Title of this talk:

Intuitionistic Propositional Logic For Children and Meta-Children, or: How Archetypal Are Finite Planar Heyting Algebras?

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`http://angg.twu.net/math-b.html#ebl-2017`
`http://angg.twu.net/LATEX/2017planar-has.pdf` (paper)

Some quotes:

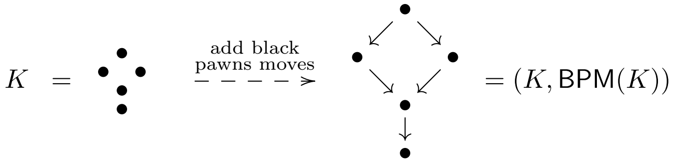
One great way to make the expression "for children" precise in mathematical titles is to *define* "children" as "people without mathematical maturity", in the sense that they are not able to understand structures that are too abstract straight away — *they need particular cases first.*

"Meta-children" are people who want to study the relation between mathematics "for children" and "for adults" and produce (meta)mathematics for adults from that.

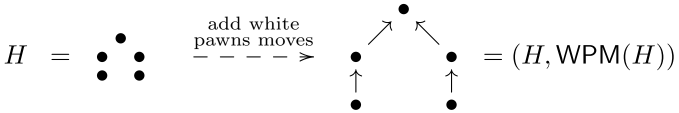
ZHAs [i.e., finite, planar HAs] help us visualize fragments of the Lindenbaum Algebra of HAs.

Bullet diagrams as directed graphs

Sometimes we want arrows going up,
sometimes we want arrows going down.

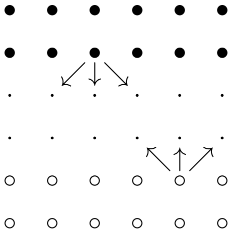


$$= \left(\left\{ \begin{array}{ccc} & (1,3), & \\ (0,2), & & (2,2), \\ & (1,1), & \\ & (0,0) & \end{array} \right\}, \left\{ \begin{array}{c} ((1,3),(0,2)),((1,3),(2,2)), \\ ((0,2),(1,1)),((2,2),(1,1)), \\ ((1,1),(0,0)) \end{array} \right\} \right)$$



Black and white pawns moves

Mnemonic:
a game with black pawns and white pawns
black pawns are solid/heavy/sink/go down
white pawns are hollow/light/float/go up



$((a, b), (c, d)) \in \mathsf{BPM}(S)$ means $(a, b), (c, d) \in S$ and $(a, b) \to (c, d)$ is a black pawn move

$((a, b), (c, d)) \in \mathsf{WPM}(S)$ means $(a, b), (c, d) \in S$ and $(a, b) \to (c, d)$ is a white pawn move

LR-coordinates

$\mathbb{N}^2 \subset \mathbb{Z}^2$ is a quarter-plane.
$\mathbb{LR} \subset \mathbb{Z}^2$ is a "quarter-plane turned 45° to the left".
$\mathbb{LR} = \{\, \langle l, r \rangle \mid l, r \in \mathbb{N} \,\}$

LR-coordinates:
$lr = \langle l, r \rangle = (0,0) + l\,\underbrace{\overrightarrow{(-1,1)}}_{\nwarrow} + r\,\underbrace{\overrightarrow{(1,1)}}_{\nearrow}$

The lower part of $\mathbb{LR}$,
in LR-coordinates and xy-coordinates:

$$\begin{array}{ccccccccc}
40 & & 31 & & 22 & & 13 & & 04 \\
 & 30 & & 21 & & 12 & & 03 & \\
 & & 20 & & 11 & & 02 & & \\
 & & & 10 & & 01 & & & \\
 & & & & 00 & & & &
\end{array}
\quad = \quad
\begin{array}{ccccccccc}
(-4,4) & & (-2,4) & & (0,4) & & (2,4) & & (4,4) \\
 & (-3,3) & & (-1,3) & & (1,3) & & (3,3) & \\
 & & (-2,2) & & (0,2) & & (2,2) & & \\
 & & & (-1,1) & & (1,1) & & & \\
 & & & & (0,0) & & & &
\end{array}$$

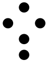
Bullet diagrams as subsets of $\mathbb{Z}^2$

