On some missing diagrams in the Elephant

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Abstract

Imagine two category theorists, Aleks and Bob, who both think very visually and who have exactly the same background. One day Aleks discovers a theorem, T_1, and sends an e-mail, E_1, to Bob, stating and proving T_1 in a purely algebraic way; then Bob is able to reconstruct by himself Aleks's diagrams for T_1 exactly as Aleks has thought them. We say that Bob has reconstructed the missing diagrams in Aleks's e-mail.

Now suppose that Carol has published a paper, P_2, with a theorem T_2. Aleks and Bob both read her paper independently, and both pretend that she thinks diagrammatically in the same way as them. They both "reconstruct the missing diagrams" in P_2 in the same way, even though Carol has never used those diagrams herself.

Here we will reconstruct, in the sense above, some of the "missing diagrams" in two factorizations of geometric morphisms in section A4 of Johnstone's "Sketches of an Elephant", and also some "missing examples". Our criteria for determining what is "missing" and how to fill out the holes are essentially the ones presented in the "Logic for Children" workshop at the UniLog 2018; they are derived from a certain definition of "children" that turned out to be especially fruitful.

One of the themes of the workshop [5] was a set of techniques for drawing diagrams for general cases and for particular cases in parallel, in a way that makes both diagrams have similar shapes, and that lets us transfer knowledge from the general to the particular and back. The term "for children" in the title of the workshop comes from some peoples' reactions to Category Theory: "I need a version for children of that!". We defined children in a certain way in order to get guidelines for how to construct a version "for children" of a categorical text; namely, "children": 1) prefer to start from particular cases and then generalize; 2) like diagrams and like finite objects that can be drawn explicitly; 3) become familiar with mathematical ideas by calculating and by checking several cases (i.e., by "playing"), rather than by proving theorems.

1 Categories with coordinates

Let's see a way to define finite categories whose objects have coordinates in $\mathbb{N}^2$ and whose arrows can be named by just their sources and targets. We call these categories ZCategories, and it's easier to start with an example. The left half of Figure 1 is a ZCategory $\mathbf{A}$ whose objects are $\mathbf{A}_0 = \{1,2,3,4,5\}$, with coordinates $c(1) = (0,2)$, $c(2) = (1,1)$, $c(3) = (2,1)$, $c(4) = (1,0)$, $c(5) = (2,0)$. The arrow $2 \to 4$ belongs to $\mathbf{A}$, but it is not shown. The right half of Figure 1 is a functor $F : \mathbf{A} \to \mathbf{Set}$ — a ZPresheaf.

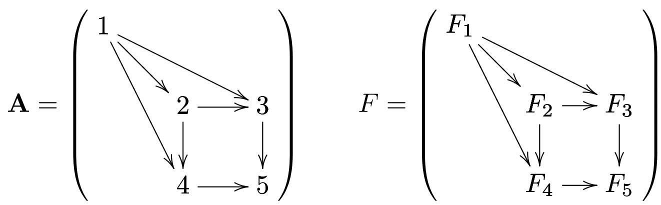
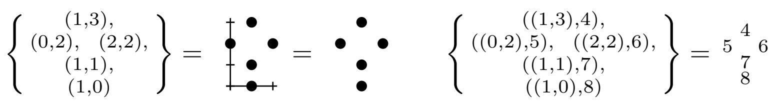


Figure 1: A ZCategory and a ZPresheaf.

A ZSet is a finite set $P \subset \mathbb{N}^2$ that touches both the x-axis and the y-axis. A ZDirectedGraph is a pair (P, A) where P is a ZSet and $A \subseteq P \times P$ is a set of arrows. We write (P, A^*) for the transitive-reflexive closure of (P, A).

The section 1 of [7] defines positional notations for ZSets and for functions with ZSets as their domains. They're like this:



The condition "...that touches both the x-axis and the y-axis" lets us draw ZSets as just bullets, omitting the axes.

