Five applications of the "Logic for Children" project to Category Theory

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http://angg.twu.net/math-b.html#ebl-2019

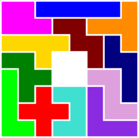
Category Theory...

...seems to be a very elegant area with “the right abstractions” and lots of diagrams, but the diagrams are usually omitted from the texts as if they were “obvious exercises”, and the motivating examples are mentioned briefly, if at all — so the comparisons between these “abstractions” and the examples are also left as exercises.

Topos Theory is a very important sub-area of CT.
When I tried to read Johnstone’s “Topos Theory” (1977)
I understood very little, even though I tried very hard.
“I need a version for children of this!!!”
(I.e., with the missing diagrams and the examples.)

My current favorite definition of "children":
They prefer to start from particular cases
and then generalize —
They like diagrams and finite objects
drawn very explicitly —
They become familiar with mathematical ideas
by calculating / checking several cases
(rather than by proving theorems)

Example: pentominos.
Let "children" play
with pentominos for a while
before showing to them
theorems and game trees!



Five applications

1. A way to develop visual intuition about Intuitionistic Propositional Logic. Models for IPL are Heyting Algebras; topologies are HAs. Look for finite topologies! Use order topologies. Bonus: use planar topologies ("ZHAs").

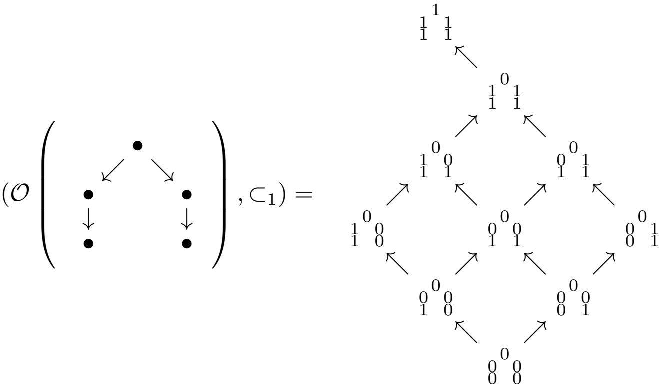
2. A way to build a topos with a given logic (when that logic is a ZHA). Solution: $\mathbf{Set}^{(P,A)}$.

3. Sheaves are related to J-operators (← old terminology) on HAs. So: a way to visualize J-operators on ZHAs ("slashings").

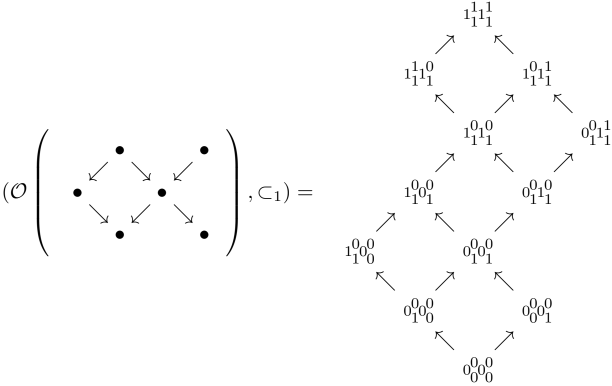
1. The sheaf associated to a J-operator. Solution: question marks; erasing followed by reconstruction yields the sheafification functor.

2. A version “for children” for parts of The Elephant — in which the “missing diagrams” are no longer missing and we can remember theorems and constructions by shape and movement. Also: motivating examples “for children”, in which everything is finite and can be drawn explicitly. “Children” develop familiarity with mathematical structures by calculating rather than by proving theorems.

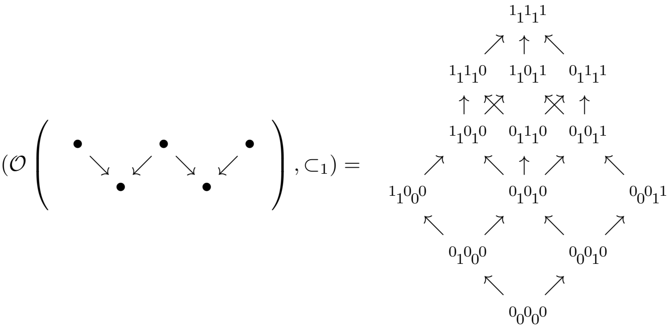
Finite topologies



Finite topologies (2)



Finite topologies (3)

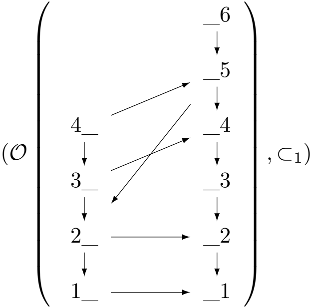


Non-planar! Why?
Answer: because the W has three independent points!