Two cases:
For a finite, non-empty $S \in \mathbb{Z}^2$,

S is a ZSet iff $S \subset \mathbb{N}^2$ and
S touches the xy-axes

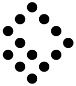
S is an LRSet iff $S \subset \mathbb{LR}$ and S
touches the lr-axes
at the point (0,0)

A ZSet:



$$= \begin{array}{ccc} & (1,3) & \\ (0,2) & & (2,2) \\ & (1,1) & \\ & (1,0) & \end{array}$$

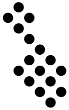
An LRSet:



$$= \begin{array}{cccccc} & & (0,6) & & & \\ & (-1,5) & & (1,5) & & \\ (-2,4) & & & & (2,4) & \\ & (-1,3) & & & & (3,3) \\ (-2,2) & & (0,2) & & (2,2) & \\ & (-1,1) & & (1,1) & & \\ & & (0,0) & & & \end{array} = \begin{array}{cccccc} & & 33 & & & \\ & 32 & & 23 & & \\ 31 & & & & 13 & \\ & 21 & & & & 03 \\ 20 & & 11 & & 02 & \\ & 10 & & 01 & & \\ & & 00 & & & \end{array}$$

(h, L, R): height, left wall, right wall

ZHAs (planar Heyting Algebras) are LRSets obeying extra conditions...
A ZHA is "everything between a left wall and a right wall" —
The left wall has one point for each y
The left wall is made of points of the form $(L(y), y)$ (same for "right")



$\leftarrow - - \rightarrow$

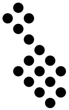
$(-3,9)$	$L(9) = -3$	$R(9) = -3$
$(-4,8)$ $(-2,8)$	$L(8) = -4$	$R(8) = -2$
$(-3,7)$	$L(7) = -3$	$R(7) = -3$
$(-2,6)$	$L(6) = -2$	$R(6) = -2$
$(-1,5)$	$L(5) = -1$	$R(5) = -1$
$(-2,4)$ $(0,4)$	$L(4) = -2$	$R(4) = 0$
$(-3,3)$ $(-1,3)$ $(1,3)$	$L(3) = -3$	$R(3) = 1$
$(-2,2)$ $(0,2)$	$L(2) = -2$	$R(2) = 0$
$(-1,1)$ $(1,1)$	$L(1) = -1$	$R(1) = 1$
$(0,0)$	$L(0) = 0$	$R(0) = 0$

Top point: $(-3, 9)$
Height: 9
$h = 9, \quad L : \{0, \ldots, 9\} \to \mathbb{Z}, \quad R : \{0, \ldots, 9\} \to \mathbb{Z},$
The ZHA is everything in $\mathbb{LR}$ between the left and the right wall

(h, L, R): height, left wall, right wall (2)

Numbers, sets and lists feel very concrete to “children”, so:

$$(h, L, R) = \left(9, \left\{ \begin{matrix} (9,-3), \\ (8,-4), \\ (7,-3), \\ (6,-2), \\ (5,-1), \\ (4,-2), \\ (3,-3), \\ (2,-2), \\ (1,-1), \\ (0,0) \end{matrix} \right\}, \left\{ \begin{matrix} (9,-3), \\ (8,-2), \\ (7,-3), \\ (6,-2), \\ (5,-1), \\ (4,0), \\ (3,1), \\ (2,0), \\ (1,1), \\ (0,0) \end{matrix} \right\} \right) \quad \overset{\text{generates}}{\dashrightarrow}$$



(h, L, R): height, left wall, right wall (3)

A triple (h, L, R) is a height-left-right-wall (“HLRW”) iff:
1) $h \in \mathbb{N}$
2) $L : \{0, \ldots, h\} \to \mathbb{Z}$
3) $R : \{0, \ldots, h\} \to \mathbb{Z}$
4) $L(y + 1) = L(y) \pm 1$ always
5) $R(y + 1) = R(y) \pm 1$ always
6) $L(0) = R(0) = 0$
7) $L(y) \le R(y)$ always
8) $L(h) = R(h)$

The ZHA generated by (h, L, R) is:
$\mathsf{ZHAG}(h, L, R) =$
$\quad \{\, (x, y) \in \mathbb{LR} \mid y \le h, L(y) \le x \le R(y) \,\}$

Formal definition of a ZHA:

> A *ZHA* is a set of the form $\mathsf{ZHAG}(h, L, R)$,
> for some HLRW (h, L, R).