A ZCategory $\mathbf{B}$ is a category plus a structure $((P, A), c)$, called its drawing instructions, obeying: 1) (P, A) is a ZDirectedGraph; 2) $c : \mathbf{B}_0 \to P$ is a bijection between the objects of $\mathbf{B}$ and the ZSet P; 3) for any objects $D, E \in \mathbf{B}$ the hom-set $\mathrm{Hom}_{\mathbf{B}}(D, E)$ is singleton when $(c(D), c(E)) \in A^*$, and is empty when $(c(D), c(E)) \notin A^*$. The conditions 1–3 imply that a ZCategory is a finite preorder category; the coordinates say where each object is to be drawn, and the set A says which arrows are to be drawn explicitly; the other arrows are said to be implicit.

A ZTopos is a functor category of the form $\mathbf{Set}^{\mathbf{B}}$, where $\mathbf{B}$ is a ZCategory. Objects of a ZTopos $\mathbf{Set}^{\mathbf{B}}$ inherit the drawing instructions from $\mathbf{B}$, as the F in the example above.

We call the objects of a ZTopos ZPresheaves. Note that a presheaf P on $\mathbf{B}$ is an element of $\mathbf{Set}^{\mathbf{B}^{\mathrm{op}}}$, which means that for each arrow $D \to E$ in $\mathbf{B}$ the presheaf P returns an arrow $P(D \to E) : PE \to PD$ in $\mathbf{Set}$; ZPresheaves don't have this reversal of direction.

2 Internal Diagrams

Internal diagrams are a tool that lets us lower the level of abstraction. They merge ideas from the standard notation for declaring functions with the way we used to draw functions in school, using arrows between the elements of blob-sets. Look at Figure 2, at the left. Compare its '$\mathbb{N} \to \mathbb{R}$' in the upper line (the external view), with the '$n \mapsto \sqrt{n}$' in the lower line (the internal view); the $n \mapsto \sqrt{n}$ shows a (generic) element and its image. The middle part of Figure 2 shows the external view at the bottom and an internal view at the top; note that all elements in the blobs for $\mathbb{N}$ and $\mathbb{R}$ are named, but only a few of the elements are shown (compare with [4], p.3); the arrows like $3 \mapsto \sqrt{3}$ and $4 \mapsto 2$, that show elements and their images, are substitution instances of the generic $n \mapsto \sqrt{n}$, maybe after some calculations (or "reductions" in λ-calculus terminology). The right part of Figure 2 shows an adjunction $L \dashv R$ between categories $\mathbf{A}$ and $\mathbf{B}$, drawn in our favourite "shape" (see [9], where all this is explained in detail): with the functor L going left and the functor R going right. We don't draw blobs to stress that $B, LA, LRB \in \mathbf{B}$ and $A, RB, RLA \in \mathbf{A}$, and we draw "generic" unit and counit maps.

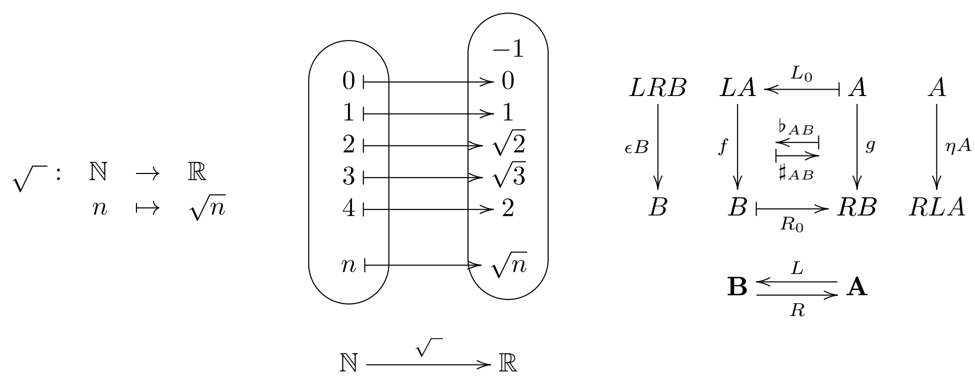


Figure 2: The standard notation for defining a function;
An internal view and the external view of the function $\sqrt{\ }$;
An internal view and the external view of an adjunction $L \dashv R$.

