uparrow \\ \forall v \in \mathcal{U}.\; \mathcal{S}\; \textsf{r-}J\textsf{-covers}\; v \end{pmatrix} .
\end{array}$$

Here we say that a sieve $\mathcal{S}$ on u *J-covers* u if $\mathcal{S} \in J(u)$, and that a sieve $\mathcal{S}$ *r-J-covers* v if its restriction to ${\downarrow}v$, $\mathcal{S} \cap {\downarrow}v$, J-covers v — i.e., if $\mathcal{S} \cap {\downarrow}v \in J(v)$.

Here our notational conventions are that $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{U}$ are sieves, and that $\mathcal{U}$ is a J-covering sieve on u. Other calligraphic capitals, like $\mathcal{A}$, may denote subsets of $\mathbf{D}_0$ that don't need to be downward-closed.

This diagram shows all these notional conventions, plus a few more:

$$\begin{array}{ccccccccc}
u & \in & \mathbf{D} \\
\downarrow & & \cup \\
v & \in & \mathcal{S} & \in & \Omega(u) & = & \mathsf{D}({\downarrow}u) & \subset & \mathcal{P}({\downarrow}u) \\
& & & & \cup \\
w & \in & \mathcal{U} & \in & J(u) \\
& & & & \cup \\
& & & & 1_{\top}(u) & = & \{\, \mathcal{S} \in \Omega(u) \mid \mathcal{S} = {\downarrow}u \,\}
\end{array}$$

4.8 Every $J(u)$ is a filter

Remember that if $\mathbf{P}$ is an upward-directed poset with terminal object $\top$ and binary meet $\wedge$ then a subset $\mathbf{F} \subset \mathbf{P}$ is a filter if it contains $\top$ and is closed upwards and by binary meets. More formally, $\mathbf{F}$ is a filter in $\mathbf{P}$ if:

$$\begin{array}{ll} & \top \in \mathbf{F}, \\ \forall \mathcal{R}, \mathcal{S} \in \mathbf{P}. & \begin{pmatrix} \mathcal{S} \\ \textsf{above} \\ \mathcal{R} \end{pmatrix} \to \begin{pmatrix} \mathcal{S} \in \mathbf{F} \\ \uparrow \\ \mathcal{R} \in \mathbf{F} \end{pmatrix}, \\ \forall \mathcal{R}, \mathcal{S} \in \mathbf{P}. & \begin{pmatrix} \mathcal{R}, \mathcal{S} \in \mathbf{F} \\ \downarrow \\ \mathcal{R} \wedge \mathcal{S} \in \mathbf{F} \end{pmatrix}. \end{array}$$

Also, a filter $\mathbf{F} \subset \mathbf{P}$ is *principal* when it contains the meet of all its elements; when this happens we have $\mathbf{F} = {\uparrow_{\mathbf{P}}} \{\bigwedge \mathbf{F}\} = {\uparrow} \bigwedge \mathbf{F}$, and we say that $\mathbf{F}$ is generated by its bottom element $\bigwedge \mathbf{F}$. When $\mathbf{P}$ is a finite poset all the filters on $\mathbf{P}$ are principal.

Theorem 4.8.1. If J is a Grothendieck topology on a $\mathbf{Set}^{\mathbf{D}}$ then every $J(u)$ is a filter on $\Omega(u)$.

Proof. We can re-state this as 1) $J(u)$ contains the top element of $\Omega(u)$, 2) $J(u)$ is closed upwards, 3) $J(u)$ is closed by binary meets, and we can re-state that again in a more visual way as:

$$\begin{array}{lll} 1) & & {\downarrow} u \in J(u), \\ 2) & \forall \mathcal{R}, \mathcal{S} \in \Omega(u). & \begin{pmatrix} \mathcal{S} \\ \cup \\ \mathcal{R} \end{pmatrix} \to \begin{pmatrix} \mathcal{S} \in J(u) \\ \uparrow \\ \mathcal{R} \in J(u) \end{pmatrix}, \\ 3) & \forall \mathcal{R}, \mathcal{S} \in \Omega(u). & \begin{pmatrix} \mathcal{R}, \mathcal{S} \in J(u) \\ \downarrow \\ \mathcal{R} \cap \mathcal{S} \in J(u) \end{pmatrix}. \end{array}$$

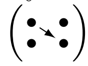
Part (1) is an obvious consequence of $\{{\downarrow} u\} = 1_{\top}(u) \subset J(u) \subset \Omega(u)$. The proofs of (2) and (3) are quite technical and difficult to understand intuitively, so we will present them in Natural Deduction form and let the reader check that every step is correct.

