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Prawitz's example: original version
In [Prawitz], this example of a translation (from natural language) is used to introduce Natural Deduction:

An informal derivation of $\forall x.\exists y.(Pxy \& Qxy)$ from the two assumptions
(1) $\forall x.\forall y.Pxy$
(2) $\forall x.\forall y.(Pxy \supset Qxy)$
may run somewhat as follows:
From (1), we obtain:
(3) $\exists y.Pay$
Let us therefore assume
(4) Pab.
From (2), we have:
(5) $Pab \supset Qab$
and from (4) and (5)
(6) Qab.
Hence, from (4) and (6), we obtain
(7) $Pab \& Qab$
and from (7) we get
(8) $\exists y.(Pay \& Qay)$

Now, (8) is obtained from assumption (4), but the argument is independent of the particular value of the parameter b that satisfies (4). In view of (3), we therefore have:

(9) (8) is independent of the assumption (4).

Because of (9), (8) depends only on (1) and (2) and thus holds on these assumptions for any arbitrary value of a. Hence, the desired result:

(10) $\forall x.\exists y.(Pxy \& Qxy)$.

The corresponding natural deduction is given below; the numerals refer to steps in the informal argument above (rather than to the way the assumptions are discharged).

$$\dfrac{\dfrac{\dfrac{{\scriptstyle (1)}\ \forall x \exists y Pxy}{{\scriptstyle (3)}\ \exists y Pay} \qquad \dfrac{\dfrac{{\scriptstyle (4)}\ Pab \qquad \dfrac{{\scriptstyle (4)}\ Pab \qquad \dfrac{\dfrac{\forall x \forall y (Pxy \supset Qxy)\ {\scriptstyle (2)}}{\forall y (Pay \supset Qay)}}{Pab \supset Qab\ {\scriptstyle (5)}}}{Qab\ {\scriptstyle (6)}}}{Pab \& Qab\ {\scriptstyle (7)}}}{\exists y (Pay \& Qay)\ {\scriptstyle (8)}}}{\exists y (Pay \& Qay)\ {\scriptstyle (9)}}}{\forall x \exists y (Pxy \& Qyx)\ {\scriptstyle (10)}}$$

Prawitz's example: proper subtrees
We will use the same letters for free and bound variables, and we'll often abbreviate 'Pab' and 'Qab' as just 'P' and 'Q'.
In our notation, with all discharges indicated, that tree becomes:

$$\dfrac{\dfrac{\dfrac{[a]^2 \quad \forall a.\exists b.P}{\exists b.P}\,(\forall\mathrm{E}) \qquad \dfrac{\dfrac{[P]^1 \quad \dfrac{[P]^1 \quad \dfrac{[b]^1 \quad \dfrac{[a]^2 \quad \forall a.\forall b.P \supset Q}{\forall b.P \supset Q}\,(\forall\mathrm{E})}{P \supset Q}\,(\forall\mathrm{E})}{Q}}{P\&Q}}{\exists b.P\&Q}\,(\exists\mathrm{I})}{\exists b.P\&Q}\,(\exists\mathrm{E});1}{\forall a.\exists b.P\&Q}\,(\forall\mathrm{I});2$$

Definition: a subtree of an ND derivation is improper when it contains a bar that discharges hypotheses (say, "$(\exists\mathrm{E});1$" above) but it doesn't contain all of the leaves associated to that discharge (in that case, $[P]^1$, $[P]^1$, and $[b]^1$).

It is easy to attribute a meaning (a "semantics") for proper subtrees in which all the hypotheses and the conclusion have the same free variables.
For example, this subtree,

$$\dfrac{\exists b.Pab \quad \forall b.Pab \supset Qab}{\exists b.Pab\&Qab}$$

corresponds to this inclusion between subsets of A:

$$\begin{aligned} &\{\, b \mid \exists b.Pab \quad \text{and} \quad \forall b.Pab \supset Qab \,\} \\ &\subseteq \{\, b \mid \exists b.Pab\&Pqb \,\} \end{aligned}$$

But how do we attribute a semantics for proper subtrees where the sets of free variables of the hypotheses and the conclusion are not all equal?
Even worse: how can we interpret hypotheses like 'b' (or '$f(a)$'), that are *terms* (values for variables), instead of "truths"?
This seems to make no sense in "subset semantics"...