3 Geometric morphisms (and how to draw them)

A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ is an adjunction $(f^* \dashv f_*)$ between toposes $\mathcal{E}$ and $\mathcal{F}$ plus the assurance that f^* is exact; a ZGM is a geometric morphism generated by a functor $f : \mathbf{A} \to \mathbf{B}$ between ZCategories (a ZFunctor), in the following sense. A functor $f : \mathbf{A} \to \mathbf{B}$ induces a geometric morphism $f : \mathbf{Set}^{\mathbf{A}} \to \mathbf{Set}^{\mathbf{B}}$ between ZToposes; where f^* is defined "by composition". The right adjoint f_* can be calculated by the Kan extension formula, but in small examples it is better to calculate it directly by guess-and-test. Figure 3 shows how we draw a geometric morphism and a ZGM; Figure 4 shows why we are interested in ZGMs: if we choose ZCategories **A** and **B** we can replace several objects of our diagram for ZGMs by their internal views, and this gives us a way to "understand" the adjunction and the unit and counit maps.

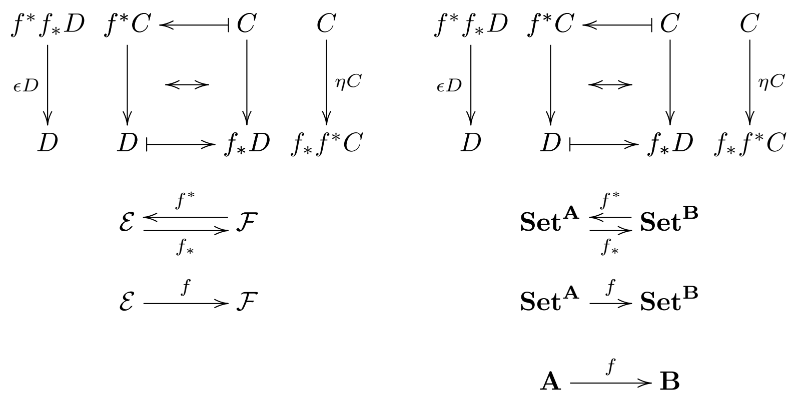


Figure 3: A geometric morphism and a ZGM.

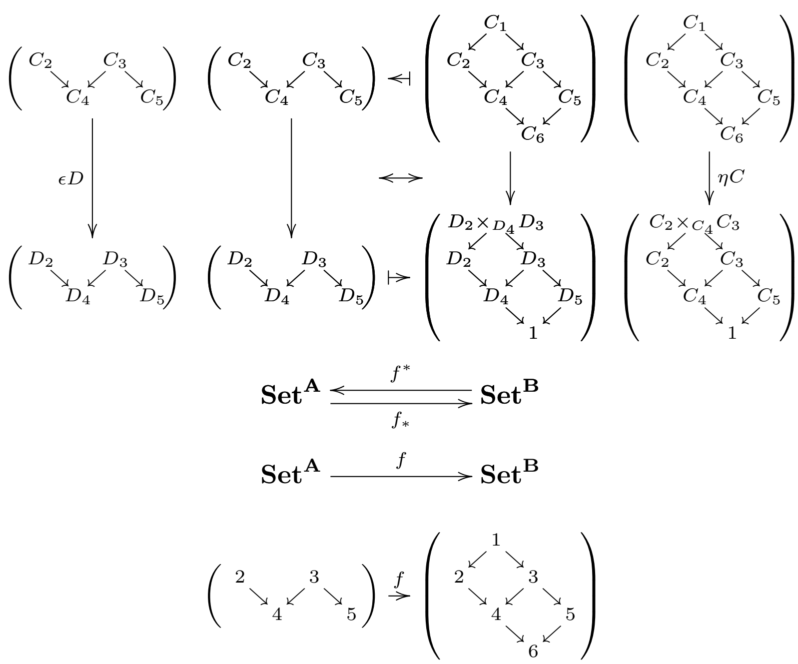


Figure 4: A ZGM in a particular case.

