APEX CALCULUS

Version 4.0

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4: Applications of the Derivative

In Chapter 3, we learned how the first and second derivatives of a function influence its graph. In this chapter we explore other applications of the derivative.

4.1 Newton's Method

Solving equations is one of the most important things we do in mathematics, yet we are surprisingly limited in what we can solve analytically. For instance, equations as simple as $x^5+x+1=0$ or $\cos x = x$ cannot be solved by algebraic methods in terms of familiar functions. Fortunately, there are methods that can give us *approximate* solutions to equations like these. These methods can usually give an approximation correct to as many decimal places as we like. In Section 1.5 we learned about the Bisection Method. This section focuses on another technique (which generally works faster), called Newton's Method.

Newton's Method is built around tangent lines. The main idea is that if x is sufficiently close to a root of $f(x)$, then the tangent line to the graph at $(x, f(x))$ will cross the x-axis at a point closer to the root than x.

We start Newton's Method with an initial guess about roughly where the root is. Call this x_0. (See Figure 4.1.1(a).) Draw the tangent line to the graph at $(x_0, f(x_0))$ and see where it meets the x-axis. Call this point x_1. Then repeat the process – draw the tangent line to the graph at $(x_1, f(x_1))$ and see where it meets the x-axis. (See Figure 4.1.1(b).) Call this point x_2. Repeat the process again to get x_3, x_4, etc. This sequence of points will often converge rather quickly to a root of f.

We can use this *geometric* process to create an *algebraic* process. Let's look at how we found x_1. We started with the tangent line to the graph at $(x_0, f(x_0))$. The slope of this tangent line is $f'(x_0)$ and the equation of the line is

$$y = f'(x_0)(x-x_0)+f(x_0).$$

This line crosses the x-axis when $y=0$, and the x–value where it crosses is what we called x_1. So let $y=0$ and replace x with x_1, giving the equation:

$$0 = f'(x_0)(x_1-x_0)+f(x_0).$$

Now solve for x_1:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

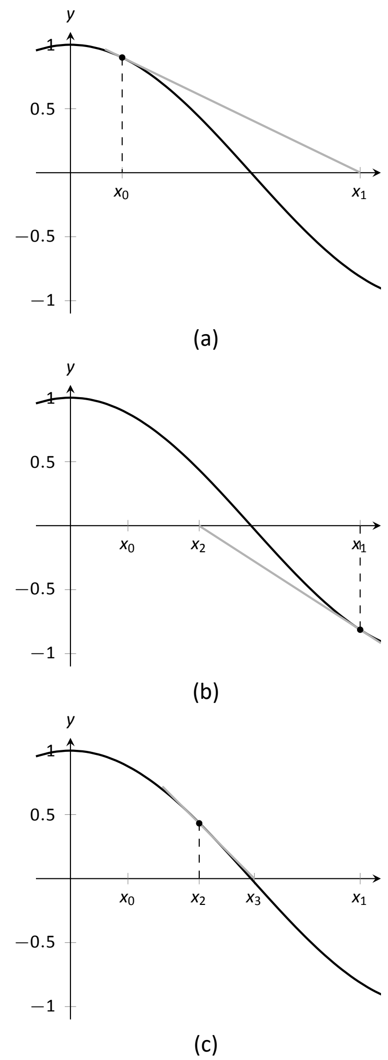


Figure 4.1.1: Demonstrating the geometric concept behind Newton's Method. Note how x_3 is very close to a solution to $f(x)=0$.

Since we repeat the same geometric process to find x_2 from x_1, we have

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, given an approximation x_n, we can find the next approximation, x_{n+1} as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

We summarize this process as follows.

Key Idea 4.1.1 Newton's Method

Let f be a differentiable function on an interval I with a root in I. To approximate the value of the root, accurate to d decimal places:

1. Choose a value x_0 as an initial approximation of the root. (This is often done by looking at a graph of f.)
2. Create successive approximations iteratively; given an approximation x_n, compute the next approximation x_{n+1} as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

3. Stop the iterations when successive approximations do not differ in the first d places after the decimal point.

Note: Newton's Method is not infallible. The sequence of approximate values may not converge, or it may converge so slowly that one is "tricked" into thinking a certain approximation is better than it actually is. These issues will be discussed at the end of the section.

Let's practice Newton's Method with a concrete example.

Example 4.1.1 Using Newton's Method

Approximate the real root of $x^3 - x^2 - 1 = 0$, accurate to the first 3 places after the decimal, using Newton's Method and an initial approximation of $x_0 = 1$.

Solution To begin, we compute $f'(x) = 3x^2 - 2x$. Then we apply the

Notes:

Newton's Method algorithm, outlined in Key Idea 4.1.1.

$$x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1^3 - 1^2 - 1}{3 \cdot 1^2 - 2 \cdot 1} = 2,$$
$$x_2 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{2^3 - 2^2 - 1}{3 \cdot 2^2 - 2 \cdot 2} = 1.625,$$
$$x_3 = 1.625 - \frac{f(1.625)}{f'(1.625)} = 1.625 - \frac{1.625^3 - 1.625^2 - 1}{3 \cdot 1.625^2 - 2 \cdot 1.625} \approx 1.48579.$$
$$x_4 = 1.48579 - \frac{f(1.48579)}{f'(1.48579)} \approx 1.46596$$
$$x_5 = 1.46596 - \frac{f(1.46596)}{f'(1.46596)} \approx 1.46557$$

We performed 5 iterations of Newton's Method to find a root accurate to the first 3 places after the decimal; our final approximation is 1.465. The exact value of the root, to six decimal places, is 1.465571; It turns out that our x_5 is accurate to more than just 3 decimal places.

A graph of $f(x)$ is given in Figure 4.1.2. We can see from the graph that our initial approximation of $x_0 = 1$ was not particularly accurate; a closer guess would have been $x_0 = 1.5$. Our choice was based on ease of initial calculation, and shows that Newton's Method can be robust enough that we do not have to make a very accurate initial approximation.

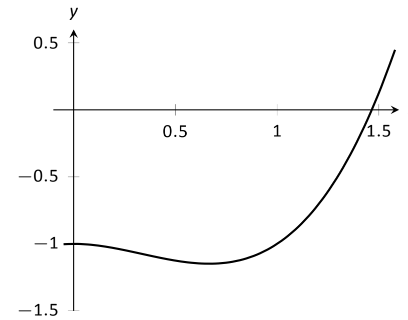


Figure 4.1.2: A graph of $f(x) = x^3 - x^2 - 1$ in Example 4.1.1.

We can automate this process on a calculator that has an `Ans` key that returns the result of the previous calculation. Start by pressing `1` and then `Enter`. (We have just entered our initial guess, $x_0 = 1$.) Now compute

$$\texttt{Ans} - \frac{f(\texttt{Ans})}{f'(\texttt{Ans})}$$

by entering the following and repeatedly press the `Enter` key:

```
Ans-(Ans^3-Ans^2-1)/(3*Ans^2-2*Ans)
```

Each time we press the `Enter` key, we are finding the successive approximations, x_1, x_2, …, and each one is getting closer to the root. In fact, once we get past around x_7 or so, the approximations don't appear to be changing. They actually are changing, but the change is far enough to the right of the decimal point that it doesn't show up on the calculator's display. When this happens, we can be pretty confident that we have found an accurate approximation.

Using a calculator in this manner makes the calculations simple; many iterations can be computed very quickly.

Notes:

Example 4.1.2 Using Newton's Method to find where functions intersect

Use Newton's Method to approximate a solution to $\cos x = x$, accurate to 5 places after the decimal.

Solution Newton's Method provides a method of solving $f(x) = 0$; it is not (directly) a method for solving equations like $f(x) = g(x)$. However, this is not a problem; we can rewrite the latter equation as $f(x) - g(x) = 0$ and then use Newton's Method.