This is the proof that $J(u)$ is upwards-closed:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[v \in \mathcal{R}]^1 \quad \mathcal{R} \subset \mathcal{S}}{v \in \mathcal{S}} \quad \mathcal{S} \in \Omega(u)}{{\downarrow}v \subset \mathcal{S}}}{\mathcal{S} \cap {\downarrow}v = {\downarrow}v} \quad \dfrac{\dfrac{[v \in \mathcal{R}]^1 \quad \dfrac{\mathcal{R} \in \Omega(u)}{\mathcal{R} \subset \mathbf{D}}}{v \in \mathbf{D}} \quad \mathsf{hasmax}_J}{{\downarrow}v \in J(v)}}{\mathcal{S} \cap {\downarrow}v \in J(v)}}{\mathcal{S}\ \text{r-}J\text{-covers}\ v}}{\forall v \in \mathcal{R}.\mathcal{S}\ \text{r-}J\text{-covers}\ v}\ 1 \quad [\mathcal{R} \in J(u)]^2 \quad \mathcal{S} \in \Omega(u) \quad \mathsf{trans}_J}{\mathcal{S} \in J(u)}}{\mathcal{R} \in J(u) \to \mathcal{S} \in J(u)}\ 2$$

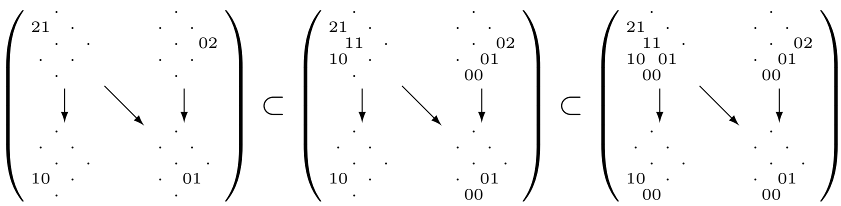
And this is the proof that $J(u)$ is closed by binary meets:

$$\dfrac{\mathcal{R} \in J(u) \quad \overline{\overline{\mathcal{R} \cap \mathcal{S} \in \Omega(u)}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\mathcal{R} \in J(u)}{\mathcal{R} \subset {\downarrow}u} \quad \dfrac{\dfrac{\dfrac{\mathcal{S} \in J(u)}{\mathcal{S}\ J\text{-covers}\ u} \quad \mathsf{stab}_J}{\forall v \in {\downarrow}u.\mathcal{S}\ \text{r-}J\text{-covers}\ v}}{\forall v \in {\downarrow}u.\mathcal{S} \cap {\downarrow}v \in J(v)}}{\forall v \in \mathcal{R}.\mathcal{S} \cap {\downarrow}v \in J(v)} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[v \in \mathcal{R}]^1}{{\downarrow}v \subset {\downarrow}\mathcal{R}} \quad \dfrac{\mathcal{R} \in J(u)}{{\downarrow}\mathcal{R} = \mathcal{R}}}{{\downarrow}v \subset \mathcal{R}}}{\mathcal{R} \cap {\downarrow}v = {\downarrow}v}}{\mathcal{R} \cap \mathcal{S} \cap {\downarrow}v = \mathcal{S} \cap {\downarrow}v}}{\mathcal{S} \cap {\downarrow}v = (\mathcal{R} \cap \mathcal{S}) \cap {\downarrow}v}}{\forall v \in \mathcal{R}.\mathcal{S} \cap {\downarrow}v = (\mathcal{R} \cap \mathcal{S}) \cap {\downarrow}v}\ 1}{\forall v \in \mathcal{R}.(\mathcal{R} \cap \mathcal{S}) \cap {\downarrow}v \in J(v)}}{\forall v \in \mathcal{R}.(\mathcal{R} \cap \mathcal{S})\ \text{r-}J\text{-covers}\ v} \quad \mathsf{trans}_J}{\mathcal{R} \cap \mathcal{S} \in J(u)}$$

Remember that we established (in sec.4) that $\mathbf{D}$ always stands for a *finite* downward-directed poset. So in a $\mathbf{Set}^{\mathbf{D}}$ all '$\Omega(u)$'s are finite upward-directed posets, and all filters on each $\Omega(u)$ are principal and can recovered from their bottom elements. This gives us a very compact way to represent Grothendieck topologies. For example, if $\mathbf{D} =$  and

$$\bigwedge J(2\_) = 10, \qquad \bigwedge J(\_2) = 00,$$
$$\bigwedge J(1\_) = 10, \qquad \bigwedge J(\_1) = 00$$

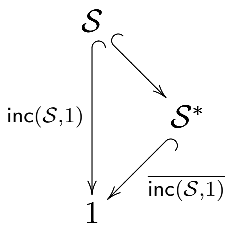
then $1_\top \subset J \subset \Omega(u)$ is:

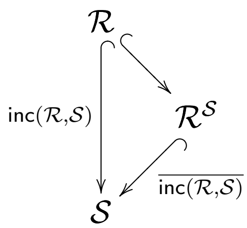


4.9 Every closure operator $\overline{(\cdot)}$ induces a nucleus $(\cdot)^*$

In this section we will suppose that our topos $\mathbf{Set}^{\mathbf{D}} = \mathbf{E}$ has a closure operator $\overline{(\cdot)}$.