4 Planar Heyting Algebras and 2-Column Graphs

The preprints [7] and [8] explain how to use 2-column graphs ("*2CGs*") to develop visual intuition about intuitionistic logic (the first one) and sheaves (the second one).

The central construction in [7] can be stated as: every 2CG is associated to a Planar Heyting Algebra (a "ZHA", defined in section 4 of [7]) and vice-versa, and the central construction in [8] is: every 2CG with question marks is associated to a ZHA with a J-operator and vice-versa. This can be represented as:

$$\begin{array}{rcl} (P,A) & \leftrightsquigarrow & H \\ ((P,A),Q) & \leftrightsquigarrow & (H,J) \end{array}$$

where the '$\leftrightsquigarrow$' is pronounced "is associated to". Formally, $(P,A) \leftrightsquigarrow H$ when H is isomorphic to the order topology $\mathcal{O}_A(P)$, and $((P,A),Q) \leftrightsquigarrow (H,J)$ when $(P,A) \leftrightsquigarrow H$ and besides that the equivalence relation that Q induces on $\mathcal{O}_A(P)$ is the same as the equivalence that the J-operator J induces on H.

Here's why this is relevant to us. We can regard a 2CG (P,A) as a ZCategory. The logic of the ZTopos $\mathbf{Set}^{(P,A)}$ — i.e., its Sub(1) — is exactly $\mathcal{O}_A(P)$. Also, each sheaf $\mathbf{sh}_j(\mathbf{Set}^{(P,A)})$ on $\mathbf{Set}^{(P,A)}$ corresponds to a set of question marks $Q \subseteq P$ and vice-versa.

These constructions are easy to understand if we have a concrete example, so look at Figure 6.

The representation of the characteristic function of a subset $S \subseteq P$, χ_S, is the diagram that we obtain from (P,A) by replacing each point of P by 1 when it belongs to S and by 0 when not. We say that S obeys the condition of an arrow $(\alpha \to \beta) \in A$ when $\chi_S(\alpha) \leq \chi_S(\beta)$, and that S violates the condition of an $(\alpha \to \beta) \in A$ when $\chi_S(\alpha) = 1$ and $\chi_S(\beta) = 0$. A subset $S \subseteq P$ is A-open when it obeys the conditions of all arrows in A. $\mathcal{O}_A(P)$ is the set of all A-open subsets of P.

A specification for a 2CG is a 4-uple (l, r, R, L); it generates a 2CG (P,A) with $P := \mathsf{pile}(lr)$, and with sets of intercolumn arrows R (going right) and L (going left). The set A of arrows of (P,A) is $R \cup L$ plus all the intracolumn arrows that point one step down. The 2CG (P,A) in Figure 5 is generated by this specification:

$$(4, 6, \left\{ \begin{array}{l} 4\_\!\_{\to}\_\!\_5, \\ 3\_\!\_{\to}\_\!\_4, \\ 2\_\!\_{\to}\_\!\_2, \\ 1\_\!\_{\to}\_\!\_1, \end{array} \right\}, \{\, 2\_\!\_{\leftarrow}\_\!\_5 \,\})$$

We write $\mathsf{pile}(ab)$ for the set $\{a\_\!\_, \ldots, 1\_\!\_,\ \_\!\_1, \ldots, \_\!\_b\}$; $\chi_{\mathsf{pile}(a,b)}$ is a pile of a '1's in the left column and b '1's in the right column. A $\mathsf{pile}(ab)$ automatically obeys the conditions of the vertical arrows in A (as they all point one step down), and it is possible to translate the intercolumn arrows into conditions in the 'ab's (sec.15 of [7]). The specification above becomes this:

$$\mathcal{O}_A(P) = \{\, \mathsf{pile}(ab) \mid a \in \{0,\ldots,4\}, b \in \{0,\ldots,6\}, \begin{pmatrix} a{\geq}4 \to b{\geq}5 \wedge \\ a{\geq}3 \to b{\geq}4 \wedge \\ a{\geq}2 \to b{\geq}2 \wedge \\ a{\geq}1 \to b{\geq}1 \end{pmatrix} \wedge (\, a{\geq}2 \leftarrow b{\geq}5 \,) \,\}$$

$$\begin{array}{rcl} (P,A) & \leftrightsquigarrow & H \\ ((P,A),Q) & \leftrightsquigarrow & (H,J) \end{array}$$

Figure 5: The correspondence between 2CGs and ZHAs: the general case.