To understand this we need to introduce other translations.

Quantifiers: judgment rules
In [Jacobs], sec. 4.1, the rules for the quantifiers for first-order logic are stated in terms of "judgments", as below:
(his notation is very different, though -)

$$\frac{a,b;Pa\vdash Qab}{a;Pa\vdash\forall b.Qab}\;(\forall\mathrm{I}) \qquad \frac{a\vdash b \quad a;Pa\vdash\forall b.Qab}{a;Pa\vdash Qab}\;(\forall\mathrm{E})$$

$$\frac{a\vdash b \quad a;Pa\vdash Qab}{a;Pa\vdash\exists b.Qab}\;(\exists\mathrm{I}) \qquad \frac{a;Pa\vdash\exists b.Qab \quad a,b;Qab,Ra\vdash Sa}{a;Pa,Ra\vdash Sa}\;(\exists\mathrm{E})$$

Each judgment of the form '$a;Pa\vdash Qa$' can be understood as an inclusion $\{\,a\mid Pa\,\}\subseteq\{\,a\mid Qa\,\}$.
Judgments of the form $a\vdash b$ are functions $A\to B$
(or sections of a dependent projections, as we will see later).

In (∀E) and (∃I) there seems to be a missing 'b' in one of the hypotheses/conclusions; that b is taken to be the image of a by $a\vdash b$.

Here's how to translate the "judgment rules" to Natural Deduction...

$$\dfrac{\begin{array}{c}[b]^2 \quad Pa\\ \vdots\\ Qab\end{array}}{\forall b.Qab}\;(\forall\mathrm{I});2 \qquad \dfrac{b \quad \begin{array}{c}Pa\\ \vdots\\ \forall b.Qab\end{array}}{Qab}\;(\forall\mathrm{E})$$

$$\dfrac{\begin{array}{c}b \quad Pa\\ \vdots\\ Qab\end{array}}{\exists b.Qab}\;(\exists\mathrm{I}) \qquad \dfrac{\begin{array}{c}Pa\\ \vdots\\ \exists b.Qab\end{array} \quad \begin{array}{c}[b]^1 \quad [Qab]^1 \quad Ra\\ \vdots\\ Sa\end{array}}{Sa}\;(\exists\mathrm{E});1$$

in the ND form the free variables of each subtree are not shown - they must be inferred.

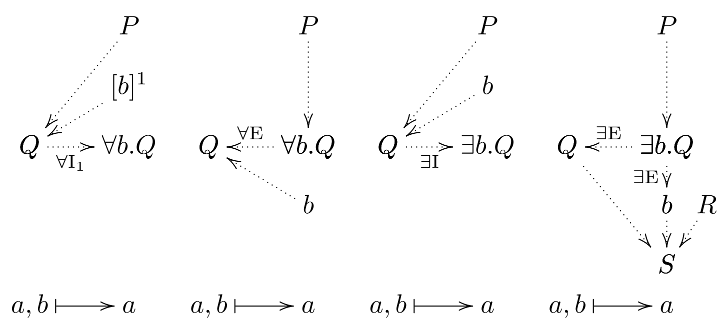
Quantifiers: diagrammatic rules
Now let's draw these rules in a diagrammatic form:

$$\frac{a,b;Pa \vdash Qab}{a;Pa \vdash \forall b.Qab}\ (\forall\mathrm{I}) \qquad \dfrac{\begin{matrix}[b]^2 \quad Pa\\ \vdots \\ Qab\end{matrix}}{\forall b.Qab}\ (\forall\mathrm{I});2 \qquad \dfrac{\begin{matrix} & Pa\\ & \vdots \\ b & \forall b.Qab\end{matrix}}{Qab}\ (\forall\mathrm{E})$$