(Theorem: every ZHA is a Heyting Algebra)

Heyting Algebras

A Heyting Algebra (a "HA") is a structure

$H = (\Omega, \leq_H, \top_H, \bot_H, \land_H, \lor_H, \to_H)$

in which:
1) Ω is a set (the "set of truth values")
2) $\leq_H$ is a (strict) partial order on Ω
3) $\top_H$ is the top element
4) $\bot_H$ is the bottom element
5) $(P \leq_H (Q \land_H R)) \leftrightarrow ((P \leq_H Q) \land (P \leq_H R))$
6) $((P \lor_H Q) \leq_H R) \leftrightarrow ((P \leq_H R) \land (Q \leq_H R))$
7) $(P \leq_H (Q \to_H R)) \leftrightarrow ((P \land_H Q) \leq_H R)$

Sometimes we add operations '$\neg$' and $\leftrightarrow$ to a HA H:

$H = (\Omega, \leq_H, \top_H, \bot_H, \land_H, \lor_H, \to_H, \neg_H, \leftrightarrow_H)$

where:
8) $\neg_H P \quad := \quad P \to_H \bot_H$
9) $P \leftrightarrow_H Q \quad := \quad (P \to_H Q) \land_H (Q \to_H P)$

Two Heyting Algebras

Numbers, sets and lists feel very concrete to “children”, so...
Classical logic:

$$\mathsf{CL} = (\Omega_{\mathsf{CL}}, \top_{\mathsf{CL}}, \bot_{\mathsf{CL}}, \land_{\mathsf{CL}}, \lor_{\mathsf{CL}}, \to_{\mathsf{CL}}, \leftrightarrow_{\mathsf{CL}}, \neg_{\mathsf{CL}}) =$$

$$\left(\left\{ \begin{matrix} 0, \\ 1 \end{matrix} \right\}, 1, 0, \left\{ \begin{matrix} ((0,0),0), \\ ((0,1),0), \\ ((1,0),0), \\ ((1,1),1) \end{matrix} \right\}, \left\{ \begin{matrix} ((0,0),0), \\ ((0,1),1), \\ ((1,0),1), \\ ((1,1),1) \end{matrix} \right\}, \left\{ \begin{matrix} ((0,0),1), \\ ((0,1),1), \\ ((1,0),0), \\ ((1,1),1) \end{matrix} \right\}, \left\{ \begin{matrix} ((0,0),1), \\ ((0,1),0), \\ ((1,0),0), \\ ((1,1),1) \end{matrix} \right\}, \left\{ \begin{matrix} (0,1), \\ (1,0) \end{matrix} \right\} \right)$$

A 3-valued logic:

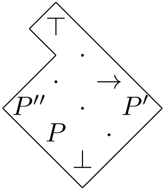
$$\mathsf{L}_3 = (\Omega_{\mathsf{L}_3}, \top_{\mathsf{L}_3}, \bot_{\mathsf{L}_3}, \land_{\mathsf{L}_3}, \lor_{\mathsf{L}_3}, \to_{\mathsf{L}_3}, \leftrightarrow_{\mathsf{L}_3}, \neg_{\mathsf{L}_3}) =$$