So we rewrite $\cos x = x$ as $\cos x - x = 0$. Written this way, we are finding a root of $f(x) = \cos x - x$. We compute $f'(x) = -\sin x - 1$. Next we need a starting value, x_0. Consider Figure 4.1.3, where $f(x) = \cos x - x$ is graphed. It seems that $x_0 = 0.75$ is pretty close to the root, so we will use that as our x_0. (The figure also shows the graphs of $y = \cos x$ and $y = x$, drawn with dashed lines. Note how they intersect at the same x value as when $f(x) = 0$.)

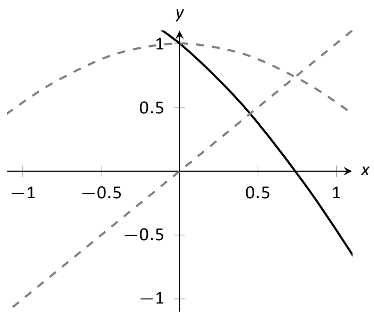


Figure 4.1.3: A graph of $f(x) = \cos x - x$ used to find an initial approximation of its root.

We now compute x_1, x_2, etc. The formula for x_1 is

$$x_1 = 0.75 - \frac{\cos(0.75) - 0.75}{-\sin(0.75) - 1} \approx 0.7391111388.$$

Apply Newton's Method again to find x_2:

$$x_2 = 0.7391111388 - \frac{\cos(0.7391111388) - 0.7391111388}{-\sin(0.7391111388) - 1} \approx 0.7390851334.$$

We can continue this way, but it is really best to automate this process. On a calculator with an Ans key, we would start by pressing 0.75, then `Enter`, inputting our initial approximation. We then enter:

```
Ans - (cos(Ans)-Ans)/(-sin(Ans)-1).
```

Repeatedly pressing the `Enter` key gives successive approximations. We quickly find:

$$x_3 = 0.7390851332$$
$$x_4 = 0.7390851332.$$

Our approximations x_2 and x_3 did not differ for at least the first 5 places after the decimal, so we could have stopped. However, using our calculator in the manner described is easy, so finding x_4 was not hard. It is interesting to see how we found an approximation, accurate to as many decimal places as our calculator displays, in just 4 iterations.

If you know how to program, you can translate the following pseudocode into your favorite language to perform the computation in this problem.

Notes:

```
x = .75
while true
    oldx = x
    x = x - (cos(x)-x)/(-sin(x)-1)
    print x
    if abs(x-oldx) < .0000000001
        break
```

This code calculates x_1, x_2, etc., storing each result in the variable `x`. The previous approximation is stored in the variable `oldx`. We continue looping until the difference between two successive approximations, `abs(x-oldx)`, is less than some small tolerance, in this case, `.0000000001`.

Convergence of Newton's Method

What should one use for the initial guess, x_0? Generally, the closer to the actual root the initial guess is, the better. However, some initial guesses should be avoided. For instance, consider Example 4.1.1 where we sought the root to $f(x) = x^3 - x^2 - 1$. Choosing $x_0 = 0$ would have been a particularly poor choice. Consider Figure 4.1.4, where $f(x)$ is graphed along with its tangent line at $x = 0$. Since $f'(0) = 0$, the tangent line is horizontal and does not intersect the x–axis. Graphically, we see that Newton's Method fails.

We can also see analytically that it fails. Since

$$x_1 = 0 - \frac{f(0)}{f'(0)}$$

and $f'(0) = 0$, we see that x_1 is not well defined.

This problem can also occur if, for instance, it turns out that $f'(x_5) = 0$. Adjusting the initial approximation x_0 by a very small amount will likely fix the problem.

It is also possible for Newton's Method to not converge while each successive approximation is well defined. Consider $f(x) = x^{1/3}$, as shown in Figure 4.1.5. It is clear that the root is $x = 0$, but let's approximate this with $x_0 = 0.1$. Figure 4.1.5(a) shows graphically the calculation of x_1; notice how it is farther from the root than x_0. Figures 4.1.5(b) and (c) show the calculation of x_2 and x_3, which are even farther away; our successive approximations are getting worse. (It turns out that in this particular example, each successive approximation is twice as far from the true answer as the previous approximation.)

There is no "fix" to this problem; Newton's Method simply will not work and another method must be used.

While Newton's Method does not always work, it does work "most of the time," and it is generally very fast. Once the approximations get close to the root,

Notes:

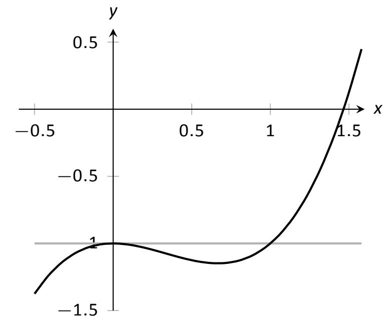


Figure 4.1.4: A graph of $f(x) = x^3 - x^2 - 1$, showing why an initial approximation of $x_0 = 0$ with Newton's Method fails.

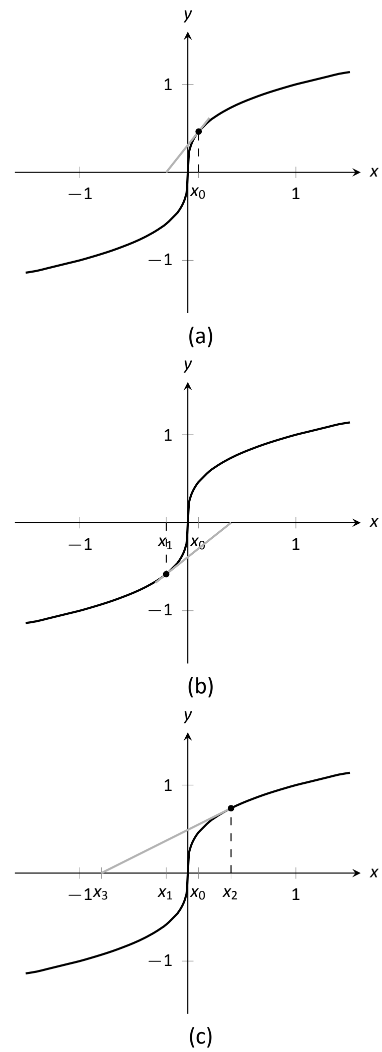


Figure 4.1.5: Newton's Method fails to find a root of $f(x) = x^{1/3}$, regardless of the choice of x_0.

Newton's Method can as much as double the number of correct decimal places with each successive approximation. A course in Numerical Analysis will introduce the reader to more iterative root finding methods, as well as give greater detail about the strengths and weaknesses of Newton's Method.

Notes:

Exercises 4.1

Terms and Concepts

1. T/F: Given a function $f(x)$, Newton's Method produces an exact solution to $f(x) = 0$.

2. T/F: In order to get a solution to $f(x) = 0$ accurate to d places after the decimal, at least $d + 1$ iterations of Newtons' Method must be used.

Problems

In Exercises 3 – 8, the roots of $f(x)$ are known or are easily found. Use 5 iterations of Newton's Method with the given initial approximation to approximate the root. Compare it to the known value of the root.

3. $f(x) = \cos x$, $x_0 = 1.5$

4. $f(x) = \sin x$, $x_0 = 1$

5. $f(x) = x^2 + x - 2$, $x_0 = 0$

6. $f(x) = x^2 - 2$, $x_0 = 1.5$

7. $f(x) = \ln x$, $x_0 = 2$

8. $f(x) = x^3 - x^2 + x - 1$, $x_0 = 1$

In Exercises 9 – 12, use Newton's Method to approximate all roots of the given functions accurate to 3 places after the decimal. If an interval is given, find only the roots that lie in that interval. Use technology to obtain good initial approximations.

9. $f(x) = x^3 + 5x^2 - x - 1$

10. $f(x) = x^4 + 2x^3 - 7x^2 - x + 5$

11. $f(x) = x^{17} - 2x^{13} - 10x^8 + 10$ on $(-2, 2)$

12. $f(x) = x^2 \cos x + (x - 1)\sin x$ on $(-3, 3)$

In Exercises 13 – 16, use Newton's Method to approximate when the given functions are equal, accurate to 3 places after the decimal. Use technology to obtain good initial approximations.