If $\mathcal{S}$ is a truth-value in $\mathbf{E}$ then we will denote the closure of $\mathcal{S} \hookrightarrow 1$ by $\mathcal{S}^* \hookrightarrow 1$, and if $\mathcal{R}$ is a truth-value "smaller than $\mathcal{S}$", in the sense that we have an inclusion $\mathcal{R} \hookrightarrow \mathcal{S}$, then we will denote the closure of the inclusion $\mathcal{R} \hookrightarrow \mathcal{S}$ by $\mathcal{R}^{\mathcal{S}} \hookrightarrow \mathcal{S}$. Formally,





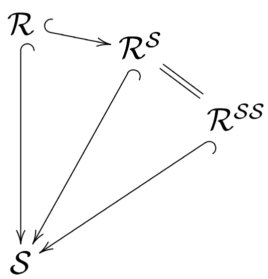
$$\begin{array}{rrcl} (\cdot)^* : & H & \to & H \\ & \mathcal{S} & \to & \mathrm{dom}(\overline{\mathsf{inc}(\mathcal{S},1)}) \\ (\cdot)^{\mathcal{S}} : & {\downarrow}\mathcal{S} & \to & {\downarrow}\mathcal{S} \\ & \mathcal{R} & \to & \mathrm{dom}(\overline{\mathsf{inc}(\mathcal{R},\mathcal{S})}) \end{array}$$

A *nucleus* on a Heyting Algebra H is an operation $(\cdot)^* : H \to H$ that obeys M1, M2, and M3 below, and if $\mathcal{S} \in H$ then an operation $(\cdot)^{\mathcal{S}} : {\downarrow}\mathcal{S} \to {\downarrow}\mathcal{S}$ is a *modality on the down-set* ${\downarrow}\mathcal{S}$ if it obeys MD1, MD2, and MD4 below:

M1) $\mathcal{S} \subset \mathcal{S}^*$,	MD1) $\mathcal{R} \subset \mathcal{R}^{\mathcal{S}}$,
M2) $\mathcal{S}^* = \mathcal{S}^{**}$,	MD2) $\mathcal{R}^{\mathcal{S}} = \mathcal{R}^{\mathcal{S}\mathcal{S}}$,
M3) $\mathcal{R}^* \cap \mathcal{S}^* = (\mathcal{R} \cap \mathcal{S})^*$,	MD3) $\mathcal{Q}^{\mathcal{S}} \cap \mathcal{R}^{\mathcal{S}} = (\mathcal{Q} \cap \mathcal{R})^{\mathcal{S}}$.

Theorem 4.9.1. The operation $(\cdot)^*$ induced by the closure operator $\overline{(\cdot)}$ is a modality on H, and the operation $(\cdot)^{\mathcal{S}}$ is a modality on the down-set ${\downarrow}\mathcal{S}$.

Proof. We will only prove MD1, MD2, and MD3. The proofs for MD1 and MD2 are essentially the same as the proofs for C1 and C2 in sec.2; the diagram is the one at the left below. The proof of MD3 is based on the proof of C4 in sec.2. Let $c : \mathcal{Q} \hookrightarrow \mathcal{S}$ and $d : \mathcal{R} \hookrightarrow \mathcal{S}$. We know from C4 that $\chi_{(\overline{c} \cap \overline{d})}$ $= \chi_{\overline{c \cap d}}$, so $\mathrm{dom}(\overline{c} \cap \overline{d}) = \mathcal{Q}^{\mathcal{S}} \cap \mathcal{R}^{\mathcal{S}}$ and $\mathrm{dom}(\overline{c \cap d}) = (\mathcal{Q} \cap \mathcal{R})^{\mathcal{S}}$ are the same object.



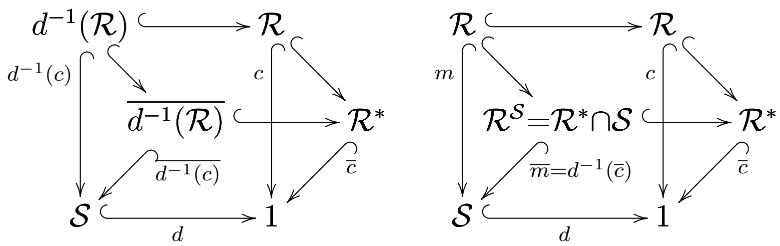
$$\begin{array}{rcl} \overline{c} & : & \mathcal{Q}^{\mathcal{S}} \hookrightarrow \mathcal{S} \\ \overline{d} & : & \mathcal{R}^{\mathcal{S}} \hookrightarrow \mathcal{S} \\ \overline{c} \cap \overline{d} & : & (\mathcal{Q}^{\mathcal{S}} \cap \mathcal{R}^{\mathcal{S}}) \hookrightarrow \mathcal{S} \\ c \cap d & : & (\mathcal{Q} \cap \mathcal{R}) \hookrightarrow \mathcal{S} \\ \overline{c \cap d} & : & (\mathcal{Q} \cap \mathcal{R})^{\mathcal{S}} \hookrightarrow \mathcal{S} \\ \mathrm{dom}(\overline{c} \cap \overline{d}) & = & \mathcal{Q}^{\mathcal{S}} \cap \mathcal{R}^{\mathcal{S}} \\ \mathrm{dom}(\overline{c \cap d}) & = & (\mathcal{Q} \cap \mathcal{R})^{\mathcal{S}} \end{array}$$

4.10 Every nucleus $(\cdot)^*$ induces a topology j

Let's start with a lemma.