$$\frac{a \vdash b \quad a;Pa \vdash \forall b.Qab}{a;Pa \vdash Qab}\ (\forall\mathrm{E})$$

$$\frac{a \vdash b \quad a;Pa \vdash Qab}{a;Pa \vdash \exists b.Qab}\ (\exists\mathrm{I}) \qquad \dfrac{\begin{matrix}b \quad Pa\\ \vdots \\ Qab\end{matrix}}{\exists b.Qab}\ (\exists\mathrm{I}) \qquad \dfrac{\begin{matrix}Pa & [b]^1 \quad [Qab]^1 & Ra\\ \vdots & \vdots & \\ \exists b.Qab & Sa & \end{matrix}}{Sa}\ (\exists\mathrm{E});1$$

$$\frac{a;Pa \vdash \exists b.Qab \quad a,b;Qab,Ra \vdash Sa}{a;Pa,Ra \vdash Sa}\ (\exists\mathrm{E})$$



Each proposition will be draw over (the list of) is free variables. We draw 'b' over 'a' for reasons that will be become clear later (briefly: in the system with dependent types the type for b will be B_a, which depends on a).

Let's translate the example from [Prawitz] to diagrammatic form.
We get a DAG over $a, b \mapsto b \mapsto *$, and we can translate the notion of "proper subtree" into a corresponding notion for DAGs.

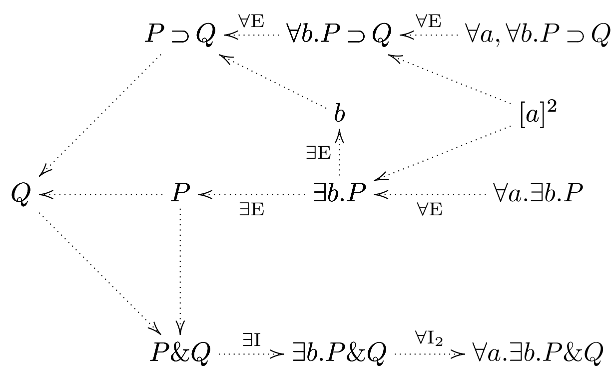
A sub-DAG is "proper" when it is made of a subset of the vertices and arrows of the original DAG (ignore the base $a, b \mapsto b \mapsto *$ - think of it as being just the shadow of what's above it) such that:

- If an arrow $(\alpha \to \beta) \in D'$ then the vertices α and β belong to D';
- D' has exactly one final node (its conclusion);
- If $(\alpha \mapsto \gamma)$ and $(\beta \mapsto \gamma)$ belong to D, and if $(\alpha \mapsto \gamma) \in D'$, then $(\beta \mapsto \gamma) \in D'$;
- If D' contains a discharging arrow then it contains all the associated discharged nodes.

Dotted diagrams

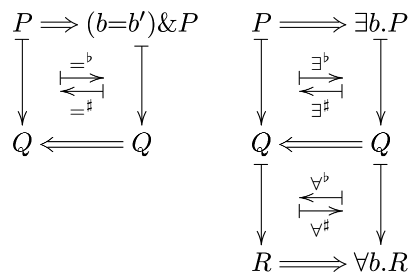
$$\cfrac{\cfrac{\cfrac{[a]^2 \quad \forall a.\exists b.P}{\exists b.P}\ (\forall\mathrm{E}) \qquad \cfrac{\cfrac{[P]^1 \quad \cfrac{[P]^1 \quad \cfrac{[b]^1 \quad \cfrac{[a]^2 \quad \forall a.\forall b.P \supset Q}{\forall b.P \supset Q}\ (\forall\mathrm{E})}{P \supset Q}\ (\forall\mathrm{E})}{Q}}{P\&Q}}{\exists b.P\&Q}\ (\exists\mathrm{I})}{\exists b.P\&Q}\ (\exists\mathrm{E});1}{\forall a.\exists b.P\&Q}\ (\forall\mathrm{I});2$$

$$\cfrac{\cfrac{A}{a}\ [\mathrm{v}]^3 \qquad \cfrac{\cfrac{\cfrac{A}{a}\ [\mathrm{v}]^3}{B}}{b}\ [\mathrm{v}]^1}{\cfrac{\mathrm{W}[P]}{P}}\ [\mathrm{v}]^2$$