13. $f(x) = x^2$, $g(x) = \cos x$

14. $f(x) = x^2 - 1$, $g(x) = \sin x$

15. $f(x) = e^{x^2}$, $g(x) = \cos x$

16. $f(x) = x$, $g(x) = \tan x$ on $[-6, 6]$

17. Why does Newton's Method fail in finding a root of $f(x) = x^3 - 3x^2 + x + 3$ when $x_0 = 1$?

18. Why does Newton's Method fail in finding a root of $f(x) = -17x^4 + 130x^3 - 301x^2 + 156x + 156$ when $x_0 = 1$?

4.2 Related Rates

When two quantities are related by an equation, knowing the value of one quantity can determine the value of the other. For instance, the circumference and radius of a circle are related by $C = 2\pi r$; knowing that $C = 6\pi$in determines the radius must be 3in.

Note: This section relies heavily on implicit differentiation, so referring back to Section 2.6 may help.

The topic of **related rates** takes this one step further: knowing the *rate* at which one quantity is changing can determine the *rate* at which another changes.

We demonstrate the concepts of related rates through examples.

Example 4.2.1 Understanding related rates

The radius of a circle is growing at a rate of 5in/hr. At what rate is the circumference growing?

Solution The circumference and radius of a circle are related by $C = 2\pi r$. We are given information about how the length of r changes with respect to time; that is, we are told $\frac{dr}{dt} = 5$in/hr. We want to know how the length of C changes with respect to time, i.e., we want to know $\frac{dC}{dt}$.

Implicitly differentiate both sides of $C = 2\pi r$ with respect to t:

$$\begin{aligned} C &= 2\pi r \\ \frac{d}{dt}\big(C\big) &= \frac{d}{dt}\big(2\pi r\big) \\ \frac{dC}{dt} &= 2\pi\frac{dr}{dt}. \end{aligned}$$

As we know $\frac{dr}{dt} = 5$in/hr, we know

$$\frac{dC}{dt} = 2\pi 5 = 10\pi \approx 31.4\text{in/hr}.$$

Consider another, similar example.

Example 4.2.2 Finding related rates

Water streams out of a faucet at a rate of $2\text{in}^3/\text{s}$ onto a flat surface at a constant rate, forming a circular puddle that is $1/8$in deep.

1. At what rate is the area of the puddle growing?
2. At what rate is the radius of the circle growing?

Notes:

Solution

1. We can answer this question two ways: using "common sense" or related rates. The common sense method states that the volume of the puddle is growing by $2\text{in}^3/\text{s}$, where

$$\text{volume of puddle} = \text{area of circle} \times \text{depth}.$$

   Since the depth is constant at $1/8$in, the area must be growing by $16\text{in}^2/\text{s}$.

   This approach reveals the underlying related–rates principle. Let V and A represent the Volume and Area of the puddle. We know $V = A \times \frac{1}{8}$. Take the derivative of both sides with respect to t, employing implicit differentiation.

$$\begin{aligned} V &= \frac{1}{8}A \\ \frac{d}{dt}\big(V\big) &= \frac{d}{dt}\left(\frac{1}{8}A\right) \\ \frac{dV}{dt} &= \frac{1}{8}\frac{dA}{dt} \end{aligned}$$

   As $\frac{dV}{dt} = 2$, we know $2 = \frac{1}{8}\frac{dA}{dt}$, and hence $\frac{dA}{dt} = 16$. Thus the area is growing by $16\text{in}^2/\text{s}$.

2. To start, we need an equation that relates what we know to the radius. We just learned something about the surface area of the circular puddle, and we know $A = \pi r^2$. We should be able to learn about the rate at which the radius is growing with this information.

   Implicitly derive both sides of $A = \pi r^2$ with respect to t:

$$\begin{aligned} A &= \pi r^2 \\ \frac{d}{dt}\big(A\big) &= \frac{d}{dt}\big(\pi r^2\big) \\ \frac{dA}{dt} &= 2\pi r\frac{dr}{dt} \end{aligned}$$

   Our work above told us that $\frac{dA}{dt} = 16\text{in}^2/\text{s}$. Solving for $\frac{dr}{dt}$, we have

$$\frac{dr}{dt} = \frac{8}{\pi r}.$$

   Note how our answer is not a number, but rather a function of r. In other words, *the rate at which the radius is growing depends on how big the*

Notes:

circle already is. If the circle is very large, adding 2in^3 of water will not make the circle much bigger at all. If the circle is dime–sized, adding the same amount of water will make a radical change in the radius of the circle.

In some ways, our problem was (intentionally) ill–posed. We need to specify a current radius in order to know a rate of change. When the puddle has a radius of 10in, the radius is growing at a rate of

$$\frac{dr}{dt} = \frac{8}{10\pi} = \frac{4}{5\pi} \approx 0.25\text{in/s}.$$

Example 4.2.3 Studying related rates

Radar guns measure the rate of distance change between the gun and the object it is measuring. For instance, a reading of "55mph" means the object is moving away from the gun at a rate of 55 miles per hour, whereas a measurement of "$-$25mph" would mean that the object is approaching the gun at a rate of 25 miles per hour.

If the radar gun is moving (say, attached to a police car) then radar readouts are only immediately understandable if the gun and the object are moving along the same line. If a police officer is traveling 60mph and gets a readout of 15mph, he knows that the car ahead of him is moving away at a rate of 15 miles an hour, meaning the car is traveling 75mph. (This straight–line principle is one reason officers park on the side of the highway and try to shoot straight back down the road. It gives the most accurate reading.)

Suppose an officer is driving due north at 30 mph and sees a car moving due east, as shown in Figure 4.2.1. Using his radar gun, he measures a reading of 20mph. By using landmarks, he believes both he and the other car are about 1/2 mile from the intersection of their two roads.

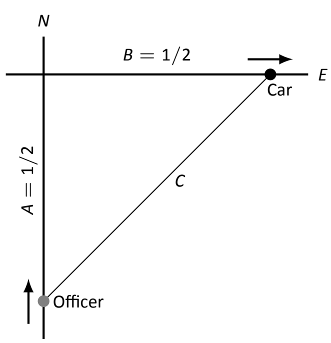


Figure 4.2.1: A sketch of a police car (at bottom) attempting to measure the speed of a car (at right) in Example 4.2.3.

If the speed limit on the other road is 55mph, is the other driver speeding?

SOLUTION Using the diagram in Figure 4.2.1, let's label what we know about the situation. As both the police officer and other driver are 1/2 mile from the intersection, we have $A = 1/2$, $B = 1/2$, and through the Pythagorean Theorem, $C = 1/\sqrt{2} \approx 0.707$.

We know the police officer is traveling at 30mph; that is, $\frac{dA}{dt} = -30$. The reason this rate of change is negative is that A is getting smaller; the distance between the officer and the intersection is shrinking. The radar measurement is $\frac{dC}{dt} = 20$. We want to find $\frac{dB}{dt}$.

We need an equation that relates B to A and/or C. The Pythagorean Theorem

Notes:

is a good choice: $A^2 + B^2 = C^2$. Differentiate both sides with respect to t:

$$A^2 + B^2 = C^2$$
$$\frac{d}{dt}\left(A^2 + B^2\right) = \frac{d}{dt}\left(C^2\right)$$
$$2A\frac{dA}{dt} + 2B\frac{dB}{dt} = 2C\frac{dC}{dt}$$

We have values for everything except $\frac{dB}{dt}$. Solving for this we have

$$\frac{dB}{dt} = \frac{C\frac{dC}{dt} - A\frac{dA}{dt}}{B} \approx 58.28\text{mph}.$$

The other driver appears to be speeding slightly.

Note: Example 4.2.3 is both interesting and impractical. It highlights the difficulty in using radar in a non–linear fashion, and explains why "in real life" the police officer would follow the other driver to determine their speed, and not pull out pencil and paper.
The principles here are important, though. Many automated vehicles make judgments about other moving objects based on perceived distances, radar–like measurements and the concepts of related rates.