Finite topologies (4): 2CGs and ZHAs

A 2-column graph never has three independent points.

Trick: $\mathsf{pile}(25) = \{2\_\_, 1\_\_,\ \ \_\_1, \_\_2, \_\_3, \_\_4, \_\_5\}$.



=

```
   46
 45  36
   35  26
 34  25
   24
 23  14
22  13  04
 12  03
11  02
 01
00
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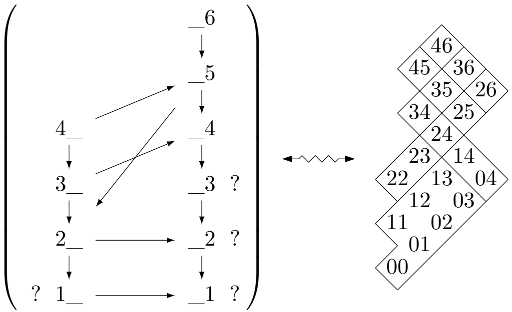
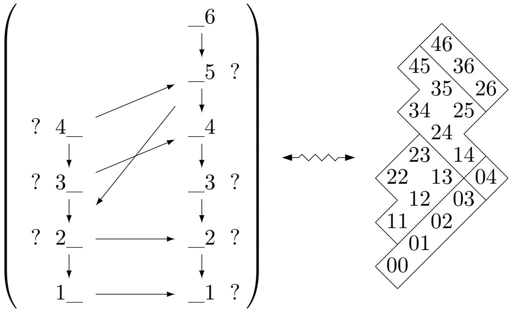
A famous J-operator: (13∨)



A strange J-operator



Logic in a ZHA (visually!!!)

Notation: a 2-column graph is (P, A) — (points, arrows) —
its order topology is $\mathcal{O}_A(P)$,
and a Planar Heyting Algebra (a ZHA) is $H \subset \mathbb{Z}^2$.
The correspondence is written as $(P, A) \leftrightsquigarrow H$
and formally it means $\mathcal{O}_A(P) \cong H$.
There are two ways to define $\top, \bot \in H$ and $\land, \lor, \to \in \top$...

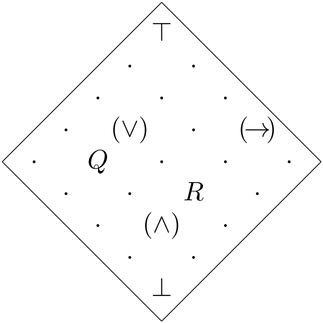
1) Via topology, in $\mathcal{O}_A(P)$:
$\top := P$,
$\bot := \emptyset$,
$Q \land R := Q \cap R$,
$Q \lor R := Q \cup R$,
$R \to S := \mathsf{Int}((T \backslash R) \cup S)$

Logic in a ZHA (visually!!!) (2)

2) Via order. For example:

$$\begin{array}{rcl}
(Q \to (R \to S)) & \leftrightarrow & ((Q \land R) \to S) \\
(Q \leq (R \to S)) & \leftrightarrow & ((Q \land R) \leq S) \\
\{\, Q \in H \mid Q \leq (R \to S) \,\} & = & \{\, Q \in H \mid (Q \land R) \leq S \,\} \\
(R \to S) & = & \sup \{\, Q \in H \mid (Q \land R) \leq S \,\} \\
 & = & \bigcup \{\, Q \in H \mid (Q \land R) \leq S \,\}
\end{array}$$

Logic in a ZHA (visually!!!) (3)

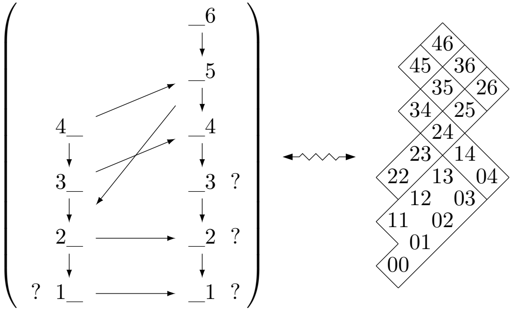


J-operators

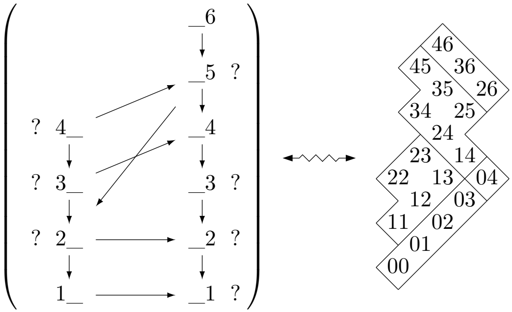
A J-operator on a Heyting Algebra H is an operation $J : H \to H$, abbreviated as '$\cdot^*$', obeying $P \leq P^* = P^{**}$ and $(P \wedge Q)^* = P^* \wedge Q^*$.
Some famous J-operators: $(\neg\neg)$, $(A\vee)$, $(A \to)$ (for $A \in H$).

A J-operator induces an equivalence relation:
$P \sim_J Q$ iff $P^* = Q^*$.
For many years I didn't have ANY visual intuition on what J-operators were, or could be.
When we play with J-operators on ZHAs we discover that:
Each equivalence class $[P]^J$ has a top element, a bottom element, and all element in between; and P^* is always the top element of $[P]^J$...