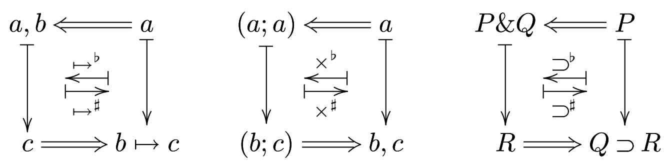


$$a, b \longmapsto a \longmapsto *$$

Names for some adjunctions



$$a, b \longmapsto a, b, b' \qquad a, b \longmapsto a$$



Rules for the quantifiers

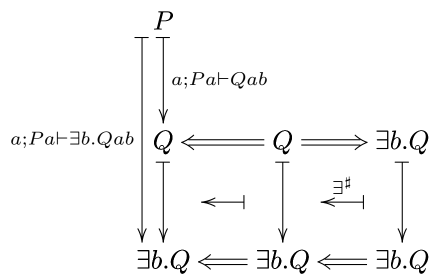
$$\frac{a, b; P \vdash Q}{a; P \vdash \forall b.Q}\;(\forall\mathrm{I}) \quad := \quad (\forall^\sharp)$$

$$\frac{a \vdash b \quad a; P \vdash \forall b.Q}{a; P \vdash Q}\;(\forall\mathrm{E}) \quad := \quad \frac{a \vdash b \quad \dfrac{a; P \vdash \forall b.Q}{a, b; P \vdash Q}\;(\forall^\flat)}{a; P \vdash Q}\;\mathrm{s}$$

$$\frac{a \vdash b \quad a; P \vdash Q}{a; P \vdash \exists b.Q}\;(\exists\mathrm{I}) \quad := \quad \frac{a; P \vdash Q \quad \dfrac{a \vdash b \quad \dfrac{\dfrac{}{a; \exists b.Q \vdash \exists b.Q}}{a, b; Q \vdash \exists b.Q}\;(\exists^\sharp)}{a; Q \vdash \exists b.Q}\;\mathrm{s}}{a; P \vdash \exists b.Q}\;\circ$$

$$\frac{a; P \vdash \exists b.Q \quad a, b; Q, R \vdash S}{a; P, R \vdash S}\;(\exists\mathrm{E}) \quad := \quad \frac{\dfrac{a; P \vdash \exists b.Q \quad \dfrac{\dfrac{a, b; Q, R \vdash S}{a, b; Q \vdash R \supset S}\;(\supset^\sharp)}{a; \exists b.Q \vdash R \supset S}\;(\exists^\sharp)}{a; P \vdash R \supset S}\;\mathrm{s}}{a; P, R \vdash S}\;(\supset^\flat)$$

Introduction of the existential



$$a \xmapsto{a \vdash b} a, b \longmapsto a$$

$$\frac{a \vdash b \quad a; Pa \vdash Qab}{a; Pa \vdash \exists b.Qab}\ (\exists \mathrm{I}) \qquad \frac{\begin{array}{c} a; Pa \\ \vdots \\ a; Qab \end{array}}{a; \exists b.Qab}$$

Elimination of the existential

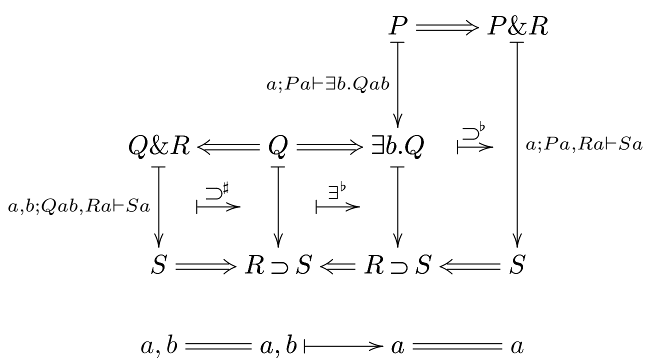
In Natural Deduction:

$$\dfrac{\begin{array}{c} a; Pa \\ \vdots \\ a; \exists b.Qab \end{array} \qquad \begin{array}{c} [a,b;Qab]^1 \quad \dfrac{a;Ra}{a,b;Ra} \\ \vdots \\ a,b;Sa \end{array}}{a;Sa}\ (\exists\mathrm{E});1$$

