Using Planar Heyting Algebras to develop visual intuition about Intuitionistic Propositional Logic

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How I started working on this

Many years ago I was studying Topos Theory (mostly by myself) and Logic (mostly with Luiz Carlos — Proof Theory and tableaux)... I needed to understand a bit of Intuitionistic (Propositional) Logic for toposes, but I didn't have any intuition about it...

For example:

Which one of these implications is false?
And how do you remember that?

$$\begin{array}{ll}
(\neg\neg P) \to P & \leftarrow \text{has a countermodel} \\
(\neg\neg P) \leftarrow P & \leftarrow \text{is a theorem} \\
\neg(P \land Q) \leftarrow (\neg P \lor \neg Q) & \leftarrow \text{is a theorem} \\
\neg(P \land Q) \to (\neg P \lor \neg Q) & \leftarrow \text{has a countermodel} \\
\neg(P \lor Q) \leftarrow (\neg P \land \neg Q) & \leftarrow \text{is a theorem} \\
\neg(P \lor Q) \to (\neg P \land \neg Q) & \leftarrow \text{is a theorem}
\end{array}$$

One way: remember which.
Or better: remember the proofs of the ones that can be proved.
Another way: remember the countermodel.
Or better: have a way to test countermodels very quickly.
(Possible countermodels!)

Logic for Children

Fast forward many years. I organized with a friend a workshop called "Logic for Children" in the UniLog 2018 in Vichy... this is from one of the announcements:

The "children" in "logic for children" means "people without mathematical maturity", which in its turn means people who:

- have trouble with very abstract definitions,
- prefer to start from particular cases (and then generalize),
- handle diagrams better than algebraic notations,
- like to use diagrams and analogies (...)

Logic for Children (2)

If we say that categorical definitions are “for adults” — because they may be very abstract — and that particular cases, diagrams, and analogies are “for children”, then our intent with this workshop becomes easy to state. “Children” are willing to use ”tools for children” to do mathematics, even if they will have to translate everything to a language “for adults” to make their results dependable and publishable, and even if the bridge between their tools “for children” and “for adults” is somewhat defective, i.e., if the translation only works on simple cases...

Logic for Children (3)

We are interested in that *bridge* between maths “for adults” and “for children” in several areas. Maths ”for children” are hard to publish, even informally as notes (see **this thread** in the Categories mailing list), so often techniques are rediscovered over and over, but kept restricted to the “oral culture” of the area.

Our main intents with this workshop are:

- to discuss (over coffe breaks!) the techniques of the “bridge” that we currently use in seemingly ad-hoc ways,
- to systematize and “mechanize” these techniques to make them quicker to apply,

- to find ways to publish those techniques - in journals or elsewhere,
- to connect people in several areas working in related ideas, and to create repositories of online resources.

The Gödel translation
From the Wikipedia:

Let A be a propositional intuitionistic formula. A modal formula $T(A)$ is defined by induction on the complexity of A:

$$
\begin{aligned}
&T(P) = \Box P \quad \text{for any propositional variable } P,\\
&T(\bot) = \bot,\\
&T(A \wedge B) = T(A) \wedge T(B),\\
&T(A \vee B) = T(A) \vee T(B),\\
&T(A \to B) = \Box(T(A) \to T(B)),
\end{aligned}
$$

As negation is in intuitionistic logic defined by $A \to \bot$, we also have $T(\neg A) = \Box\neg T(A)$.

The Gödel translation (2)
From the Wikipedia (cont.):

T is called the *Gödel translation* or *Gödel–McKinsey–Tarski translation*. The translation is sometimes presented in slightly different ways: for example, one may insert $\Box$ before every subformula. All such variants are provably equivalent in S4.

An example:

$$T(\underbrace{(\underbrace{\neg\underbrace{\neg\underbrace{P}_{\Box P}}_{\Box\neg\Box P}}_{\Box\neg\Box\neg\Box P}) \to \underbrace{P}_{\Box P}}_{\Box((\Box\neg\Box\neg\Box P)\to(\Box P)}) = \Box((\Box\neg\Box\neg\Box P) \to (\Box P)$$

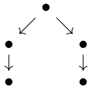
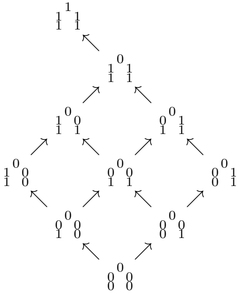
The Gödel translation (3)

Let A be a formula in IPL.
Then A is a theorem in IPL iff $T(A)$ is a theorem in S4.
A is a non-theorem of IPL
$\leftrightarrow T(A)$ has a countermodel in S4
$\leftrightarrow$ there is a Kripke model (W, R, v) that falsifies $T(A)$

Suppose that A has propositional variables P and Q.
Suppose that we've fixed (W, R).
I realized that I did not need to test all valuations —
I could test only the valuations in which $P = \Box P$ and
$Q = \Box P$, i.e., the 'P's and 'Q's that are "stable" by $\Box$...