So we only need to draw the equivalence classes!

A famous J-operator: $(13\vee)$

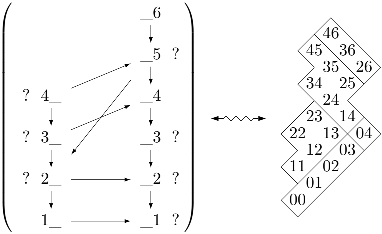


A strange J-operator



Question marks

Every set of question marks $Q \subseteq P$ in (P, A) induces an equivalence relation on $H \cong \mathcal{O}_A(P)$. Two subsets $S, S' \subseteq P$ are Q-equivalent when S and S' only differ in points of Q, i.e.: $S \backslash Q = S' \backslash Q$. Here $Q = \{4\_, 3\_, 2\_, \quad \_1, \_2, \_3, \_5, \}$, and:
$\mathsf{pile}(22) \sim_Q \mathsf{pile}(23) \not\sim_Q \mathsf{pile}(24)$,
$12^* = 23, \quad 22 \sim_J \mathsf{pile23} \not\sim_J 24.$



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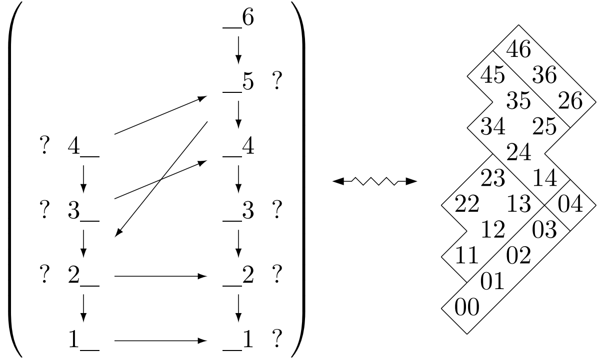
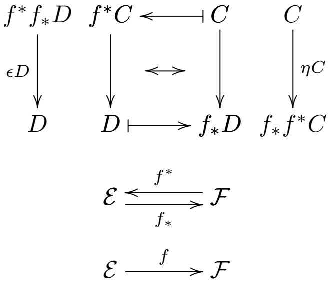


Toposes, geometric morphisms, internal diagrams

Internal diagrams are a tool to lower the lever of abstraction.

This is a geometric morphism between toposes.



Toposes, geometric morphisms, internal diagrams (2)
Let $\mathbf{A}$ and $\mathbf{B}$ be 2CGs regarded as categories.
Then a functor $f : \mathbf{A} \to \mathbf{B}$ induces a geometric morphism...

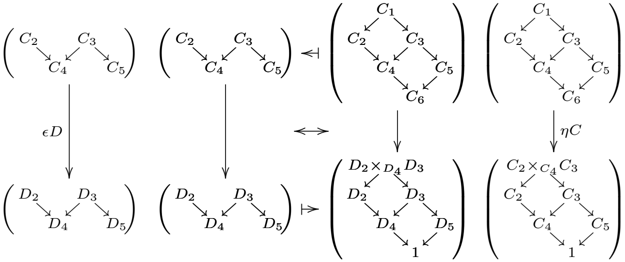
$$\begin{array}{ccccccc} f^*f_*D & & f^*C & \longleftarrow\!\shortmid & C & & C \\ \downarrow{\scriptstyle \epsilon D} & & \downarrow & \leftrightarrow & \downarrow & & \downarrow{\scriptstyle \eta C} \\ D & & D & \longmapsto & f_*D & & f_*f^*C \end{array}$$

$$\mathbf{Set}^{\mathbf{A}} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathbf{Set}^{\mathbf{B}}$$

$$\mathbf{Set}^{\mathbf{A}} \xrightarrow{f} \mathbf{Set}^{\mathbf{B}}$$

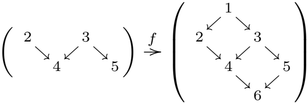
$$\mathbf{A} \xrightarrow{\quad f \quad} \mathbf{B}$$

And if we draw the internal views of $\mathbf{A}$, $\mathbf{B}$, C, D...



$$\mathbf{Set}^{\mathbf{A}} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathbf{Set}^{\mathbf{B}}$$

$$\mathbf{Set}^{\mathbf{A}} \xrightarrow{\;f\;} \mathbf{Set}^{\mathbf{B}}$$



A factorization