Lemma 4.10.1. The operations $(\cdot)^*$ and $(\cdot)^{\mathcal{S}}$ induced by the closure operator $\overline{(\cdot)}$ are related in this way: $\mathcal{R}^{\mathcal{S}} = \mathcal{R}^* \cap \mathcal{S}$.

Proof. This is a corollary of Theorem 2.3. Let's call our inclusions $m : \mathcal{R} \hookrightarrow \mathcal{S}$, $d : \mathcal{S} \hookrightarrow 1$, and $c : \mathcal{R} \hookrightarrow 1$. Our diagrams are:

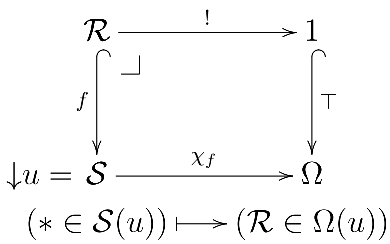


Theorem 4.10.2.
a) If $f : \mathcal{R} \hookrightarrow \mathcal{S}$ and $\mathcal{S} = {\downarrow}u$ then $\chi_f(* \in \mathcal{S}(u)) = (\mathcal{R} \in \Omega(u))$.
b) If $g : \mathcal{R}^{\mathcal{S}} \hookrightarrow \mathcal{S}$ and $\mathcal{S} = {\downarrow}u$ then $\chi_g(* \in \mathcal{S}(u)) = (\mathcal{R}^{\mathcal{S}} \in \Omega(u))$.
c) If $h : \mathcal{R} \hookrightarrow \mathcal{S}$ and $\mathcal{S} = {\downarrow}u$ then $\chi_{\overline{h}}(* \in \mathcal{S}(u)) = (\mathcal{R}^{\mathcal{S}} \in \Omega(u))$.
d) If $h : \mathcal{R} \hookrightarrow \mathcal{S}$ and $\mathcal{S} = {\downarrow}u$ then $(j \circ \chi_h)(* \in \mathcal{S}(u)) = (\mathcal{R}^{\mathcal{S}} \in \Omega(u))$.
e) If $h : \mathcal{R} \hookrightarrow \mathcal{S}$ and $\mathcal{S} = {\downarrow}u$ then $(j \circ \chi_h)(* \in \mathcal{S}(u)) = (\mathcal{R}^* \cap \mathcal{S} \in \Omega(u))$.
f) If $h : \mathcal{R} \hookrightarrow \mathcal{S}$ and $\mathcal{S} = {\downarrow}u$ then $j(\mathcal{R} \in \Omega(u)) = (\mathcal{R}^* \cap \mathcal{S} \in \Omega(u))$.
g) If $\mathcal{S} = {\downarrow}u$ then $j(\mathcal{R} \in \Omega(u)) = (\mathcal{R}^* \cap \mathcal{S} \in \Omega(u))$.
h) $j(\mathcal{R} \in \Omega(u)) = (\mathcal{R}^* \cap {\downarrow}u \in \Omega(u))$.
i) $j(u)(\mathcal{R}) = \mathcal{R}^* \cap {\downarrow}u$.
j) $j = \lambda u \in \mathbf{D}.\lambda \mathcal{R} \in \Omega(u).(\mathcal{R}^* \wedge {\downarrow}u)$.

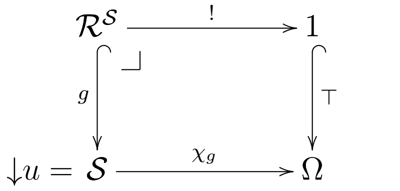
Proof. Item (a) is a consequence of the formula in section 4.5:

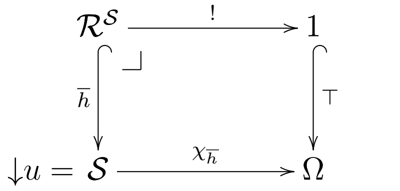
$$\begin{array}{rcl}
\chi_f(b \in B(u)) & = & (\mathsf{CST}(A \cap ({\downarrow}(b \in B(u))) \in \Omega(u)) \\
\chi_f(* \in \mathcal{S}(u)) & = & (\mathsf{CST}(\mathcal{R} \cap ({\downarrow}(* \in \mathcal{S}(u))) \in \Omega(u)) \\
& = & (\mathsf{CST}(\mathcal{R} \cap {\downarrow}u) \in \Omega(u)) \\
& = & (\mathsf{CST}(\mathcal{R}) \in \Omega(u)) \\
& = & (\mathcal{R} \in \Omega(u))
\end{array}$$

We can visualize it as:



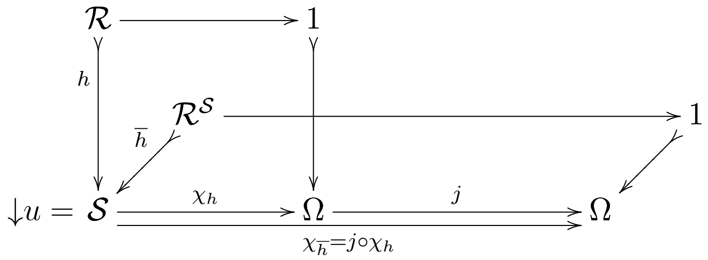
The items (b) and (c) are substitution instances of (a) and (b). These substituions become easy to understand if we draw their diagrams:





$(* \in \mathcal{S}(u)) \longmapsto (\mathcal{R}^{\mathcal{S}} \in \Omega(u))$ $\qquad$ $(* \in \mathcal{S}(u)) \longmapsto (\mathcal{R}^{\mathcal{S}} \in \Omega(u))$

To draw a diagram for (d), (e), (f), (g), (h), (i), and (j) we transform the arrow $\overline{h}$ in the diagram for (c) into a arrow going southwest in a diagram like the ones in sec.2.2:



$$(* \in \mathcal{S}(u)) \longmapsto (\mathcal{R}^{\mathcal{S}} \in \Omega(u)) = (\mathcal{R}^* \cap \mathcal{S} \in \Omega(u))$$

$$(* \in \mathcal{S}(u)) \longmapsto (\mathcal{R} \in \Omega(u))$$

$$(\mathcal{R} \in \Omega(u)) \longmapsto (\mathcal{R}^* \cap \mathcal{S} \in \Omega(u))$$

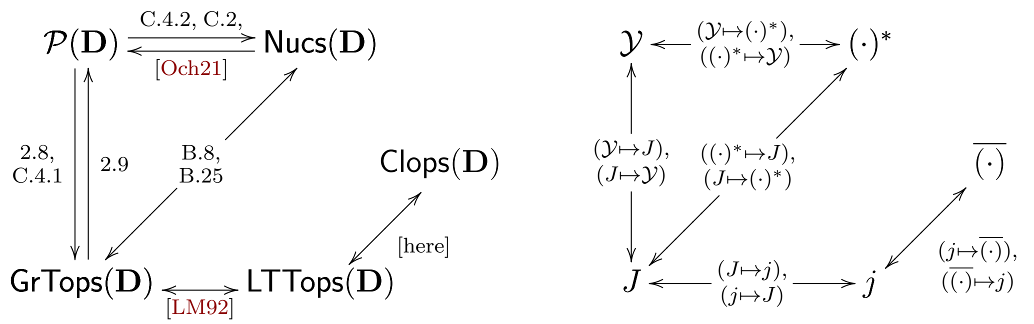
TODO: Write the proof of (e)→(f).

4.11 Some bijections

Let $\mathbf{D}$ be a finite 2-column graph. As usual, let $\mathbf{E}$ be the topos $\mathbf{Set}^{\mathbf{D}}$, and let $H = \mathsf{Subsets}(1_{\mathbf{E}})$ be its Heyting Algebra of truth-values. Let's denote the set of subsets of $\mathbf{D}_0$ by $\mathcal{P}(\mathbf{D})$, the set of nuclei on $\mathbf{Set}^{\mathbf{D}}$ by $\mathsf{Nucs}(\mathbf{D})$, the set of Grotendieck topologies on $\mathbf{Set}^{\mathbf{D}}$ by $\mathsf{GrTops}(\mathbf{D})$, the set of Lawvere-Tierney topologies on $\mathbf{Set}^{\mathbf{D}}$ by $\mathsf{LTTops}(\mathbf{D})$, and the set of closure operators on $\mathbf{Set}^{\mathbf{D}}$ by $\mathsf{Clops}(\mathbf{D})$.

We have bijections between the sets $\mathcal{P}(\mathbf{D})$, $\mathsf{Nucs}(\mathbf{D})$, and $\mathsf{GrTops}(\mathbf{D})$. They are proved, together with the commutatity of the triangle, in [Lin14] and [Och21]. The bijection between $\mathsf{GrTops}(\mathbf{D})$ and $\mathsf{LTTops}(\mathbf{D})$ is proved in [LM92], theorem V.1.2 and section V.4, and the bijection between $\mathsf{LTTops}(\mathbf{D})$ and $\mathsf{Clops}(\mathbf{D})$ has brief proofs in several standard books and a detailed proof in sections 2–3 here.

The diagram below shows all these bijections and where more information about them can be found. The right half shows how we will refer to the here; for example, $(J \leftrightarrow j)$ is the bijection between $\mathsf{GrTops}(\mathbf{D})$ and $\mathsf{LTTops}(\mathbf{D})$, and we call its components $(J \mapsto j)$ and $(j \mapsto J)$.