$$\left(\left\{ \begin{matrix} 00, \\ 01, \\ 11 \end{matrix} \right\}, 11, 00, \left\{ \begin{matrix} ((00,00),00), \\ ((00,01),00), \\ ((00,11),00), \\ ((01,00),00), \\ ((01,01),01), \\ ((01,11),01), \\ ((11,00),00), \\ ((11,01),01), \\ ((11,11),11) \end{matrix} \right\}, \left\{ \begin{matrix} ((00,00),00), \\ ((00,01),01), \\ ((00,11),11), \\ ((01,00),01), \\ ((01,01),01), \\ ((01,11),11), \\ ((11,00),11), \\ ((11,01),11), \\ ((11,11),11) \end{matrix} \right\}, \left\{ \begin{matrix} ((00,00),11), \\ ((00,01),11), \\ ((00,11),11), \\ ((01,00),00), \\ ((01,01),11), \\ ((01,11),11), \\ ((11,00),00), \\ ((11,01),01), \\ ((11,11),11) \end{matrix} \right\}, \left\{ \begin{matrix} ((00,00),11), \\ ((00,01),00), \\ ((00,11),00), \\ ((01,00),00), \\ ((01,01),11), \\ ((01,11),01), \\ ((11,00),00), \\ ((11,01),01), \\ ((11,11),11) \end{matrix} \right\}, \left\{ \begin{matrix} (00,11), \\ (01,00), \\ (11,00) \end{matrix} \right\} \right)$$

From the handouts: two non-tautologies (for children)

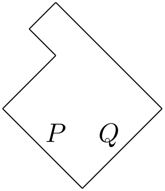
In the ZHA H, with the valuation v, we have:

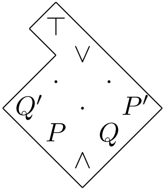
$$H = \begin{array}{ccccc} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array}$$



$$\underbrace{(\underbrace{\neg\underbrace{\neg\underbrace{P}_{10}}_{02}}_{20}) \to \underbrace{P}_{10}}_{12}$$

$v =$





$$\underbrace{\underbrace{\neg(\underbrace{\underbrace{P}_{10}\land\underbrace{Q}_{01}}_{00})}_{32} \to \underbrace{(\underbrace{\neg\underbrace{P}_{10}}_{02}\lor\underbrace{\neg\underbrace{Q}_{01}}_{20})}_{22}}_{22}$$

...these two classical tautologies are not $=\top$ ($=32$) in v,
so they are not true in all Heyting Algebras,
and they can't be theorems of intuitionistic logic...

Intuitionistic logic (IPL) has fewer tautologies the classical logic (CPL).
How can we prove that something holds in all ZHAs (or in all HAs)?
This motivates rewriting the axioms of a HA into tree rules —

Logic in a ZHA: computing $\top, \bot, \land, \lor, \to$

Every ZHA (a subset of $\mathbb{Z}^2$) “is” a HA (a 7-uple)... ← magic
Trick: a ZHA can be extended *canonically* to a structure

$$H = (\Omega, \leq_H, \top_H, \bot_H, \land_H, \lor_H, \to_C)$$

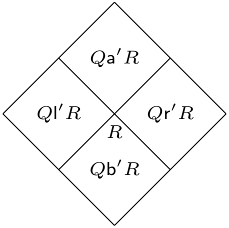
where
1) Ω is the set of points of the ZHA (“set of truth-values”)
2) $ab \leq_H cd$ iff $a \leq c$ and $b \leq d$
3) $\top_H$ is the top element
4) $\bot_H$ is the bottom element (i.e., 00)
5) $ab \land_H cd = \min(a, c) \min(b, d)$
6) $ab \lor_H cd = \max(a, c) \max(b, d)$
7) $\to_C$ is the “(quickly) computable implication”:

$$Q \to_C R \;:=\; \left(\begin{array}{llll} \mathsf{if} & Q\mathsf{b}R & \mathsf{then} & \top \\ \mathsf{elseif} & Q\mathsf{l}R & \mathsf{then} & \mathsf{ne}(R) \\ \mathsf{elseif} & Q\mathsf{r}R & \mathsf{then} & \mathsf{nw}(R) \\ \mathsf{elseif} & Q\mathsf{a}R & \mathsf{then} & R \\ \mathsf{end} & & & \end{array} \right) = \left(\begin{array}{llll} \mathsf{if} & Q\mathsf{b}'R & \mathsf{then} & \top \\ \mathsf{elseif} & Q\mathsf{l}'R & \mathsf{then} & \mathsf{ne}(R) \\ \mathsf{elseif} & Q\mathsf{r}'R & \mathsf{then} & \mathsf{nw}(R) \\ \mathsf{elseif} & Q\mathsf{a}'R & \mathsf{then} & R \\ \mathsf{end} & & & \end{array} \right)$$

where b, l, r, a abbreviate
below, leftof, rightof, above
and b′, l′, r′, a′ are...