Example 4.2.4 Studying related rates

A camera is placed on a tripod 10ft from the side of a road. The camera is to turn to track a car that is to drive by at 100mph for a promotional video. The video's planners want to know what kind of motor the tripod should be equipped with in order to properly track the car as it passes by. Figure 4.2.2 shows the proposed setup.

How fast must the camera be able to turn to track the car?

Solution We seek information about how fast the camera is to *turn*; therefore, we need an equation that will relate an angle θ to the position of the camera and the speed and position of the car.

Figure 4.2.2 suggests we use a trigonometric equation. Letting x represent the distance the car is from the point on the road directly in front of the camera, we have

$$\tan\theta = \frac{x}{10}. \qquad (4.1)$$

As the car is moving at 100mph, we have $\frac{dx}{dt} = -100$mph (as in the last example, since x is getting smaller as the car travels, $\frac{dx}{dt}$ is negative). We need to convert the measurements so they use the same units; rewrite -100mph in terms of ft/s:

$$\frac{dx}{dt} = -100\frac{\text{m}}{\text{hr}} = -100\frac{\text{m}}{\text{hr}}\cdot 5280\frac{\text{ft}}{\text{m}}\cdot\frac{1}{3600}\frac{\text{hr}}{\text{s}} = -146.\overline{6}\text{ft/s}.$$

Now take the derivative of both sides of Equation (4.1) using implicit differenti-

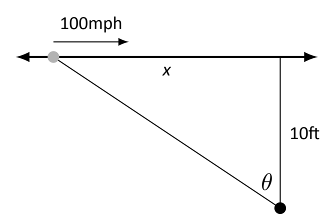


Figure 4.2.2: Tracking a speeding car (at left) with a rotating camera.

Notes:

ation:

$$\begin{aligned}
\tan\theta &= \frac{x}{10}\\
\frac{d}{dt}\big(\tan\theta\big) &= \frac{d}{dt}\left(\frac{x}{10}\right)\\
\sec^2\theta\,\frac{d\theta}{dt} &= \frac{1}{10}\frac{dx}{dt}\\
\frac{d\theta}{dt} &= \frac{\cos^2\theta}{10}\frac{dx}{dt} \qquad (4.2)
\end{aligned}$$

We want to know the fastest the camera has to turn. Common sense tells us this is when the car is directly in front of the camera (i.e., when $\theta = 0$). Our mathematics bears this out. In Equation (4.2) we see this is when $\cos^2\theta$ is largest; this is when $\cos\theta = 1$, or when $\theta = 0$.

With $\frac{dx}{dt} \approx -146.67$ft/s, we have

$$\frac{d\theta}{dt} = -\frac{1\text{rad}}{10\text{ft}}146.67\text{ft/s} = -14.667\text{radians/s}.$$

We find that $\frac{d\theta}{dt}$ is negative; this matches our diagram in Figure 4.2.2 for θ is getting smaller as the car approaches the camera.

What is the practical meaning of -14.667radians/s? Recall that 1 circular revolution goes through 2π radians, thus 14.667rad/s means $14.667/(2\pi) \approx 2.33$ revolutions per second. The negative sign indicates the camera is rotating in a clockwise fashion.

We introduced the derivative as a function that gives the slopes of tangent lines of functions. This chapter emphasizes using the derivative in other ways. Newton's Method uses the derivative to approximate roots of functions; this section stresses the "rate of change" aspect of the derivative to find a relationship between the rates of change of two related quantities.

In the next section we use Extreme Value concepts to *optimize* quantities.

Notes:

Exercises 4.2

Terms and Concepts

1. T/F: Implicit differentiation is often used when solving "related rates" type problems.

2. T/F: A study of related rates is part of the standard police officer training.

Problems

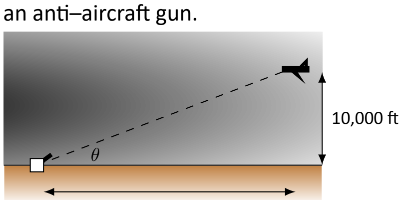
3. Water flows onto a flat surface at a rate of 5cm^3/s forming a circular puddle 10mm deep. How fast is the radius growing when the radius is:
   (a) 1 cm?
   (b) 10 cm?
   (c) 100 cm?

4. A circular balloon is inflated with air flowing at a rate of 10cm^3/s. How fast is the radius of the balloon increasing when the radius is:
   (a) 1 cm?
   (b) 10 cm?
   (c) 100 cm?

5. Consider the traffic situation introduced in Example 4.2.3. How fast is the "other car" traveling if the officer and the other car are each 1/2 mile from the intersection, the other car is traveling *due west*, the officer is traveling north at 50mph, and the radar reading is -80mph?

6. Consider the traffic situation introduced in Example 4.2.3. Calculate how fast the "other car" is traveling in each of the following situations.
   (a) The officer is traveling due north at 50mph and is 1/2 mile from the intersection, while the other car is 1 mile from the intersection traveling west and the radar reading is -80mph?
   (b) The officer is traveling due north at 50mph and is 1 mile from the intersection, while the other car is 1/2 mile from the intersection traveling west and the radar reading is -80mph?

7. An F-22 aircraft is flying at 500mph with an elevation of 10,000ft on a straight–line path that will take it directly over an anti–aircraft gun.

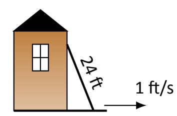


   How fast must the gun be able to turn to accurately track the aircraft when the plane is:
   (a) 1 mile away?
   (b) 1/5 mile away?
   (c) Directly overhead?

8. An F-22 aircraft is flying at 500mph with an elevation of 100ft on a straight–line path that will take it directly over an anti–aircraft gun as in Exercise 7 (note the lower elevation here).

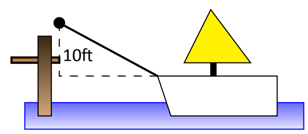
   How fast must the gun be able to turn to accurately track the aircraft when the plane is:
   (a) 1000 feet away?
   (b) 100 feet away?
   (c) Directly overhead?

9. A 24ft. ladder is leaning against a house while the base is pulled away at a constant rate of 1ft/s.



   At what rate is the top of the ladder sliding down the side of the house when the base is:
   (a) 1 foot from the house?
   (b) 10 feet from the house?
   (c) 23 feet from the house?
   (d) 24 feet from the house?

10. A boat is being pulled into a dock at a constant rate of 30ft/min by a winch located 10ft above the deck of the boat.

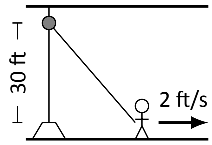


   At what rate is the boat approaching the dock when the boat is:
   (a) 50 feet out?
   (b) 15 feet out?
   (c) 1 foot from the dock?
   (d) What happens when the length of rope pulling in the boat is less than 10 feet long?

11. An inverted cylindrical cone, 20ft deep and 10ft across at the top, is being filled with water at a rate of 10ft^3/min. At what rate is the water rising in the tank when the depth of the water is:
   (a) 1 foot?
   (b) 10 feet?
   (c) 19 feet?

   How long will the tank take to fill when starting at empty?

12. A rope, attached to a weight, goes up through a pulley at the ceiling and back down to a worker. The man holds the rope at the same height as the connection point between rope and weight.



Suppose the man stands directly next to the weight (i.e., a total rope length of 60 ft) and begins to walk away at a rate of 2ft/s. How fast is the weight rising when the man has walked:

(a) 10 feet?

(b) 40 feet?

How far must the man walk to raise the weight all the way to the pulley?

13. Consider the situation described in Exercise 12. Suppose the man starts 40ft from the weight and begins to walk away at a rate of 2ft/s.

(a) How long is the rope?

(b) How fast is the weight rising after the man has walked 10 feet?

(c) How fast is the weight rising after the man has walked 30 feet?

(d) How far must the man walk to raise the weight all the way to the pulley?

14. A hot air balloon lifts off from ground rising vertically. From 100 feet away, a 5' woman tracks the path of the balloon. When her sightline with the balloon makes a 45° angle with the horizontal, she notes the angle is increasing at about 5°/min.

(a) What is the elevation of the balloon?

(b) How fast is it rising?