These are the components of the bijections in the triangle. The references like "C.4.2" point to sections, theorems, and definitions in [Lin14].

$$
\begin{array}{rrcll}
(\mathcal{Y} \mapsto (\cdot)^*): & \mathcal{S}^* & = & \mathcal{Y} \to \mathcal{S} & \text{C.4.2, C.2} \\
((\cdot)^* \mapsto \mathcal{Y}): & \mathcal{Y} & = & \{\, u \in \mathbf{D} \mid ({\downarrow}u)^* \neq ({\downarrow}^\circ u)^* \,\} & \text{[Och21]} \\
(\mathcal{Y} \mapsto J): & J(u) & = & \{\, \mathcal{S} \in \Omega(u) \mid \mathcal{Y} \cap {\downarrow}u \subseteq \mathcal{S} \,\} & \text{2.8, C.4.1} \\
(J \mapsto \mathcal{Y}): & X_J & = & \{\, u \in \mathbf{D} \mid J(u) = \{{\downarrow}u\} \,\} & \text{2.9} \\
((\cdot)^* \mapsto J): & J(u) & = & \{\, \mathcal{S} \in \Omega(u) \mid u \in \mathcal{S}^* \,\} & \text{B.8, B.25} \\
(J \mapsto (\cdot)^*): & \mathcal{S}^* & = & \{\, u \in \mathbf{D} \mid \mathcal{S} \cap {\downarrow}u \in J(u) \,\} & \text{B.8, B.25}
\end{array}
$$

These are the components of the bijection $(J \leftrightarrow j)$:

$$
\begin{array}{rrcl}
(J \mapsto j): & j & = & \chi_{(J \hookrightarrow \Omega)} \\
(j \mapsto J): & J & = & \mathrm{dom}(\sigma(j))
\end{array}
$$

These are the components of the bijection $(j \leftrightarrow \overline{(\cdot)})$:

$$\begin{array}{rcl} (j \mapsto \overline{(\cdot)}): & \overline{(f : A \hookrightarrow B)} & = & \sigma(j \circ \chi_f) \\ (\overline{(\cdot)} \mapsto j): & j & = & \chi_{\overline{(1_\top \hookrightarrow \Omega)}} \end{array}$$

We have also defined these two operations in sections 4.9 and 4.10:

$$\begin{array}{rcl} (\overline{(\cdot)} \mapsto (\cdot)^*): & \mathcal{S}^* & = & \mathrm{dom}(\overline{(\mathcal{S} \hookrightarrow 1)}) \\ ((\cdot)^* \mapsto j): & j(u)(\mathcal{R}) & = & \mathcal{R}^* \cap {\downarrow} u \end{array}$$

We did not prove that our formulas for them yield exactly the corresponding components of the bijections obtained by composing the bijections that we know. Let state that as two conjectures:

Conjecture 4.11.1. $((\cdot)^* \mapsto j) = ((\cdot)^* \mapsto J); (J \mapsto j)$.

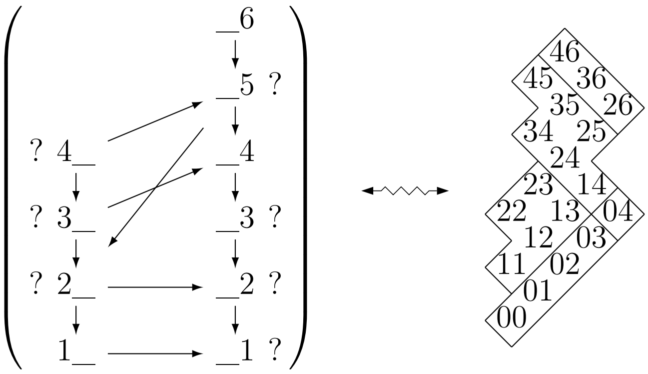
Conjecture 4.11.2. $(\overline{(\cdot)} \mapsto (\cdot)^*) = (\overline{(\cdot)} \mapsto j); (j \mapsto J); (J \mapsto (\cdot)^*)$.

4.12 Valid 4-uples

One way to start to get some visual intuition on what nuclei and topologies "mean" is to choose a 2-column graph $\mathbf{D}$ and then choose either a $(\cdot)^*$, a $\mathcal{Y}$, a J, or a j on it, and then calculate the other three elements of the 4-uple $((\cdot)^*, \mathcal{Y}, J, j)$ from it using the bijections from the previous section. We will always draw these 4-uples in this shape:

$$\begin{array}{ccc} \mathcal{Y} & \leftrightarrow & (\cdot)^* \\ \updownarrow & \swarrow\!\!\nearrow & \\ J & \leftrightarrow & j \end{array}$$

The main result of [PH2] is a way to visualize the bijection $(\mathcal{Y} \leftrightarrow (\cdot)^*)$ when $\mathbf{D}$ is a 2-column graph. We will need to change its notation a bit. In [PH2] the notation for a 2CG with question marks is $((P, A), Q)$ and the notation for a ZHA with a nucleus is (H, J); here we will use $((\mathbf{P}_0, A), \mathcal{Q})$ and $(H, (\cdot)^*)$. The set of arrows A of the 2CG is a subset of $\mathbf{P}_0 \times \mathbf{P}_0$, and $\mathcal{Q}$ is the set of points of $\mathbf{P}_0$ "with question marks". Here is an example of a $((\mathbf{P}_0, A), \mathcal{Q})$ with its associated $(H, (\cdot)^*)$:



$$((\mathbf{P}_0, A), \mathcal{Q}) \quad \leftrightsquigarrow \quad (H, (\cdot)^*)$$

We will regard two-digit numbers as "piles" of elements of $\mathbf{P}_0$:

$$ab \quad = \quad \mathsf{pile}(ab) \quad = \quad \{a\_, \ldots, 1\_, \ \_1, \ldots, \_b\}$$