Logic in a ZHA: computing $\top, \bot, \land, \lor, \to$ (2)

...and b', l', r', a' are variants of below, leftof, $\mathsf{rightof}$, above that divide the ZHA into four disjoint regions:



We have:

$$P \leq_H \left(\begin{array}{llll} \mathsf{if} & Q\mathsf{b}'R & \mathsf{then} & \top \\ \mathsf{elseif} & Q\mathsf{l}'R & \mathsf{then} & \mathsf{ne}(R) \\ \mathsf{elseif} & Q\mathsf{r}'R & \mathsf{then} & \mathsf{nw}(R) \\ \mathsf{elseif} & Q\mathsf{a}'R & \mathsf{then} & R \\ \mathsf{end} \end{array} \right) \quad \text{iff} \quad \left(\begin{array}{lcl} Q\mathsf{b}'R & \to & P \leq_H \top \\ Q\mathsf{l}'R & \to & P \leq_H \mathsf{ne}(R) \\ Q\mathsf{r}'R & \to & P \leq_H \mathsf{nw}(R) \\ Q\mathsf{a}'R & \to & P \leq_H R \end{array} \right)$$

The proof is tedious but easy, and it shows that $Q \to_C R$ obeys:

$(P \leq_H (Q {\to_C} R)) \leftrightarrow ((P \land_H Q) \leq_H R)$

and so $(Q \to_C R) = (Q \to_H R)$.

The big picture

$$\begin{tikzcd}
 & & \text{U}\lambda \arrow[d] & & \text{weirder, for adults} \arrow[dddd, dashed, leftrightarrow] \\
\text{HAs} & & \lambda 1 + (U{\to}U \approx U) \arrow[r] & \text{CCCUs} & \\
\text{ZHAs} \arrow[u] \arrow[d] \arrow[drr] & \text{IPL} \arrow[ul] \arrow[l] \arrow[r] \arrow[dl] \arrow[dd] & \lambda 1 \arrow[u] \arrow[d] & \text{CCCs} \arrow[u] \arrow[l] \arrow[dl] & \\
(X, \mathcal{O}(X)) \arrow[rr] \arrow[d] & & \text{Sets} & & \\
(X, \mathcal{P}(X)) \arrow[r] \arrow[urr] & \text{CPL} & & & \text{for children}
\end{tikzcd}$$

'$\longrightarrow$': "can be interpreted in"
$\lambda 1$: simply-typed λ-calculus
$U\lambda$: untyped λ-calculus
$U{\to}U \approx U$: we have an object U such that $U{\to}U$ is isomorphic to U
CCCU: a cartesian-closed category with an object $U{\to}U \approx U$

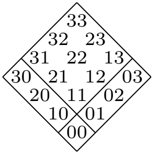
IPL distinguishes P and $\neg\neg P$; CPL does not; IPL has more models than CPL.

J-operators

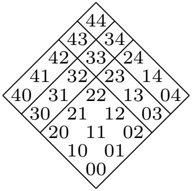
A *J-operator* is a function $\cdot^* : H \to H$ such that
$P \leq P^* = P^{**}$ and $P^* \wedge Q^* = (P \wedge Q)^*$.
(They are important in topos theory: they induce sheaves.)

Examples:

$$\begin{array}{rcl} P^* & := & \neg\neg P \\ 20^* & = & 30 \\ 31^* & = & 33 \end{array}$$



$$\begin{array}{rcl} P^* & := & 22 \vee P \\ 20^* & = & 22 \\ 31^* & = & 32 \end{array}$$



Trick (visual): P^* is the top element
in the equivalence class of P.

The “fences” divide the ZHA into equivalence classes ($P \sim Q$ iff $P^* = Q^*$)
Theorem: a J-operator takes each P to the top element in its class.
Theorem (hard): J-operators correspond to slashings by diagonal cuts
without cuts stopping midway (see the paper, secs 17–25).

Lindenbaum(-Tarski) algebras

...are non-strict partial orders ← i.e., reflexive/transitive relations
on the set of all expressions (wffs) ← what about smaller sets?
of a logic.