In Sequent Calculus:

$$\dfrac{a;Pa \vdash \exists b.Qab \quad a,b;Qab,Ra \vdash Sa}{a;Pa,Ra \vdash Sa}\ (\exists\mathrm{E})$$

Categorically:



A derivation from Prawitz

Prawitz, p.19:

$$\dfrac{\dfrac{\dfrac{\forall x\exists yPxy}{\exists yPay} \qquad \dfrac{\dfrac{Pab \qquad \dfrac{Pab \qquad \dfrac{\dfrac{\forall x\forall y(Pxy\supset Pyx)}{\forall y(Pay\supset Pya)}}{Pab\supset Pba}}{Pba}}{Pab\&Pba}}{\exists y(Pab\&Pba)}}{\exists y(Pay\&Pya)}}{\forall x\exists y(Pxy\&Pyx)}$$

$$\dfrac{\dfrac{\dfrac{[a]^3 \quad \forall x\exists yPxy}{\exists yPay}\;(\forall\mathrm{E}) \qquad \dfrac{\dfrac{[Pab]^2 \qquad \dfrac{[Pab]^2 \qquad \dfrac{[b]^1 \qquad \dfrac{[a]^3 \quad \forall x\forall y(Pxy\supset Pyx)}{\forall y(Pay\supset Pya)}\;(\forall\mathrm{E})}{Pab\supset Pba}\;(\forall\mathrm{E})}{Pba}}{Pab\&Pba}}{\exists y(Pay\&Pya)}\;(\exists\mathrm{I});1}{\exists y(Pay\&Pya)}\;(\exists\mathrm{E});2}{\forall x\exists y(Pxy\&Pyx)}\;(\forall\mathrm{I});3$$

$$\dfrac{\dfrac{\dfrac{\forall x\exists yPxy}{a;\exists yPay}\;(\forall\mathrm{E}) \qquad \dfrac{\dfrac{[a,b;Pab]^2 \qquad \dfrac{[a,b;Pab]^2 \qquad \dfrac{\dfrac{\forall x\forall y(Pxy\supset Pyx)}{a;\forall y(Pay\supset Pya)}\;(\forall\mathrm{E})}{a,b;Pab\supset Pba}\;(\forall\mathrm{E})}{a,b;Pba}}{a,b;Pab\&Pba}}{a;\exists y(Pay\&Pya)}\;(\exists\mathrm{I});1}{a;\exists y(Pay\&Pya)}\;(\exists\mathrm{E});2}{\forall x\exists y(Pxy\&Pyx)}\;(\forall\mathrm{I});3$$

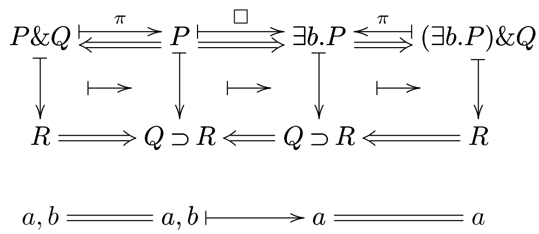
A derivation from Prawitz (2)

$$\dfrac{\dfrac{\dfrac{\forall a.\exists b.P}{a;\exists b.P}\,(\forall\text{E}) \qquad \dfrac{\dfrac{[a,b;P]^2 \qquad \dfrac{[a,b;P]^2 \qquad \dfrac{\dfrac{\forall a.\forall b.P\supset Q}{a;\forall b.P\supset Q}\,(\forall\text{E})}{a,b;P\supset Q}\,(\forall\text{E})}{a,b;Q}}{a,b;P\&Q}}{a;\exists b.P\&Q}\,(\exists\text{I});1}{a;\exists b.P\&Q}\,(\exists\text{E});2}{\forall a.\exists b.P\&Q}\,(\forall\text{I});3$$