A set of question marks $\mathcal{Q}$ induces an equivalence relation $\sim_\mathcal{Q}$ on H: we have $ab \sim_\mathcal{Q} cd$ when ab and cd become indistinguishable when we erase the information on the question marks; formally, $ab \sim_\mathcal{Q} cd$ means $\mathsf{pile}(ab) \backslash \mathcal{Q} = \mathsf{pile}(cd) \backslash \mathcal{Q}$. The operation $(\cdot)^*$ takes each $ab \in H$ to the topmost element in its region — in the example above we have $12^* = 23$ — and it also induces an equivalence relation: $ab \sim_{(\cdot)^*} cd$ when $ab^* = cd^*$. We say that a set of question marks $\mathcal{Q}$ "is associated to" a nucleus $(\cdot)^*$ when $(\sim_\mathcal{Q}) = (\sim_{(\cdot)^*})$.

Let's now expand the definitions of the two components of the bijection $(\mathcal{Y} \leftrightarrow (\cdot)^*)$. The "Heyting Implication" '$\to$' defined in [Lin14, def.C.2] is exactly the intuitionistic implication from [PH1, sec.16], that is calculated using the topological interior, and ${\downarrow}^\circ u$ is a shorthand for ${\downarrow}u \backslash \{u\}$. It turns out that $\mathcal{Y}$ is exactly the set of points of $\mathbf{P}_0$ without questions marks, i.e., $\mathcal{Q} = \mathbf{P}_0 \backslash \mathcal{Y}$ and $\mathcal{Y} = \mathbf{P}_0 \backslash \mathcal{Q}$:

$$\begin{array}{rrcl}
(\mathcal{Y} \mapsto (\cdot)^*): & \mathcal{S}^* & = & \mathcal{Y} \to \mathcal{S} \\
 & & = & \mathsf{int}(\{\, u \in \mathbf{P} \mid u \in \mathcal{Y} \to u \in \mathcal{S} \,\}) \\
 & & = & \mathsf{int}(\{\, u \in \mathbf{P} \mid u \notin \mathcal{Y} \vee u \in \mathcal{S} \,\}) \\
 & & = & \mathsf{int}(\{\, u \in \mathbf{P} \mid u \in (\mathbf{P}_0 \backslash \mathcal{Y}) \vee u \in \mathcal{S} \,\}) \\
 & & = & \mathsf{int}(\{\, u \in \mathbf{P} \mid u \in (\mathbf{P}_0 \backslash \mathcal{Y}) \cup \mathcal{S} \,\}) \\
 & & = & \mathsf{int}((\mathbf{P}_0 \backslash \mathcal{Y}) \cup \mathcal{S}) \\
 & & = & \mathsf{int}(\mathcal{Q} \cup \mathcal{S}) \\
((\cdot)^* \mapsto \mathcal{Y}): & \mathcal{Y} & = & \{\, u \in \mathbf{D} \mid ({\downarrow}u)^* \neq ({\downarrow}^\circ u)^* \,\}
\end{array}$$

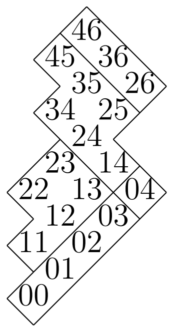
We have:

$$\begin{array}{rrcl}
(\mathcal{Q} \mapsto (\cdot)^*): & \mathcal{S}^* & = & \mathsf{int}(\mathcal{Q} \cup \mathcal{S}) \\
((\cdot)^* \mapsto \mathcal{Q}): & \mathcal{Q} & = & \{\, u \in \mathbf{D} \mid ({\downarrow}u)^* = ({\downarrow}^\circ u)^* \,\}
\end{array}$$

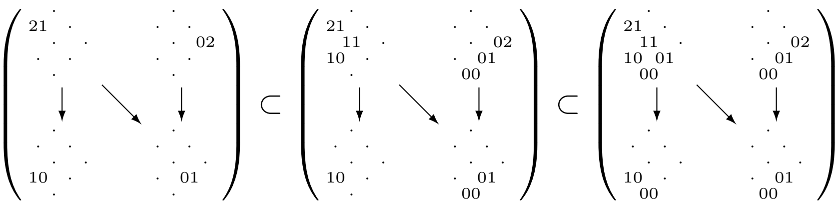
Let's see some examples to make this more concrete. Suppose that $\mathcal{Q}$ is the set of question marks of the example above. Then we can calculate 12^* by doing $\mathsf{int}(\mathcal{Q} \cup \mathsf{pile}(12))$; the '$\mathsf{int}$' discards the points 4_, 3_, and _5 from $\mathcal{Q} \cup \mathsf{pile}(12)$, and $12^* = \mathsf{int}(\mathcal{Q} \cup \mathsf{pile}(12)) = \mathsf{pile}(23) = 23$. Now let's start by the $(\cdot)^*$ of the example and try to obtain $\mathcal{Q}$. Let $u = 3\_$. Then ${\downarrow}u = 34$, ${\downarrow}^\circ u = 24$, and 34 and 24 are in the same region of the ZHA, so $3\_ \in \mathcal{Q}$. Let's try $u = \_4$. Then ${\downarrow}u = 04$, ${\downarrow}^\circ u = 03$, and 04 and 03 are in different regions the ZHA, so $\_4 \notin \mathcal{Q}$.