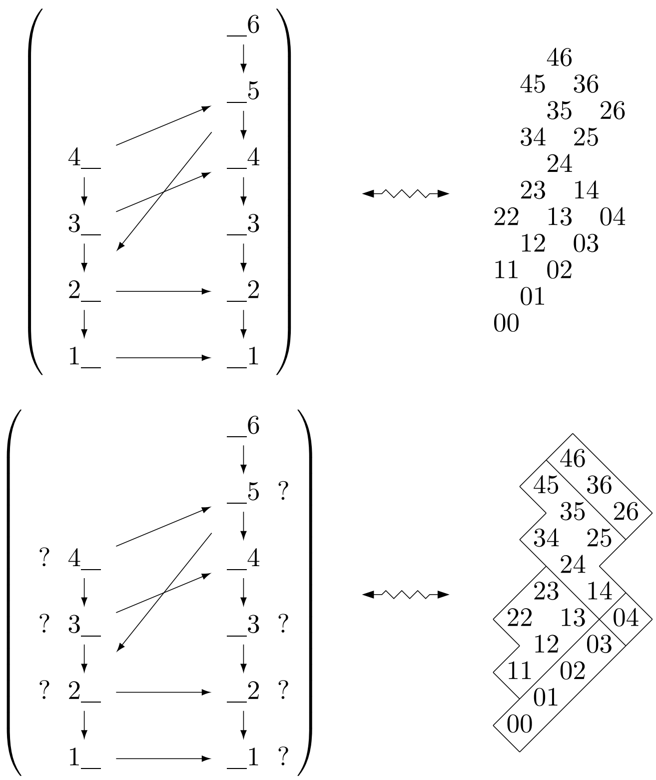


Figure 6: The correspondence between 2CGs and ZHAs: an example.

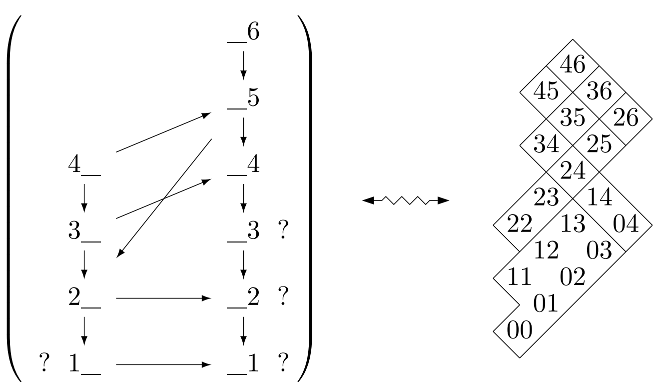


Figure 7: The closed operator (13∨) (right) and its associated question marks (left).

that lets us draw the ZHA $H \cong \mathcal{O}_A(P)$ very quickly — and lets us check the '$\leftsquigarrow\!\rightsquigarrow$' in the top of Figure 6.

In the lower half of Figure 6 the set of question marks is $Q = \{4\_\_, 3\_\_, 2\_\_, \_\_1, \_\_2, \_\_3, \_\_5, \}$. The subsets $S, S' \subseteq P$ are Q-equivalent when S and S' only differ in points of Q, i.e., $S \backslash Q = S' \backslash Q$. In the example, $\mathsf{pile}(22) \sim_Q \mathsf{pile}(23) \not\sim_Q \mathsf{pile}(24)$.

A J-operator on a Heyting Algebra H is an operation $J : H \to H$, abbreviated as '$\cdot^*$', obeying $P \leq P^* = P^{**}$ and $(P \wedge Q)^* = P^* \wedge Q^*$. The "slashing" in the ZHA of the lower half of Figure 6 induces a J-operator that takes each element of H to the top element of its equivalence class: $J(12) = 23$. We say that $A, B \in H$ are J-equivalent when $A^* = B^*$. In the example, $22 \sim_J 23 \not\sim_J 24$. It is not hard to see that in this case $(\sim_Q) = (\sim_P)$.