15. A company that produces landscaping materials is dumping sand into a conical pile. The sand is being poured at a rate of 5ft^3/sec; the physical properties of the sand, in conjunction with gravity, ensure that the cone's height is roughly 2/3 the length of the diameter of the circular base.

How fast is the cone rising when it has a height of 30 feet?

4.3 Optimization

In Section 3.1 we learned about extreme values – the largest and smallest values a function attains on an interval. We motivated our interest in such values by discussing how it made sense to want to know the highest/lowest values of a stock, or the fastest/slowest an object was moving. In this section we apply the concepts of extreme values to solve "word problems," i.e., problems stated in terms of situations that require us to create the appropriate mathematical framework in which to solve the problem.

We start with a classic example which is followed by a discussion of the topic of optimization.

Example 4.3.1 Optimization: perimeter and area

A man has 100 feet of fencing, a large yard, and a small dog. He wants to create a rectangular enclosure for his dog with the fencing that provides the maximal area. What dimensions provide the maximal area?

Solution One can likely guess the correct answer – that is great. We will proceed to show how calculus can provide this answer in a context that proves this answer is correct.

It helps to make a sketch of the situation. Our enclosure is sketched twice in Figure 4.3.1, either with green grass and nice fence boards or as a simple rectangle. Either way, drawing a rectangle forces us to realize that we need to know the dimensions of this rectangle so we can create an area function – after all, we are trying to maximize the area.

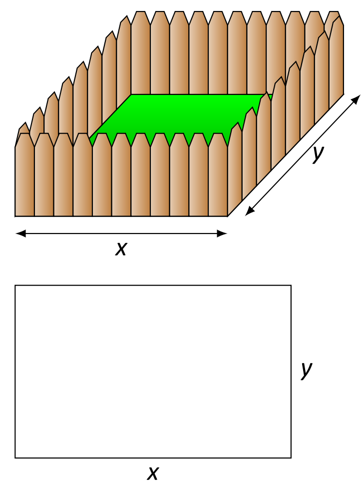


Figure 4.3.1: A sketch of the enclosure in Example 4.3.1.

We let x and y denote the lengths of the sides of the rectangle. Clearly,

$$\text{Area} = xy.$$

We do not yet know how to handle functions with 2 variables; we need to reduce this down to a single variable. We know more about the situation: the man has 100 feet of fencing. By knowing the perimeter of the rectangle must be 100, we can create another equation:

$$\text{Perimeter} = 100 = 2x + 2y.$$

We now have 2 equations and 2 unknowns. In the latter equation, we solve for y:

$$y = 50 - x.$$

Now substitute this expression for y in the area equation:

$$\text{Area} = A(x) = x(50 - x).$$

Note we now have an equation of one variable; we can truly call the Area a function of x.

Notes:

This function only makes sense when $0 \leq x \leq 50$, otherwise we get negative values of area. So we find the extreme values of $A(x)$ on the interval $[0, 50]$.

To find the critical points, we take the derivative of $A(x)$ and set it equal to 0, then solve for x.

$$\begin{aligned} A(x) &= x(50 - x) \\ &= 50x - x^2 \\ A'(x) &= 50 - 2x \end{aligned}$$

We solve $50 - 2x = 0$ to find $x = 25$; this is the only critical point. We evaluate $A(x)$ at the endpoints of our interval and at this critical point to find the extreme values; in this case, all we care about is the maximum.

Clearly $A(0) = 0$ and $A(50) = 0$, whereas $A(25) = 625\text{ft}^2$. This is the maximum. Since we earlier found $y = 50 - x$, we find that y is also 25. Thus the dimensions of the rectangular enclosure with perimeter of 100 ft. with maximum area is a square, with sides of length 25 ft.

This example is very simplistic and a bit contrived. (After all, most people create a design then buy fencing to meet their needs, and not buy fencing and plan later.) But it models well the necessary process: create equations that describe a situation, reduce an equation to a single variable, then find the needed extreme value.

"In real life," problems are much more complex. The equations are often *not* reducible to a single variable (hence multi–variable calculus is needed) and the equations themselves may be difficult to form. Understanding the principles here will provide a good foundation for the mathematics you will likely encounter later.

We outline here the basic process of solving these optimization problems.

Key Idea 4.3.1 Solving Optimization Problems

1. Understand the problem. Clearly identify what quantity is to be maximized or minimized. Make a sketch if helpful.
2. Create equations relevant to the context of the problem, using the information given. (One of these should describe the quantity to be optimized. We'll call this the *fundamental equation.*)
3. If the fundamental equation defines the quantity to be optimized as a function of more than one variable, reduce it to a single variable function using substitutions derived from the other equations.

(continued)...

Notes:

Key Idea 4.3.1 Solving Optimization Problems – Continued

4. Identify the domain of this function, keeping in mind the context of the problem.
5. Find the extreme values of this function on the determined domain.
6. Identify the values of all relevant quantities of the problem.

We will use Key Idea 4.3.1 in a variety of examples.

Example 4.3.2 Optimization: perimeter and area

Here is another classic calculus problem: A woman has a 100 feet of fencing, a small dog, and a large yard that contains a stream (that is mostly straight). She wants to create a rectangular enclosure with maximal area that uses the stream as one side. (Apparently her dog won't swim away.) What dimensions provide the maximal area?

Solution We will follow the steps outlined by Key Idea 4.3.1.

1. We are maximizing *area*. A sketch of the region will help; Figure 4.3.2 gives two sketches of the proposed enclosed area. A key feature of the sketches is to acknowledge that one side is not fenced.
2. We want to maximize the area; as in the example before,
$$\text{Area} = xy.$$
This is our fundamental equation. This defines area as a function of two variables, so we need another equation to reduce it to one variable.
We again appeal to the perimeter; here the perimeter is
$$\text{Perimeter} = 100 = x + 2y.$$
Note how this is different than in our previous example.
3. We now reduce the fundamental equation to a single variable. In the perimeter equation, solve for y: $y = 50 - x/2$. We can now write Area as
$$\text{Area} = A(x) = x(50 - x/2) = 50x - \frac{1}{2}x^2.$$
Area is now defined as a function of one variable.

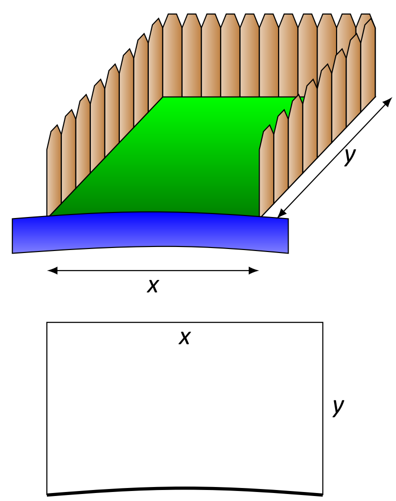


Figure 4.3.2: A sketch of the enclosure in Example 4.3.2.

Notes:

4. We want the area to be nonnegative. Since $A(x) = x(50 - x/2)$, we want $x \geq 0$ and $50 - x/2 \geq 0$. The latter inequality implies that $x \leq 100$, so $0 \leq x \leq 100$.

5. We now find the extreme values. At the endpoints, the minimum is found, giving an area of 0.

   Find the critical points. We have $A'(x) = 50 - x$; setting this equal to 0 and solving for x returns $x = 50$. This gives an area of

   $$A(50) = 50(25) = 1250.$$

6. We earlier set $y = 50 - x/2$; thus $y = 25$. Thus our rectangle will have two sides of length 25 and one side of length 50, with a total area of 1250 ft^2.

Keep in mind as we do these problems that we are practicing a *process*; that is, we are learning to turn a situation into a system of equations. These equations allow us to write a certain quantity as a function of one variable, which we then optimize.

Example 4.3.3 Optimization: minimizing cost

A power line needs to be run from a power station located on the beach to an offshore facility. Figure 4.3.3 shows the distances between the power station to the facility.

It costs \$50/ft. to run a power line along the land, and \$130/ft. to run a power line under water. How much of the power line should be run along the land to minimize the overall cost? What is the minimal cost?