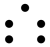
Stable truth-values

And I realized that I could draw the set of the stable truth-values, and it would be a finite topology — an order topology... my topologies would usually be planar, they were Heyting Algebras, and $\Box P$ would be the interior of the set P...

$(H, \mathsf{BPM}(H)) =$  $(\mathcal{O}(H), \subset_1) =$ 

Kripke frames
A Kripke frame is a pair (W, R), with $R \subseteq W \times W$.
(W for "worlds", R for "accessibility relation").
I will write it as (P, A), for "points" and "arrows".
I pair (P, A) with $A \subseteq P \times P$ is (also) a directed graph.
Convention: (P, A^*) is the transitive-reflexive closure of (P, A).
A Kripke frame *for S4* is a pair (P, A^*).

I established some conventions to let me draw Kripke frames for S4 very compactly — "ahead" would be "down", and the arrows would be implicit. For example, this is my notation for a certain Kripke frame for S4:



Lambda-Calculus, Logics and Translations
These ideas became the basis for a hands-on seminar course called "Lambda-Calculus, Logics and Translations" in which the students learned many ideas from Logic, including IPL, Kripke models, S4, Natural Deduction, in a *very atypical* order...
For example:

Let Ω be the set of points of a ZHA and $\leq$ the default partial order on it. The *default meanings for* $\top, \bot, \wedge, \vee, \to, \leftrightarrow, \neg$ are these ones:

$$
\begin{aligned}
\langle a,b\rangle \leq \langle c,d\rangle &\;:=\; a \leq c \wedge b \leq d \\
\langle a,b\rangle \geq \langle c,d\rangle &\;:=\; a \geq c \wedge b \geq d
\end{aligned}
$$

$$
\begin{aligned}
\langle a,b\rangle \ \mathsf{above}\ \langle c,d\rangle &\;:=\; a \geq c \wedge b \geq d \\
\langle a,b\rangle \ \mathsf{below}\ \langle c,d\rangle &\;:=\; a \leq c \wedge b \leq d \\
\langle a,b\rangle \ \mathsf{leftof}\ \langle c,d\rangle &\;:=\; a \geq c \wedge b \leq d \\
\langle a,b\rangle \ \mathsf{rightof}\ \langle c,d\rangle &\;:=\; a \leq c \wedge b \geq d
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{valid}(\langle a,b\rangle) &\;:=\; \langle a,b\rangle \in \Omega \\
\mathsf{ne}(\langle a,b\rangle) &\;:=\; \mathsf{if\ valid}\ (\langle a,b+1\rangle)\ \mathsf{then\ ne}(\langle a,b+1\rangle)\ \mathsf{else}\ \langle a,b\rangle\ \mathsf{end} \\
r\ \mathsf{nw}\ (\langle a,b\rangle) &\;:=\; \mathsf{if\ valid}\ (\langle a+1,b\rangle)\ \mathsf{then\ nw}(\langle a+1,b\rangle)\ \mathsf{else}\ \langle a,b\rangle\ \mathsf{end}
\end{aligned}
$$

$$
\begin{aligned}
\langle a,b\rangle \wedge \langle c,d\rangle &\;:=\; \langle \mathsf{min}(a,c), \mathsf{min}(b,d)\rangle \\
\langle a,b\rangle \vee \langle c,d\rangle &\;:=\; \langle \mathsf{max}(a,c), \mathsf{max}(b,d)\rangle
\end{aligned}
$$

$$
\begin{array}{rcl}
\langle a,b\rangle \to \langle c,d\rangle & := & \begin{array}{llll}
\textsf{if} & \langle a,b\rangle \ \textsf{below}\ \langle c,d\rangle & \textsf{then} & \top \\
\textsf{elseif} & \langle a,b\rangle \ \textsf{leftof}\ \langle c,d\rangle & \textsf{then} & \textsf{ne}(\langle c,d\rangle) \\
\textsf{elseif} & \langle a,b\rangle \ \textsf{rightof}\ \langle c,d\rangle & \textsf{then} & \textsf{nw}(\langle c,d\rangle) \\
\textsf{elseif} & \langle a,b\rangle \ \textsf{above}\ \langle c,d\rangle & \textsf{then} & \langle c,d\rangle \\
\textsf{end} & & &
\end{array} \\
\top & := & \textsf{sup}(\Omega) \\
\bot & := & \langle 0,0\rangle \\
\neg\langle a,b\rangle & := & \langle a,b\rangle \to \bot \\
\langle a,b\rangle \leftrightarrow \langle c,d\rangle & := & (\langle a,b\rangle \to \langle c,d\rangle) \wedge (\langle c,d\rangle \to \langle a,b\rangle)
\end{array}
$$

Calculate, and represent in positional notation...