Figure 7 shows the "closed" J-operator $(13\vee)$, that takes each $A \in H$ to $13 \vee A$; it would be called $J_{13}(p) = 13 \vee p$ in the notation of [3], page 329, 2.18.(i).

5 Question marks and sheaves

A J-operator J in a $((P, A), Q) \leftsquigarrow\!\rightsquigarrow (H, J)$ can be interpreted in the topos $\mathbf{Set}^{(P,A)}$ as an operation $J : \mathrm{Sub}(1) \to \mathrm{Sub}(1)$ on its truth-values. This J can be extended to a local operator (see [EA4.4]) $j : \Omega \to \Omega$ in the topos $\mathbf{Set}^{(P,A)}$. The Elephant uses local operators instead of J-operators practically everywhere; our $(13\vee)$ corresponds to a "closed local operator" $c(\mathsf{pile}(13))$ in it — see [EA p.206].

Each set of question marks $Q \subseteq P$ induces an operation that erases the information on objects associated to the points of Q and an operation that "reconstructs" this information on those objects in a "natural" way. Figure 4 shows a case of this erasing and reconstruction, with $Q = \{1, 6\}$.

It is possible to show that in Figure 4 and in all similar cases the image of f_* is a sheaf and η is a sheafification functor. By "all similar cases" we mean: the points of $\mathbf{A}$ are a subset of the points of $\mathbf{B}$, and the partial order on $\mathbf{A}$ is the restriction to those points of the partial order on $\mathbf{B}$. We will need a notation for that. If $\mathbf{B}$ is drawn as the directed graph (P, A) and $S \subseteq P$, then we draw $\mathbf{B}$ as $(S, A|_S)$, where $A|_S \subseteq S \times S$ is a set of arrows on S that obeys $(A|_S)^* = (S \times S) \cap A^*$. We refer to that as a restriction of $\mathbf{A}$ to S, or as a restriction of $\mathbf{A}$ with question marks Q, where $Q = P \backslash S$.

6 Two factorizations of geometric morphisms

The Elephant presents in its sections A4.2 and A4.5 two factorizations of geometric morphisms that can be combined in a single diagram — see Figure 8. An arbitrary geometry morphism $g : \mathcal{A} \to \mathcal{D}$ can be factored in an essentially unique way as a surjection followed by an inclusion ([EA4.2.10]), and an inclusion $i : \mathcal{B} \to \mathcal{D}$ can be factored in an essentially unique way as a dense g.m.

followed by a closed g.m. ([EA4.5.20]). A canonical way to build these factorizations is by taking $\mathcal{B} := \mathcal{A}_{\mathbb{G}}$, where $\mathbb{G}$ is a certain comonad on $\mathcal{A}$ ([EA4.2.8]), and taking $\mathcal{C} := \mathbf{sh}_j(\mathcal{D})$, where j is a certain local operator on $\mathcal{D}$.

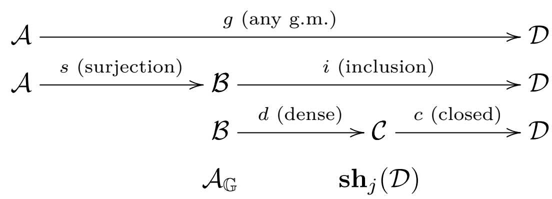


Figure 8: Two factorizations of geometric morphisms.

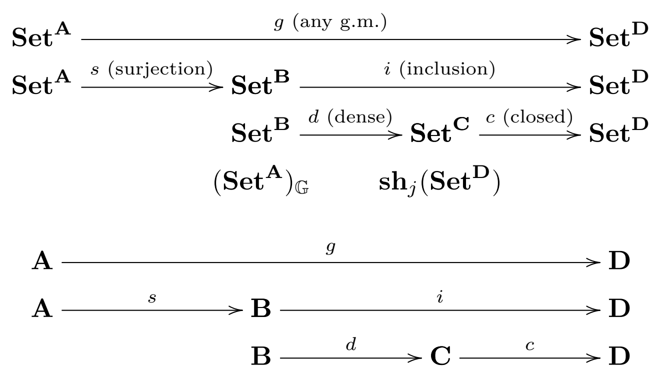


Figure 9: The same factorizations, but on ZGMs.