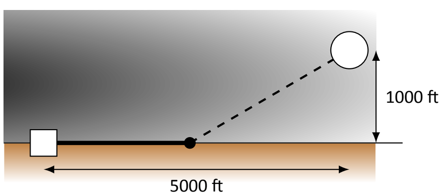


Figure 4.3.3: Running a power line from the power station to an offshore facility with minimal cost in Example 4.3.3.

SOLUTION We will follow the strategy of Key Idea 4.3.1 implicitly, without specifically numbering steps.

There are two immediate solutions that we could consider, each of which we will reject through "common sense." First, we could minimize the distance by directly connecting the two locations with a straight line. However, this requires that all the wire be laid underwater, the most costly option. Second, we could minimize the underwater length by running a wire all 5000 ft. along the beach, directly across from the offshore facility. This has the undesired effect of having the longest distance of all, probably ensuring a non–minimal cost.

The optimal solution likely has the line being run along the ground for a while, then underwater, as the figure implies. We need to label our unknown distances – the distance run along the ground and the distance run underwater. Recognizing that the underwater distance can be measured as the hypotenuse of a right triangle, we choose to label the distances as shown in Figure 4.3.4.

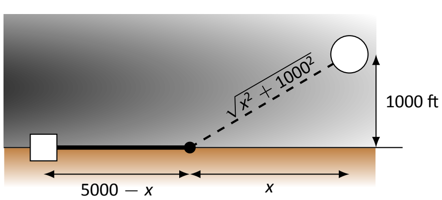


Figure 4.3.4: Labeling unknown distances in Example 4.3.3.

Notes:

By choosing x as we did, we make the expression under the square root simple. We now create the cost function.

$$\begin{aligned}
\text{Cost} &= \text{land cost} &+& \quad \text{water cost} \\
& \$50 \times \text{land distance} &+& \quad \$130 \times \text{water distance} \\
& 50(5000 - x) &+& \quad 130\sqrt{x^2 + 1000^2}.
\end{aligned}$$

So we have $c(x) = 50(5000 - x) + 130\sqrt{x^2 + 1000^2}$. This function only makes sense on the interval $[0, 5000]$. While we are fairly certain the endpoints will not give a minimal cost, we still evaluate $c(x)$ at each to verify.

$$c(0) = 380,000 \qquad c(5000) \approx 662,873.$$

We now find the critical values of $c(x)$. We compute $c'(x)$ as

$$c'(x) = -50 + \frac{130x}{\sqrt{x^2 + 1000^2}}.$$

Recognize that this is never undefined. Setting $c'(x) = 0$ and solving for x, we have:

$$\begin{aligned}
-50 + \frac{130x}{\sqrt{x^2 + 1000^2}} &= 0 \\
\frac{130x}{\sqrt{x^2 + 1000^2}} &= 50 \\
\frac{130^2x^2}{x^2 + 1000^2} &= 50^2 \\
130^2x^2 &= 50^2(x^2 + 1000^2) \\
130^2x^2 - 50^2x^2 &= 50^2 \cdot 1000^2 \\
(130^2 - 50^2)x^2 &= 50,000^2 \\
x^2 &= \frac{50,000^2}{130^2 - 50^2} \\
x &= \frac{50,000}{\sqrt{130^2 - 50^2}} \\
x &= \frac{50,000}{120} = \frac{1250}{3} \approx 416.67.
\end{aligned}$$

Evaluating $c(x)$ at $x = 416.67$ gives a cost of about \$370,000. The distance the power line is laid along land is $5000 - 416.67 = 4583.33$ ft., and the underwater distance is $\sqrt{416.67^2 + 1000^2} \approx 1083$ ft.

Notes:

In the exercises you will see a variety of situations that require you to combine problem–solving skills with calculus. Focus on the *process*; learn how to form equations from situations that can be manipulated into what you need. Eschew memorizing how to do "this kind of problem" as opposed to "that kind of problem." Learning a process will benefit one far longer than memorizing a specific technique.

The next section introduces our final application of the derivative: *differentials*. Given $y = f(x)$, they offer a method of approximating the change in y after x changes by a small amount.

Notes:

Exercises 4.3

Terms and Concepts

1. T/F: An "optimization problem" is essentially an "extreme values" problem in a "story problem" setting.

2. T/F: This section teaches one to find the extreme values of a function that has more than one variable.

Problems

3. Find the maximum product of two numbers (not necessarily integers) that have a sum of 100.

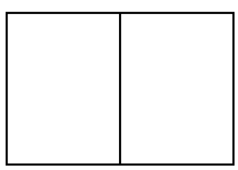
4. Find the minimum sum of two positive numbers whose product is 500.

5. Find the maximum sum of two positive numbers whose product is 500.

6. Find the maximum sum of two numbers, each of which is in $[0, 300]$ whose product is 500.

7. Find the maximal area of a right triangle with hypotenuse of length 1.

8. A rancher has 1000 feet of fencing in which to construct adjacent, equally sized rectangular pens. What dimensions should these pens have to maximize the enclosed area?



9. A standard soda can is roughly cylindrical and holds 355cm^3 of liquid. What dimensions should the cylinder be to minimize the material needed to produce the can? Based on your dimensions, determine whether or not the standard can is produced to minimize the material costs.

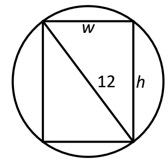
10. Find the dimensions of a cylindrical can with a volume of 206in^3 that minimizes the surface area.

    The "#10 can"is a standard sized can used by the restaurant industry that holds about 206in^3 with a diameter of 6 2/16in and height of 7in. Does it seem these dimensions were chosen with minimization in mind?

11. The United States Postal Service charges more for boxes whose combined length and girth exceeds 108" (the "length" of a package is the length of its longest side; the girth is the perimeter of the cross section, i.e., $2w + 2h$).

    What is the maximum volume of a package with a square cross section ($w = h$) that does not exceed the 108" standard?

12. The strength S of a wooden beam is directly proportional to its cross sectional width w and the square of its height h; that is, $S = kwh^2$ for some constant k.



    Given a circular log with diameter of 12 inches, what sized beam can be cut from the log with maximum strength?

13. A power line is to be run to an offshore facility in the manner described in Example 4.3.3. The offshore facility is 2 miles at sea and 5 miles along the shoreline from the power plant. It costs \$50,000 per mile to lay a power line underground and \$80,000 to run the line underwater.

    How much of the power line should be run underground to minimize the overall costs?

14. A power line is to be run to an offshore facility in the manner described in Example 4.3.3. The offshore facility is 5 miles at sea and 2 miles along the shoreline from the power plant. It costs \$50,000 per mile to lay a power line underground and \$80,000 to run the line underwater.

    How much of the power line should be run underground to minimize the overall costs?

15. A woman throws a stick into a lake for her dog to fetch; the stick is 20 feet down the shore line and 15 feet into the water from there. The dog may jump directly into the water and swim, or run along the shore line to get closer to the stick before swimming. The dog runs about 22ft/s and swims about 1.5ft/s.

    How far along the shore should the dog run to minimize the time it takes to get to the stick? (Hint: the figure from Example 4.3.3 can be useful.)

16. A woman throws a stick into a lake for her dog to fetch; the stick is 15 feet down the shore line and 30 feet into the water from there. The dog may jump directly into the water and swim, or run along the shore line to get closer to the stick before swimming. The dog runs about 22ft/s and swims about 1.5ft/s.

    How far along the shore should the dog run to minimize the time it takes to get to the stick? *(Google "calculus dog" to learn more about a dog's ability to minimize times.)*

17. What are the dimensions of the rectangle with largest area that can be drawn inside the unit circle?

4.4 Differentials

In Section 2.2 we explored the meaning and use of the derivative. This section starts by revisiting some of those ideas.

Recall that the derivative of a function f can be used to find the slopes of lines tangent to the graph of f. At $x = c$, the tangent line to the graph of f has equation

$$y = f'(c)(x - c) + f(c).$$

The tangent line can be used to find good approximations of $f(x)$ for values of x near c.

