Cálculo II
PURO-UFF
Notas sobre duas técnicas de integração:
substituição trigonométrica e frações parciais
Prof: Eduardo Ochs
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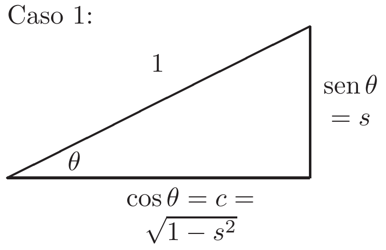
Abreviações: quando θ é uma variável,
$s = \operatorname{sen}\theta$
$c = \cos\theta$
$t = \tan\theta = \frac{\operatorname{sen}\theta}{\cos\theta} = \frac{s}{c}$
$z = \sec\theta = \frac{1}{\cos\theta} = \frac{1}{c}$

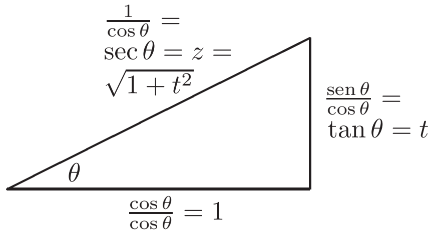
Identidades:
$t^2 = \frac{s^2}{c^2} = \frac{1-c^2}{c^2}$
$t^2c^2 = 1 - c^2$
$t^2c^2 + c^2 = 1$
$(1+t^2)c^2 = 1$
$1 + t^2 = \frac{1}{c^2} = z^2$
$z^2 = 1 + t^2$
$z = \sqrt{1+t^2}$
$t^2 = z^2 - 1$
$t = \sqrt{z^2 - 1}$

Derivadas e diferenciais:
$\frac{ds}{d\theta} = \frac{d\operatorname{sen}\theta}{d\theta} = \cos\theta = c$
$\frac{dt}{d\theta} = \frac{d}{d\theta}\frac{s}{c} = \frac{s'c - sc'}{c^2} = \frac{c^2+s^2}{c^2} = \frac{1}{c^2} = z^2 = 1 + t^2$
$\frac{dz}{d\theta} = \frac{d}{d\theta}c^{-1} = -c^{-2}c' = -c^{-2}(-s) = \frac{1}{c}\frac{s}{c} = zt$
$ds = c\,d\theta = \sqrt{1-s^2}d\theta$
$dt = z^2d\theta = (1+t^2)d\theta$
$dz = zt\,d\theta$

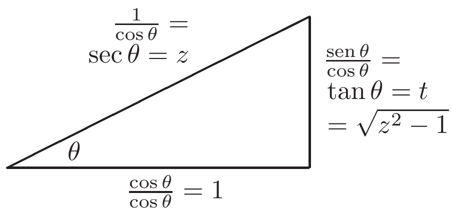
Caso 1:



Caso 2:



Caso 3:



Exemplos (adaptados do Munem, pp.492–493):

$$\begin{aligned}
\int \frac{s^2}{(1-s^2)^{3/2}}\,ds &= \int \frac{s^2}{(1-s^2)^{3/2}}(1-s^2)^{1/2}\,d\theta \\
&= \int \frac{s^2}{1-s^2}\,d\theta \\
&= \int \frac{(\operatorname{sen}\theta)^2}{(\cos\theta)^2}\,d\theta \\
&= \int (\tan\theta)^2\,d\theta
\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{t^2\sqrt{1+t^2}}\,dt &= \int \frac{1}{t^2\sqrt{1+t^2}}(1+t^2)\,d\theta \\
&= \int \frac{\sqrt{1+t^2}}{t^2}\,d\theta \\
&= \int \frac{(1/c)}{(s^2/c^2)}\,d\theta = \int \frac{1}{c}\frac{c^2}{s^2}\,d\theta = \int \frac{c}{s^2}\,d\theta
\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{z^3\sqrt{z^2-1}}\,dz &= \int \frac{1}{z^3\sqrt{z^2-1}}zt\,d\theta \\
&= \int \frac{zt}{z^3t}\,d\theta \\
&= \int \frac{1}{z2}\,d\theta \\
&= \int c^2\,d\theta
\end{aligned}$$

Um exercício de frações parciais:
usando esta notação,
$\begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = [a_2, a_1, a_0] = a_2x^2 + a_1x + a_0$,
$\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = [a_3, a_2, a_1, a_0] = a_3x^3 + a_2x^2 + a_1x + a_0$, etc,
descubra que valores de k_1, k_0, c_1, c_2 e c_3 fazem a conta abaixo fazer sentido:

$$k_1x + k_0 + \frac{c_1}{x+2} + \frac{c_2}{x+1} + \frac{c_3}{(x+1)^2} =$$

$$= \frac{k_1 \begin{bmatrix} 1 \\ 5 \\ 7 \\ 3 \\ 0 \end{bmatrix} + k_0 \begin{bmatrix} 0 \\ 1 \\ 5 \\ 7 \\ 3 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}}{[1, 5, 7, 3]}$$

$$= \frac{\begin{array}{rrl} & 10000 & x^4 \\ + & 51000 & x^3 \\ + & 75110 & x^2 \\ + & 37231 & x \\ + & 3122 & \end{array}}{x^3 + 5x^2 + 7x + 3}$$

E agora encontre uma primitiva para:

$$\int \frac{10000x^4 + 51000x^3 + 75110x^2 + 37231x + 3122}{x^3 + 5x^2 + 7x + 3}\, dx$$