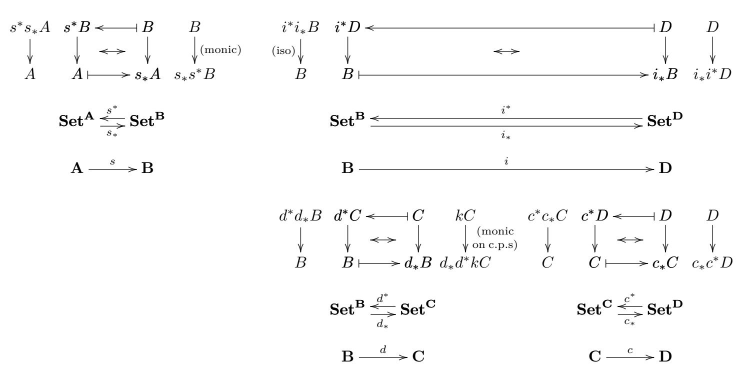


Figure 10: Conditions on the functors $\mathbf{A} \xrightarrow{s} \mathbf{B} \xrightarrow{i} \mathbf{D}$ and $\mathbf{B} \xrightarrow{d} \mathbf{C}$.

These factorizations are almost completely opaque to people who know just the basics of toposes. How can we produce a version "for children" of them in the sense of the introduction?

The trick is to start with geometric morphisms whose internal views can be drawn explicity — the ZGMs of section 2. Actually we start with the lower level of Figure 2, and with the belief that all factorizations can be performed within ZGMs.

7 Two factorizations of ZGMs

The Elephant defines surjections by a list of equivalent conditions, and the same for inclusions and dense and closed geometric morphisms. Some of these conditions — the ones drawn in Figure 10 — are very easy to test on ZGMs.

1. $\mathbf{Set}^{\mathbf{A}} \stackrel{s}{\to} \mathbf{Set}^{\mathbf{B}}$ is a surjection when for every object $B \in \mathbf{Set}^{\mathbf{B}}$ the unit map ηB is monic ([EA4.2.6 (iv)]);

2. $\mathbf{Set}^{\mathbf{B}} \stackrel{i}{\to} \mathbf{Set}^{\mathbf{D}}$ is an inclusion when for every object $B \in \mathbf{Set}^{\mathbf{B}}$ the counit map ϵB is an iso ([EA4.2.8]);

3. $\mathbf{Set}^{\mathbf{B}} \stackrel{d}{\to} \mathbf{Set}^{\mathbf{C}}$ is an dense when for every constant presheaf kC the unit map ϵkC is a monic [I can't find a reference for this now].

We also have two conditions for "dense" and "close" that are easy to state on "restrictions" in the sense of section 5 — but it's not trivial to derive them from the material in the Elephant. Let's state them anyway:

4. A restriction $\mathbf{Set}^{\mathbf{B}} \stackrel{d}{\to} \mathbf{Set}^{\mathbf{C}}$ is dense when all its question marks "have non-question marks ahead of them", i.e.: for every α in $\mathbf{C}$ such that $\alpha \in Q$ there is an arrow $\alpha \to \beta$ in $\mathbf{C}$ with $\beta \notin Q$;

5. A restriction $\mathbf{Set}^{\mathbf{C}} \stackrel{d}{\to} \mathbf{Set}^{\mathbf{D}}$ is closed when all its question marks "are at the end", i.e.: there are no arrows $\alpha \to \beta$ in $\mathbf{D}$ with $\alpha \in Q$ and $\beta \notin Q$.

We will say that a ZFunctor $\mathbf{A} \stackrel{s}{\to} \mathbf{B}$ induces a surjection when the ZGM $\mathbf{Set}^{\mathbf{A}} \stackrel{s}{\to} \mathbf{Set}^{\mathbf{B}}$ induced by it is a surjection; and the same for inclusion, dense, and closed.