For instance, we can approximate $\sin 1.1$ using the tangent line to the graph of $f(x) = \sin x$ at $x = \pi/3 \approx 1.05$. Recall that $\sin(\pi/3) = \sqrt{3}/2 \approx 0.866$, and $\cos(\pi/3) = 1/2$. Thus the tangent line to $f(x) = \sin x$ at $x = \pi/3$ is:

$$\ell(x) = \frac{1}{2}(x - \pi/3) + 0.866.$$

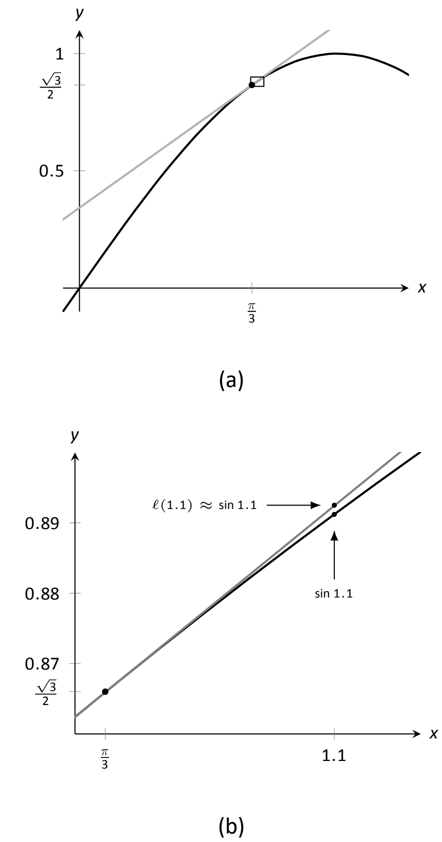


Figure 4.4.1: Graphing $f(x) = \sin x$ and its tangent line at $x = \pi/3$ in order to estimate $\sin 1.1$.

In Figure 4.4.1(a), we see a graph of $f(x) = \sin x$ graphed along with its tangent line at $x = \pi/3$. The small rectangle shows the region that is displayed in Figure 4.4.1(b). In this figure, we see how we are approximating $\sin 1.1$ with the tangent line, evaluated at 1.1. Together, the two figures show how close these values are.

Using this line to approximate $\sin 1.1$, we have:

$$\begin{aligned}\ell(1.1) &= \frac{1}{2}(1.1 - \pi/3) + 0.866\\ &= \frac{1}{2}(0.053) + 0.866 = 0.8925.\end{aligned}$$

(We leave it to the reader to see how good of an approximation this is.)

We now generalize this concept. Given $f(x)$ and an x–value c, the tangent line is $\ell(x) = f'(c)(x - c) + f(c)$. Clearly, $f(c) = \ell(c)$. Let Δx be a small number, representing a small change in x value. We assert that:

$$f(c + \Delta x) \approx \ell(c + \Delta x),$$

since the tangent line to a function approximates well the values of that function near $x = c$.

As the x-value changes from c to $c + \Delta x$, the y-value of f changes from $f(c)$ to $f(c + \Delta x)$. We call this change of y value Δy. That is:

$$\Delta y = f(c + \Delta x) - f(c).$$

Notes:

Replacing $f(c+\Delta x)$ with its tangent line approximation, we have

$$\begin{aligned}\Delta y &\approx \ell(c+\Delta x) - f(c)\\ &= f'(c)\big((c+\Delta x) - c\big) + f(c) - f(c)\\ &= f'(c)\Delta x \qquad (4.3)\end{aligned}$$

This final equation is important; it becomes the basis of the upcoming Definition and Key Idea. In short, it says that when the x-value changes from c to $c+\Delta x$, the y value of a function f changes by about $f'(c)\Delta x$.

We introduce two new variables, dx and dy in the context of a formal definition.

Definition 4.4.1 Differentials of x and y.

Let $y = f(x)$ be differentiable. The **differential of** x, denoted dx, is any nonzero real number (usually taken to be a small number). The **differential of** y, denoted dy, is

$$dy = f'(x)dx.$$

We can solve for $f'(x)$ in the above equation: $f'(x) = dy/dx$. This states that the derivative of f with respect to x is the differential of y divided by the differential of x; this is **not** the alternate notation for the derivative, $\frac{dy}{dx}$. This latter notation was chosen because of the fraction–like qualities of the derivative, but again, it is one symbol and not a fraction.

It is helpful to organize our new concepts and notations in one place.

Key Idea 4.4.1 Differential Notation

Let $y = f(x)$ be a differentiable function.

1. Let Δx represent a small, nonzero change in x value.
2. Let dx represent a small, nonzero change in x value (i.e., $\Delta x = dx$).
3. Let Δy be the change in y value as x changes by Δx; hence
$$\Delta y = f(x+\Delta x) - f(x).$$
4. Let $dy = f'(x)dx$ which, by Equation (4.3), is an *approximation* of the change in y value as x changes by Δx; $dy \approx \Delta y$.

Notes:

What is the value of differentials? Like many mathematical concepts, differentials provide both practical and theoretical benefits. We explore both here.

Example 4.4.1 Finding and using differentials
Consider $f(x) = x^2$. Knowing $f(3) = 9$, approximate $f(3.1)$.

Solution The x value is changing from $x = 3$ to $x = 3.1$; therefore, we see that $dx = 0.1$. If we know how much the y value changes from $f(3)$ to $f(3.1)$ (i.e., if we know Δy), we will know exactly what $f(3.1)$ is (since we already know $f(3)$). We can approximate Δy with dy.

$$\begin{aligned}\Delta y &\approx dy \\ &= f'(3)dx \\ &= 2 \cdot 3 \cdot 0.1 = 0.6.\end{aligned}$$

We expect the y value to change by about 0.6, so we approximate $f(3.1) \approx 9.6$.

We leave it to the reader to verify this, but the preceding discussion links the differential to the tangent line of $f(x)$ at $x = 3$. One can verify that the tangent line, evaluated at $x = 3.1$, also gives $y = 9.6$.

Of course, it is easy to compute the actual answer (by hand or with a calculator): $3.1^2 = 9.61$. (Before we get too cynical and say “Then why bother?”, note our approximation is *really* good!)

So why bother?

In “most” real life situations, we do not know the function that describes a particular behavior. Instead, we can only take measurements of how things change – measurements of the derivative.

Imagine water flowing down a winding channel. It is easy to measure the speed and direction (i.e., the *velocity*) of water at any location. It is very hard to create a function that describes the overall flow, hence it is hard to predict where a floating object placed at the beginning of the channel will end up. However, we can *approximate* the path of an object using differentials. Over small intervals, the path taken by a floating object is essentially linear. Differentials allow us to approximate the true path by piecing together lots of short, linear paths. This technique is called Euler’s Method, studied in introductory Differential Equations courses.

We use differentials once more to approximate the value of a function. Even though calculators are very accessible, it is neat to see how these techniques can sometimes be used to easily compute something that looks rather hard.

Notes:

Example 4.4.2 Using differentials to approximate a function value
Approximate $\sqrt{4.5}$.

Solution We expect $\sqrt{4.5} \approx 2$, yet we can do better. Let $f(x) = \sqrt{x}$, and let $c = 4$. Thus $f(4) = 2$. We can compute $f'(x) = 1/(2\sqrt{x})$, so $f'(4) = 1/4$.

We approximate the difference between $f(4.5)$ and $f(4)$ using differentials, with $dx = 0.5$:

$$f(4.5) - f(4) = \Delta y \approx dy = f'(4) \cdot dx = 1/4 \cdot 1/2 = 1/8 = 0.125.$$

The approximate change in f from $x = 4$ to $x = 4.5$ is 0.125, so we approximate $\sqrt{4.5} \approx 2.125$.

Differentials are important when we discuss *integration*. When we study that topic, we will use notation such as

$$\int f(x)\, dx$$

quite often. While we don't discuss here what all of that notation means, note the existence of the differential dx. Proper handling of *integrals* comes with proper handling of differentials.

In light of that, we practice finding differentials in general.