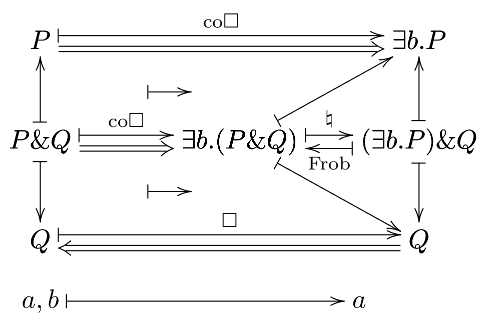
$$\dfrac{\dfrac{\dfrac{a\mapsto b,q}{a;b,q}\,(\forall\text{E}) \qquad \dfrac{\dfrac{[a,b;p]^2 \qquad \dfrac{[a,b;p]^2 \qquad \dfrac{[a;b]^1 \qquad \dfrac{[a]^3 \quad a\mapsto(b\mapsto(p\mapsto q))}{a;b\mapsto(p\mapsto q)}\,(\forall\text{E})}{a,b;p\mapsto q}\,(\forall\text{E})}{a,b;q}\,(\supset\text{E})}{a,b;p,q}\,(\&\text{I})}{a;b,p,q}\,(\exists\text{I});1}{a;b,p,q}\,(\exists\text{E});2}{a\mapsto b,p,q}\,(\forall\text{I});3$$

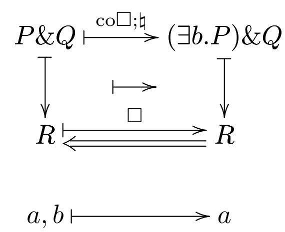
$$\dfrac{\dfrac{\dfrac{\forall a.\exists b.P}{a;\exists b.P}\,(\forall\text{E}) \qquad \begin{matrix}[a,b;P]^2 \quad \dfrac{\forall a.\forall b.P\supset Q}{a;\forall b.P\supset Q}\,(\forall\text{E})\\ \vdots \\ a;\exists b.P\&Q\end{matrix}}{a;\exists b.P\&Q}\,(\exists\text{E});2}{\forall a.\exists b.P\&Q}\,(\forall\text{I});3$$

Wild notes about exist-elim
What do I know about the $(\exists E^{\vee})$ rule from ND?



We do have a map $P\&Q \mapsto (\exists b.P)\&Q$:





I don't know how to universalize the R, though...
Ah, make the adjunction arrows bidirectional,
and start with a pair of objects...
$((a, b; P); (a; Q))$
...and then?

Notes about DTT

Dependent types (or: "dependent spaces"):
$a, b, c \vdash D$

Spaces of witnesses:
$a, b, c \vdash \mathrm{W}[P(a, b, c)]$

Sections:
$a \vdash b$

Substitutions:

$$\frac{a \vdash b \quad a, b, c \vdash D}{a, c \vdash D}$$

Arbitrary base maps
The category of display maps
Witnesses of equality
Vertical maps

Ideas about display maps:
One-step projections
Generalized projections
The category with just the projections is a poset
Sections (monics, inverse to projections)
Diagonal maps

We know how to attribute a semantics to proper trees in propositional ND, but what about ND for (intuitionistic, typed) first-order logic? Then each hypothesis, and the conclusion, may have a different set of free variables - and, worse, of the two hypotheses for the (∀E) rule,

$$\frac{b(a) \quad \forall b. P(a, b)}{P(a, b(a))} \ (\forall \mathrm{E})$$

one is a value for a variable (as a term), the other is is a proposition...

Conjecture: my categorical inter-fiber semantics for ND can be extented to a semantics for DTT.
Conjecture: in my semantics for inter-fiber ND trees, each ND tree corresponds to a structure that can convert sections (one for each hypothesis, and compatible somehow) into a section corresponding to the final conclusion.