How can we factor a ZFunctor $\mathbf{A} \stackrel{g}{\to} \mathbf{D}$ into ZFunctors $\mathbf{A} \stackrel{s}{\to} \mathbf{B} \stackrel{i}{\to} \mathbf{D}$ that induce a surjection and an inclusion, and how do we factor this $\mathbf{B} \stackrel{i}{\to} \mathbf{D}$ into $\mathbf{B} \stackrel{d}{\to} \mathbf{C} \stackrel{c}{\to} \mathbf{D}$? Here is a way to get a good part of a possible answer.

We can think that a ZFunctor $f : \mathbf{X} \to \mathbf{Y}$ does several actions. If we think that $\mathbf{X}$ is "before" and $\mathbf{Y}$ is "after", then f can, for example: create isolated objects, collapse isolated objects, collapse two ordered objects, create an arrow, create objects at the beginning of a connected set, create objects at the middle of a connected set, create objects at the end of a connected set... here are some examples that only do one of these actions each:

$$\begin{array}{rcl} (1{\to}1') & \to & (1) \\ (1 \quad 1') & \to & (1) \\ (1 \quad 2) & \to & (1{\to}2) \\ (2) & \to & (1{\to}2) \\ (1{\to}3) & \to & (1{\to}2{\to}3) \\ (1) & \to & (1 \quad 2) \\ (1) & \to & (1{\to}2) \end{array}$$

When we take $\mathbf{A} \stackrel{g}{\to} \mathbf{D}$ as each one of the seven ZFunctors and we try to factor that g as $\mathbf{A} \stackrel{s}{\to} \mathbf{B} \stackrel{d}{\to} \mathbf{C} \stackrel{c}{\to} \mathbf{D}$ we see that the first three functors factor as $(s = g, d = \mathrm{id}, c = \mathrm{id})$, the next two as $(s = \mathrm{id}, d = g, c = \mathrm{id})$, and the last two as $(s = \mathrm{id}, d = \mathrm{id}, c = g)$. The leads to a:

Conjecture 1 *Take a ZFunctor* $\mathbf{A} \stackrel{g}{\to} \mathbf{D}$*. Factor it into* $\mathbf{A} \stackrel{s}{\to} \mathbf{B} \stackrel{d}{\to} \mathbf{C} \stackrel{c}{\to} \mathbf{D}$ *in the following way: s collapses objects and creates arrows; c creates objects at the middle and at the beginnings of connected sets; d creates objects at the ends of connected sets. Then this factorization of ZFunctors induces a surjective-dense-closed factorization of ZGMs.*

8 Epilogue: timber

Sections 1, 10, 11, 12, and 16 of [6] discuss how to reconstruct theorems from incomplete versions that take very little mental space, or very little space on paper; we do something similar here.

One idea that was widely circulated after the fire in the Notre Dame cathedral was that it would be impossible to reconstruct its roof, because that would require an entire forest full of old oak trees, and that thing doesn't exist anymore... but I saw a thread on Tweeter[1] that contained these tweets: "The steeple and the beams supporting it are 160 years old, and oaks for new beams awaits at Versailles, the grown replacements for oaks cut to rebuild after the revolution." "Do you have a source on that protocol? Would love to hear more - feel free to DM." "It was a lecture I attended at Versailles on disaster recovery and long term planning. Once per century events have to be planned for in a city where structures are a thousand years old." "It was a lecture from a decade ago. I'm sorry that I don't have more details but there must be better sources than me out there. It's similar to the oaks at Oxford grown to replace the ones that rot out every 500 years, on schedule."

Sometimes we have to deal with a Topos Theory whose timber is made from Algebric Geometry oaks. Sometimes, for one reason or another, we want to be prepared to replace them by oaks from Computer Science, or with oaks coming from finite examples. The structure of the cathedral modulo these timbers can be described in several forms; some more algebraic, some more diagramatic, like the idea of "shape" in our Figures 3 and 10 and in [9].

[1] https://twitter.com/_theek_/status/1117895531563372544

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