$\mathsf{expr}_1 \leq_L \mathsf{expr}_2$ iff
we can prove $\mathsf{expr}_1 \to \mathsf{expr}_2$ in L.

Logics: CPL, IPL, IPL*
In $\mathsf{Lind}(\mathrm{IPL}^*(P, Q, R))$ we have:

$$\begin{array}{cc}
P \leq \neg\neg P & (P \lor Q)^* \leq (P^* \lor Q^*)^* \\
P \not\geq \neg\neg P & (P \lor Q)^* \geq (P^* \lor Q^*)^* \\
P \longrightarrow \neg\neg P & (P \lor Q)^* =\!= (P^* \lor Q^*)^*
\end{array}$$

$\mathsf{Lind}(L)$ is a p.o. on an infinite set
but we can look at “fragments” of it —
p.o.s on subsets of $\mathsf{Exprs}(L)$...

Standard def: the Lindenbaum algebra is the strict partial order
on the set of equivalence classes of a logic (where $P \sim Q$ iff $P \leftrightarrow Q$)

Lindenbaum algebras (2)

In $\mathsf{Lind}(\mathrm{IPL}^*(P,Q,R))$ we can prove

$$\begin{tikzcd}
& (P^*\wedge Q^*)^* \arrow[dl, equal] \arrow[d, equal] \arrow[dr, equal] & \\
(P^*\wedge Q)^* & P^*\wedge Q^* & (P\wedge Q^*)^* \\
P^*\wedge Q \arrow[u] \arrow[ur] & (P\wedge Q)^* \arrow[ul, equal] \arrow[ur, equal] & P\wedge Q^* \arrow[ul] \arrow[u] \\
& P\wedge Q \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& (P^*\vee Q^*)^* \arrow[dl, equal] \arrow[dr, equal] & \\
(P^*\vee Q)^* & P^*\vee Q^* \arrow[u] & (P\vee Q^*)^* \\
P^*\vee Q \arrow[u] \arrow[ur] & (P\vee Q)^* \arrow[ul, equal] \arrow[ur, equal] & P\vee Q^* \arrow[ul] \arrow[u] \\
& P\vee Q \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

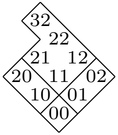
For any J-operator $\cdot^*$ obeying $P \leq P^* = P^{**}$ and $(P \wedge Q)^* = P^* \wedge Q^*$...

Valuations

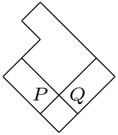
In this ZHA, with this J-operator, and this valuation,

$$H = \begin{array}{ccccc} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array}$$

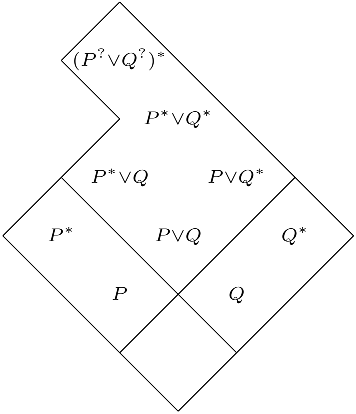
$J =$



$v =$



we have:



$\rightsquigarrow$

$$\begin{tikzcd}
 & (P^* \vee Q^*)^* & \\
(P^* \vee Q)^* \arrow[ur, equal] & P^* \vee Q^* \arrow[u] & (P \vee Q^*)^* \arrow[ul, equal] \\
P^* \vee Q \arrow[u] \arrow[ur] & (P \vee Q)^* \arrow[ul, equal] \arrow[ur, equal] & P \vee Q^* \arrow[ul] \arrow[u] \\
 & P \vee Q \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

On “archetypalness”

CPL does not distinguish P and $\neg\neg P$
IPL does distinguish P and $\neg\neg P$
How do we visualize IPL?
ZHAs help us visualize *fragments* of the Lindenbaum Algebra of HAs.

In the categorical models (“hyperdoctrines”) for first-order logic (FOL)
there are two different constructions for $R(x,y) \ := \ P(x) \land Q(x)$...
In *Internal Diagrams and Archetypal Reasoning in Category Theory* [Ochs2013]
we showed a way to use the notation of FOL, and the semantics of the
“archetypal model” (**Set**!) as tools for understanding hyperdoctrines...
Hyperdoctrines are too abstract and too hard when presented “for adults”,
but having an “archetypal model” helps a lot!...