Read sections 2–8 of “Planar Heyting Algebras for Children”.

$$\text{Let } B = \begin{matrix} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{matrix} \text{ and } C = \begin{matrix} & & & & 44 & & & & \\ & & & 43 & & 34 & & & \\ & & 42 & & 33 & & 24 & & \\ & 41 & & 32 & & 23 & & 14 & \\ 40 & & 31 & & 22 & & 13 & & 04 \\ & 30 & & 21 & & 12 & & 03 & \\ & & 20 & & 11 & & 02 & & \\ & & & 10 & & 01 & & & \\ & & & & 00 & & & & \end{matrix} .$$

Exercises

Calculate, and represent in positional notation when possible:

Calculate, and represent... (2)

a) $\lambda lr{:}B.l$
b) $\lambda lr{:}B.r$
c) $\lambda lr{:}B.(l \leq 1)$
d) $\lambda lr{:}B.(r \geq 1)$
e) $\lambda lr{:}B.lr \leq 11$
f) $\lambda lr{:}B.lr \wedge 12$
g) $\lambda lr{:}B.\,\mathsf{valid}\,(\langle l+1, r\rangle)$
h) $\lambda lr{:}B.lr\ \mathsf{leftof}\ 11$
i) $\lambda lr{:}B.lr\ \mathsf{leftof}\ 12$
j) $\lambda lr{:}B.lr\ \mathsf{above}\ 11$
k) $\lambda lr{:}B.\,\mathsf{ne}\,(lr)$
l) $\lambda lr{:}B.\,\mathsf{nw}\,(lr)$

m) $20 \to 11$
n) $02 \to 11$
o) $22 \to 11$
p) $00 \to 11$

q) $\lambda lr{:}B.\neg lr$
r) $\lambda lr{:}B.\neg\neg lr$
s) $\lambda lr{:}B.(lr = \neg\neg lr)$

t) $\lambda P{:}C.(P \to 22)$
u) $\lambda Q{:}C.(22 \to Q)$

Calculate, and represent... (3)

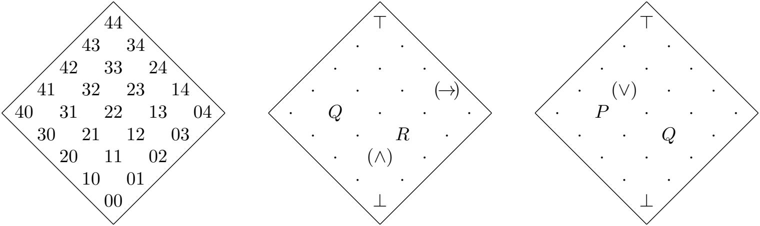
v) find X such that

$$(\lambda P{:}C.P \le X) = (\lambda P{:}C.(P \le 22) \wedge (P \le 13))).$$

w) find X such that

$$(\lambda R{:}C.X \le R) = (\lambda R{:}C.(22 \le R) \wedge (13 \le R))).$$

Let Ω be the ZDAG on the left below:



we will see that
a) if $Q = 31$ and $R = 12$ then $Q \wedge_H R = 11$,
b) if $P = 31$ and $Q = 12$ then $P \vee_H Q = 32$,
c) if $Q = 31$ and $R = 12$ then $Q \to_H R = 14$.

Let's see each case separately — but, before we start, note that in 6, 7, 8, 6', 7', 8' we work part with truth values in Ω and part with standard truth values. For example, in 6, with $P = 20$, we have:

$$\underbrace{\underbrace{(\underbrace{P}_{20} \leq_H \underbrace{(\underbrace{Q}_{31} \wedge_H \underbrace{R}_{12})}_{11})}_{0} \leftrightarrow \underbrace{(\underbrace{(\underbrace{P}_{20} \leq_H \underbrace{Q}_{31})}_{1} \wedge \underbrace{(\underbrace{P}_{20} \leq_H \underbrace{R}_{12})}_{0})}_{0}}_{1}$$