Example 4.4.3 Finding differentials
In each of the following, find the differential dy.

1. $y = \sin x$ 2. $y = e^x(x^2+2)$ 3. $y = \sqrt{x^2+3x-1}$

Solution

1. $y = \sin x$: As $f(x) = \sin x$, $f'(x) = \cos x$. Thus

$$dy = \cos(x)dx.$$

2. $y = e^x(x^2+2)$: Let $f(x) = e^x(x^2+2)$. We need $f'(x)$, requiring the Product Rule.

We have $f'(x) = e^x(x^2+2) + 2xe^x$, so

$$dy = \big(e^x(x^2+2) + 2xe^x\big)dx.$$

Notes:

3. $y = \sqrt{x^2+3x-1}$: Let $f(x) = \sqrt{x^2+3x-1}$; we need $f'(x)$, requiring the Chain Rule.
   We have $f'(x) = \dfrac{1}{2}(x^2+3x-1)^{-\frac{1}{2}}(2x+3) = \dfrac{2x+3}{2\sqrt{x^2+3x-1}}$. Thus

$$dy = \frac{(2x+3)dx}{2\sqrt{x^2+3x-1}}.$$

Finding the differential dy of $y = f(x)$ is really no harder than finding the derivative of f; we just *multiply* $f'(x)$ by dx. It is important to remember that we are not simply adding the symbol "dx" at the end.

We have seen a practical use of differentials as they offer a good method of making certain approximations. Another use is *error propagation.* Suppose a length is measured to be x, although the actual value is $x+\Delta x$ (where Δx is the error, which we hope is small). This measurement of x may be used to compute some other value; we can think of this latter value as $f(x)$ for some function f. As the true length is $x+\Delta x$, one really should have computed $f(x+\Delta x)$. The difference between $f(x)$ and $f(x+\Delta x)$ is the propagated error.

How close are $f(x)$ and $f(x+\Delta x)$? This is a difference in "y" values:

$$f(x+\Delta x) - f(x) = \Delta y \approx dy.$$

We can approximate the propagated error using differentials.

Example 4.4.4 Using differentials to approximate propagated error

A steel ball bearing is to be manufactured with a diameter of 2cm. The manufacturing process has a tolerance of ± 0.1mm in the diameter. Given that the density of steel is about 7.85g/cm^3, estimate the propagated error in the mass of the ball bearing.

Solution The mass of a ball bearing is found using the equation "mass = volume × density." In this situation the mass function is a product of the radius of the ball bearing, hence it is $m = 7.85\frac{4}{3}\pi r^3$. The differential of the mass is

$$dm = 31.4\pi r^2 dr.$$

The radius is to be 1cm; the manufacturing tolerance in the radius is ± 0.05mm, or ± 0.005cm. The propagated error is approximately:

$$\begin{aligned}\Delta m &\approx dm\\ &= 31.4\pi(1)^2(\pm 0.005)\\ &= \pm 0.493\text{g}\end{aligned}$$

Notes:

Is this error significant? It certainly depends on the application, but we can get an idea by computing the *relative error*. The ratio between amount of error to the total mass is

$$\begin{aligned}\frac{dm}{m} &= \pm\frac{0.493}{7.85\frac{4}{3}\pi} \\ &= \pm\frac{0.493}{32.88} \\ &= \pm 0.015,\end{aligned}$$

or $\pm 1.5\%$.

We leave it to the reader to confirm this, but if the diameter of the ball was supposed to be 10cm, the same manufacturing tolerance would give a propagated error in mass of ± 12.33g, which corresponds to a *percent error* of $\pm 0.188\%$. While the amount of error is much greater ($12.33 > 0.493$), the percent error is much lower.

We first learned of the derivative in the context of instantaneous rates of change and slopes of tangent lines. We furthered our understanding of the power of the derivative by studying how it relates to the graph of a function (leading to ideas of increasing/decreasing and concavity). This chapter has put the derivative to yet more uses:

- Equation solving (Newton's Method),
- Related Rates (furthering our use of the derivative to find instantaneous rates of change),
- Optimization (applied extreme values), and
- Differentials (useful for various approximations and for something called integration).

In the next chapters, we will consider the "reverse" problem to computing the derivative: given a function *f*, can we find a function whose derivative is *f*? Being able to do so opens up an incredible world of mathematics and applications.

Notes:

Exercises 4.4

Terms and Concepts

1. T/F: Given a differentiable function $y = f(x)$, we are generally free to choose a value for dx, which then determines the value of dy.

2. T/F: The symbols "dx" and "Δx" represent the same concept.

3. T/F: The symbols "dy" and "Δy" represent the same concept.

4. T/F: Differentials are important in the study of integration.

5. How are differentials and tangent lines related?

6. T/F: In real life, differentials are used to approximate function values when the function itself is not known.

Problems

In Exercises 7 – 16, use differentials to approximate the given value by hand.

7. 2.05^2

8. 5.93^2

9. 5.1^3

10. 6.8^3

11. $\sqrt{16.5}$

12. $\sqrt{24}$

13. $\sqrt[3]{63}$

14. $\sqrt[3]{8.5}$

15. $\sin 3$

16. $e^{0.1}$

In Exercises 17 – 30, compute the differential *dy*.

17. $y = x^2 + 3x - 5$

18. $y = x^7 - x^5$

19. $y = \dfrac{1}{4x^2}$

20. $y = (2x + \sin x)^2$

21. $y = x^2 e^{3x}$

22. $y = \dfrac{4}{x^4}$

23. $y = \dfrac{2x}{\tan x + 1}$

24. $y = \ln(5x)$

25. $y = e^x \sin x$

26. $y = \cos(\sin x)$

27. $y = \dfrac{x+1}{x+2}$

28. $y = 3^x \ln x$

29. $y = x \ln x - x$

30. $f(x) = \ln\big(\sec x\big)$

Exercises 31 – 34 use differentials to approximate propagated error.

31. A set of plastic spheres are to be made with a diameter of 1cm. If the manufacturing process is accurate to 1mm, what is the propagated error in volume of the spheres?

32. The distance, in feet, a stone drops in t seconds is given by $d(t) = 16t^2$. The depth of a hole is to be approximated by dropping a rock and listening for it to hit the bottom. What is the propagated error if the time measurement is accurate to $2/10^{\text{ths}}$ of a second and the measured time is:

    (a) 2 seconds?

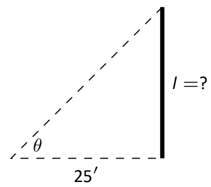
    (b) 5 seconds?

33. What is the propagated error in the measurement of the cross sectional area of a circular log if the diameter is measured at $15''$, accurate to $1/4''$?

34. A wall is to be painted that is $8'$ high and is measured to be $10',\ 7''$ long. Find the propagated error in the measurement of the wall's surface area if the measurement is accurate to $1/2''$.

Exercises 35 – 39 explore some issues related to surveying in which distances are approximated using other measured distances and measured angles. *(Hint: Convert all angles to radians before computing.)*

35. The length l of a long wall is to be approximated. The angle θ, as shown in the diagram (not to scale), is measured to be 85.2°, accurate to 1°. Assume that the triangle formed is a right triangle.



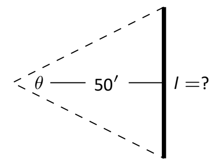
(a) What is the measured length l of the wall?

(b) What is the propagated error?

(c) What is the percent error?

36. Answer the questions of Exercise 35, but with a measured angle of 71.5°, accurate to 1°, measured from a point $100'$ from the wall.

37. The length l of a long wall is to be calculated by measuring the angle θ shown in the diagram (not to scale). Assume the formed triangle is an isosceles triangle. The measured angle is 143°, accurate to 1°.



(a) What is the measured length of the wall?

(b) What is the propagated error?

(c) What is the percent error?

38. The length of the walls in Exercises 35 – 37 are essentially the same. Which setup gives the most accurate result?

39. Consider the setup in Exercise 37. This time, assume the angle measurement of 143° is exact but the measured $50'$ from the wall is accurate to $6''$. What is the approximate percent error?