Using these methods we can easily convert a $\mathcal{Q}$ to a $(\cdot)^*$ and vice-versa; but we want to use $\mathcal{Y}$ instead of $\mathcal{Q}$ in our 4-uples, and we will draw our '$\mathcal{Y}$'s as subsets of $\mathbf{P}_0$, like this:

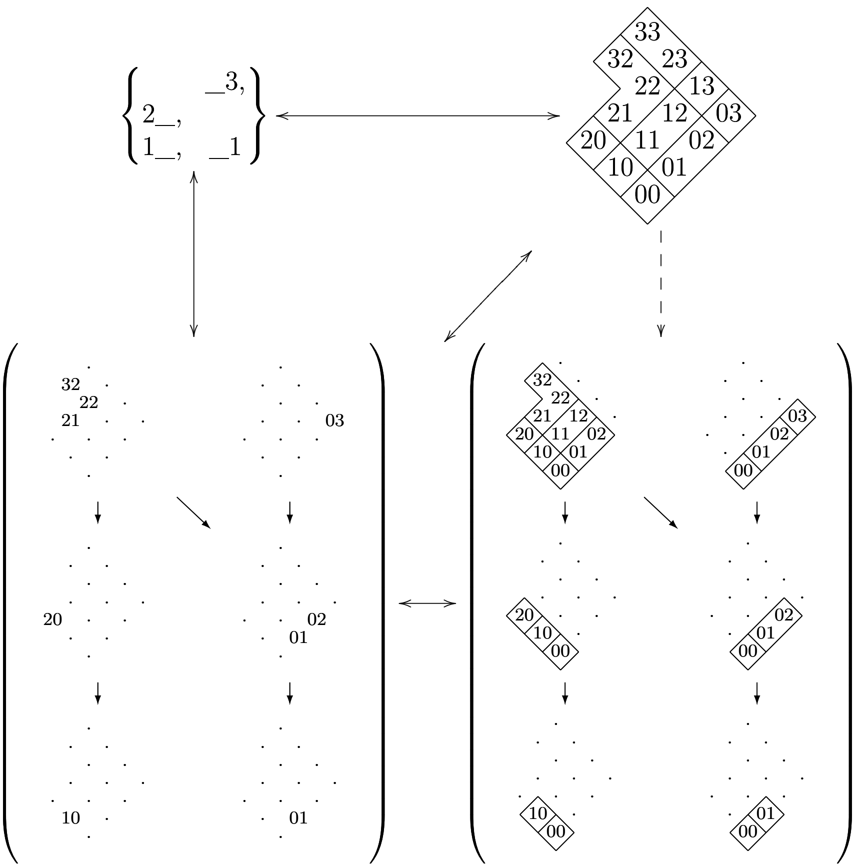
$$\left\{ \begin{array}{l} \quad \_6, \\ \\ \quad \_4, \\ \\ 1\_ \end{array} \right\} \longleftrightarrow$$



In our section 4.10 we saw that each component $j(u) : \Omega(u) \to \Omega(u)$ of a Lawvere-Tierney topology j is a modality on the down-set $\Omega(u) = \mathsf{D}(\downarrow u)$, and we have $j(u)(\mathcal{R}) = \downarrow u \wedge \mathcal{R}^* = \mathcal{R}^{\downarrow u}$; so each $j(u)$ is "$(\cdot)^*$ truncated to $\Omega(u)$". We will use this idea to draw our 'j's in a nice way — each component $j(u)$ will be drawn as a slashing on the corresponding $\Omega(u)$, and we will drawing '$\cdot$'s on all points of H that are "out of the domain", i.e., outside that $\Omega(u)$. Remember that at the end of our section 4.8 we drew $1_\top \subset J \subset \Omega$ as:



We will draw our 'j's as slashings on the components of our 'Ω', and of 'J's exactly as we drew above. Here is an example:



The diagram above is a particular case this one,

$$\begin{array}{ccc} \mathcal{Y} & \leftrightarrow & (\cdot)^* \\ \updownarrow & \swarrow\!\!\nearrow & \\ J & \leftrightarrow & j \end{array}$$

in the sense that we replaced each expression in its corners by its value on a certain particular case. If we draw many such particular cases — by choosing a 2-column graph $\mathbf{D}$, then a nucleus $(\cdot)^*$ as a slashing, at then calculating the corresponding '$\mathcal{Y}$'s, 'j', and 'J's by hand or by computer, we will see that those same patterns always occur: each element of $\mathcal{Y}$ corresponds to one of the diagonal cuts in the slashing of $(\cdot)^*$, j is always obtained by doing truncations of $(\cdot)^*$, and each $J(u)$ is made of the element in the topmost equivalence class of the corresponding $j(u)$.

(TODO: prove this last affirmation about 'J's and 'j's!)

4.13 Visualizing nuclei and topologies

TODO: Explain how to use the diagrams for particular cases of the last section to complement standard texts about Topos Theory; compare with [FavC, Section 5.5].

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