From IDARCT, sec.16:
That “archetypical language” does not need to be unambiguous (...) and does not need to be convenient for expressing all possible constructions. What is relevant is that the archetypical language, when used side-to-side with the “algebraic” language, should give us a way to reason, both intuitively and precisely, about the structure we’re working on; in particular, it should let us formulate reasonable conjectures quickly, and check them with reasonable ease...

That’s similar to using ZHAs and valuations that “distinguish enough things”!

Thank you! =)

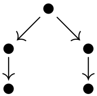
Answers to typical questions

1. ZHAs are distributive lattices
See Davey and Priestley's *Introduction to Lattices and Order, 2nd ed*, chapter 4.

2. How do I find a countermodel for a sentence?
A: Use modal tableaux and the idea below

3. Can we change "planar" to 2D, 3D, 4D, ...?
A: Yes, ZHAs are topologies on "2-column graphs"; change to 3 or more columns

$(H, \mathsf{BPM}(H)) =$



$(\mathcal{O}(H), \subset_1) =$

$$\begin{tikzcd}
& \begin{smallmatrix}&1&\\1&&1\\1&&1\end{smallmatrix} & & & \\
& & \begin{smallmatrix}&0&\\1&&1\\1&&1\end{smallmatrix} \arrow[ul] & & \\
& \begin{smallmatrix}&0&\\1&&0\\1&&1\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}&0&\\0&&1\\1&&1\end{smallmatrix} \arrow[ul] & \\
\begin{smallmatrix}&0&\\1&&0\\1&&0\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}&0&\\0&&0\\1&&1\end{smallmatrix} \arrow[ul] \arrow[ur] & & \begin{smallmatrix}&0&\\0&&1\\0&&1\end{smallmatrix} \arrow[ul] \\
& \begin{smallmatrix}&0&\\0&&0\\1&&0\end{smallmatrix} \arrow[ul] \arrow[ur] & & \begin{smallmatrix}&0&\\0&&0\\0&&1\end{smallmatrix} \arrow[ul] \arrow[ur] & \\
& & \begin{smallmatrix}&0&\\0&&0\\0&&0\end{smallmatrix} \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

4. How do we go from finite HAs to infinite HAs?
A: A starting point: try to add a line with r=2.5 to a ZHA and see what happens =)

5. Are there propositions that IPL distinguish but ZHAs do not? Or:
Are there any non-theorems of IPL that don't have countermodels in ZHAs?
A: Yes. First idea: in a ZHA we may have P, Q, R independent, (62, 53, 44)
but $P \land Q$, $P \land R$, $Q \land R$ can't be all independent... (52, 42, 43)
Let $\alpha(P,Q) := (P{\to}Q) \lor (Q{\to}P)$,
$\beta(P,Q,R) := \alpha(P,Q) \lor \alpha(P,R) \lor \alpha(Q,R)$,
$\gamma(P,Q,R) := \beta(P{\land}Q, P{\land}R, Q{\land}R)$.
Then $\gamma(P,Q,R)$ is a tautology in all ZHAs, but has a 3D countermodel.
Second idea: use the "width" of a modal logic
(See *Handbook of Modal Logic*, p.454, and Davey/Priestley p.32)

6. How do I teach these things to children:
A: I use λ-calculus and lots of *visual* exercises
See the paper and: `http://angg.twu.net/LATEX/2017-1-LA-material.pdf`
"Disciplina optativa: λ-cálculo, lógicas e traduções"

7. Future work / what are the next steps?
A: Categories, toposes and sheaves for children (ongoing work with Peter Arndt)

8. What is the shape of a full Lindenbaum Algebra in IPL?
The free HA on one generator (IPL(P)) is well-known and planar (infinite) —
The free HA on two generators (IPL(P, Q)) is uglier than the product
of two of these things...

IPL(P) is the “Rieger-Nishimura Lattice”, that is the infinite
version of this:

		77	
	76		67
75		66	
	65		56
64		55	
	54		45
53		44	
	43		34
42		33	
	32		23
31		22	
	21		12
20		11	
	10		01
		00	