The Elephant presents in its sections A4.2 and A4.5 two factorizations of geometric morphisms that can be combined in a single diagram (next slide). An arbitrary geometry morphism $g : \mathcal{A} \to \mathcal{D}$ can be factored in an essentially unique way as a surjection followed by an inclusion ([EA4.2.10]), and an inclusion $i : \mathcal{B} \to \mathcal{D}$ can be factored in an essentially unique way as a dense g.m. followed by a closed g.m. ([EA4.5.20]). A canonical way to build these factorizations is by taking $\mathcal{B} := \mathcal{A}_{\mathbb{G}}$, where $\mathbb{G}$ is a certain comonad on $\mathcal{A}$ ([EA4.2.8]), and taking $\mathcal{C} := \mathbf{sh}_j(\mathcal{D})$, where j is a certain local operator on $\mathcal{D}$.

A factorization (2)

$$\mathcal{A} \xrightarrow{\quad g \text{ (any g.m.)} \quad} \mathcal{D}$$

$$\mathcal{A} \xrightarrow{s \text{ (surjection)}} \mathcal{B} \xrightarrow{\quad i \text{ (inclusion)} \quad} \mathcal{D}$$

$$\mathcal{B} \xrightarrow{d \text{ (dense)}} \mathcal{C} \xrightarrow{c \text{ (closed)}} \mathcal{D}$$

$$\mathcal{A}_{\mathbb{G}} \qquad\qquad \mathbf{sh}_j(\mathcal{D})$$

A factorization (3)

$$\mathbf{Set}^{\mathbf{A}} \xrightarrow{\quad g \text{ (any g.m.)} \quad} \mathbf{Set}^{\mathbf{D}}$$

$$\mathbf{Set}^{\mathbf{A}} \xrightarrow{s \text{ (surjection)}} \mathbf{Set}^{\mathbf{B}} \xrightarrow{\quad i \text{ (inclusion)} \quad} \mathbf{Set}^{\mathbf{D}}$$

$$\mathbf{Set}^{\mathbf{B}} \xrightarrow{d \text{ (dense)}} \mathbf{Set}^{\mathbf{C}} \xrightarrow{c \text{ (closed)}} \mathbf{Set}^{\mathbf{D}}$$

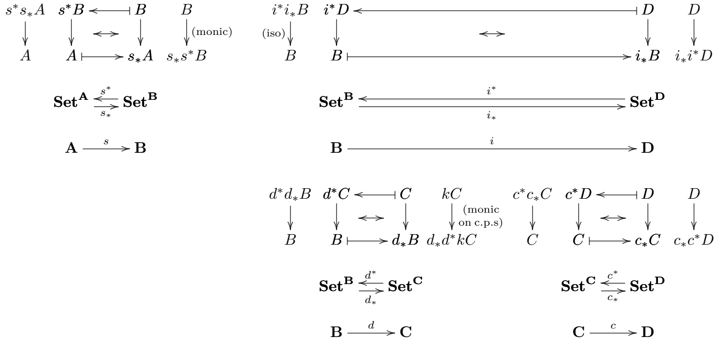
$$(\mathbf{Set}^{\mathbf{A}})_{\mathbb{G}} \qquad \mathbf{sh}_j(\mathbf{Set}^{\mathbf{D}})$$

$$\mathbf{A} \xrightarrow{\qquad g \qquad} \mathbf{D}$$

$$\mathbf{A} \xrightarrow{\quad s \quad} \mathbf{B} \xrightarrow{\quad i \quad} \mathbf{D}$$

$$\mathbf{B} \xrightarrow{\quad d \quad} \mathbf{C} \xrightarrow{\quad c \quad} \mathbf{D}$$

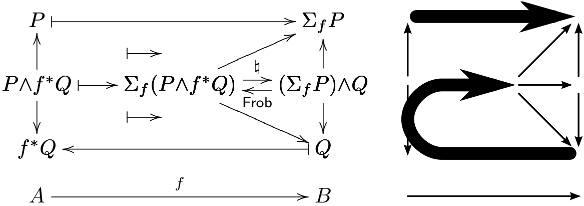
A factorization (4)



These factorizations are almost completely opaque to people who know just the basics of toposes... how can we?...

Shape and movement

This is how I remember the Frobenius Property:



(From “Internal Diagrams and Archetypal Reasoning in Category Theory” (Ochs 2013))